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Parametric Uncertainty Modeling for Application to Robust Control

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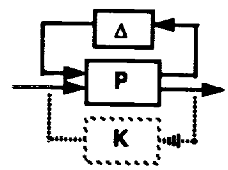
Presentation Outline

- *Parametric Uncertainty Modeling*
- *Multilinear Solution Framework*
  - *- Results*
  - *- Example*
  - *- Extension to Rational Case*
- *Concluding Remarks*
- *Further Work*

Parametric Uncertainty Modeling

Motivation:

- **Robust Control Theory & Tools**
  - Required Uncertainty Model Structure:
    - → Separated P-Δ Form:
  - Computational Efficiency Depends on Dimension of Δ Block
    - → Minimal P-Δ Model Desired:

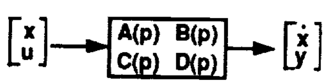


- **Practical Robust Control Applications**
  - P-Δ Model Difficult to Form for Real Parameter Variations
  - No General Systematic Approach for Minimal P-Δ Modeling
    - → Multidimensional Minimal Realization Problem
      - ⇒ Problem to be Addressed in this Paper

Parametric Uncertainty Modeling (cont)

General Problem Definition:

***Given* State Space Model of Uncertain System:**



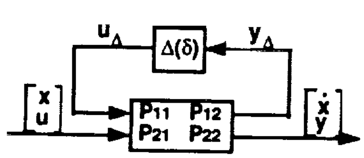
$$\dot{x} = A(p)\,x + B(p)\,u$$
$$y = C(p)\,x + D(p)\,u$$

Any Element of A(p), B(p), C(p), D(p) → Explicit Function* of p:

Uncertain Parameters: $p = [\,p_1, p_2, \cdots, p_m\,]$

$$p_{i_{min}} \le p_i \le p_{i_{max}} \longrightarrow p_i = p_{i_o} + \bar{\delta}_i = p_{i_o} + s_i\delta_i \;,\; |\,\delta_i\,| \le 1$$

***Form* a P-Δ Uncertainty Model:**



P - Constant Matrices

Δ(δ) - Uncertain Parameters

$$\Delta(\delta) = \mathrm{diag}(\,\delta_1 I_1, \delta_2 I_2, \ldots, \delta_m I_m\,)$$

Parametric Uncertainty Modeling (cont)

General Problem (cont):

Any Element of $A(p)$, $B(p)$, $C(p)$, $D(p)$ $\rightarrow$ Explicit Function* of p

*Explicit Functional Forms: — Example:

Linear Function**	$a_{ij}(p) = p_1 + p_2 a_o$
Multilinear Function	$a_{ij}(p) = p_1 + p_1 p_2 a_o$
Rational Function	$a_{ij}(p) = \dfrac{p_1 + p_2 a_o + p_2^2 p_3}{p_1 p_3 + a_1 p_4}$
$\vdots$	$\vdots$

** Formal Solution by Morton & McAfoos (1985 ACC & CDC)

$\Rightarrow$ **Many Practical Problems:**

Multilinear (Rational, ...)

Objective

***Develop**: Systematic Method for Obtaining a P-Δ Model*
***Given**: State-Space Model of a MIMO Uncertain System*
***such that**:*

- Any Element of $A(p)$, $B(p)$, $C(p)$, $D(p)$ is a *Multilinear* Function of p:

$$a_{ij}(p) = p_1 + p_1 p_2 a_o$$

- The Resulting P-Δ Model is *Minimal (or Near Minimal)*, i.e.:

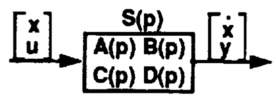
$$\Rightarrow \quad \Delta(\delta) = \mathrm{diag}(\delta_1 I_1, \delta_2 I_2, \ldots, \delta_m I_m)$$

has Minimal Dimension for the Given State-Space Model

***Extend**: Multilinear Results to Rational Case*

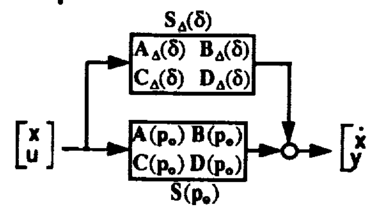
General Solution Framework

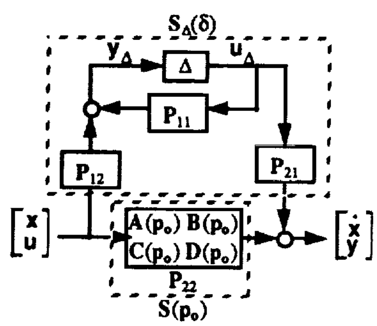
Block Diagram Perspective:



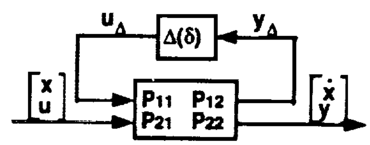
$$\begin{bmatrix}\dot{x}\\ y\end{bmatrix} = S(p)\begin{bmatrix}x\\ u\end{bmatrix} = \left(S(p_o) + S_\Delta(\delta)\right)\begin{bmatrix}x\\ u\end{bmatrix}$$

$\Rightarrow$





$\Rightarrow$



$$\begin{bmatrix}\dot{x}\\ y\end{bmatrix} = P_{22}\begin{bmatrix}x\\ u\end{bmatrix} + P_{21}\,u_\Delta$$

$$y_\Delta = P_{12}\begin{bmatrix}x\\ u\end{bmatrix} + P_{11}\,u_\Delta$$

$$u_\Delta = \Delta(\delta)\,y_\Delta$$

$$\Delta(\delta) = \mathrm{diag}(\,\delta_1 I_1,\ \delta_2 I_2,\ \ldots,\ \delta_m I_m\,)$$

General Solution Framework (cont)

Equating Given & Desired Models:

$$P_{22} = S(p_o)$$

$$\Delta(\delta) = \mathrm{diag}(\,\delta_1 I_1,\ \delta_2 I_2,\ \ldots,\ \delta_m I_m\,)$$

$$P_{21}(I - \Delta(\delta)P_{11})^{-1}\Delta(\delta)P_{12} = S_\Delta(\delta)$$

Solution of P_{21}, P_{12}, & P_{11} Matrices:

$$P_{21}(\,I - \Delta(\delta)\,P_{11})^{-1}\,\Delta(\delta)\,P_{12} = S_\Delta(\delta) = \begin{bmatrix}A_\Delta(\delta)\,B_\Delta(\delta)\\ C_\Delta(\delta)\,D_\Delta(\delta)\end{bmatrix}$$

Unknown Matrix Elements **Known Matrix Elements (Function of δ's)**

General Solution Requires: Direct Matrix Inversion

$$(\,I - \Delta(\delta)\,P_{11}\,)^{-1}$$

$\Rightarrow$ **Symbolic Matrix Inversion & Subsequent Solution <u>Difficult</u> for Many Practical Problems**

Multilinear Solution Framework

$$P_{21}(I - \Delta(\delta)\,P_{11})^{-1}\,\Delta(\delta)\,P_{12} = S_\Delta(\delta) = \begin{bmatrix} A_\Delta(\delta) & B_\Delta(\delta) \\ C_\Delta(\delta) & D_\Delta(\delta) \end{bmatrix}$$

Unknown Matrix Elements (P_{21}, P_{11}, P_{12})

Known Matrix Elements (Multilinear Function of δ's)

Finite Power Series (Exact Solution):

$$(I - \Delta(\delta)\,P_{11})^{-1} = I + (\Delta(\delta)\,P_{11}) + (\Delta(\delta)\,P_{11})^2 + \ldots + (\Delta(\delta)\,P_{11})^r$$

such that: $\boxed{(\Delta(\delta)\,P_{11})^{r+1} = 0}$ ⇒ **Requires Special Structure for P_{11}**

where: r - **Determined by Maximum Crossterm Order in A, B, C, D**

$$\Rightarrow \boxed{S_\Delta(\delta) = \begin{bmatrix} A_\Delta(\delta) & B_\Delta(\delta) \\ C_\Delta(\delta) & D_\Delta(\delta) \end{bmatrix} = P_{21}\,[\,I + \Delta(\delta)\,P_{11} + (\Delta(\delta)\,P_{11})^2 + \ldots + (\Delta(\delta)\,P_{11})^r\,]\,\Delta(\delta)P_{12}}$$

Uncertain Parameter Linear Terms **Uncertain Parameter Crossterms**

Note:

1.) **nth Order Terms** → **Repeated Parameters**

Ex.: $p_i^2 = p_i p_{i+1}$

2.) **Inverse Terms** → **Redefine Parameters**

Ex.: $\frac{1}{p_i} = \bar{p}_i$

Uncertainty Modeling Procedure

To Obtain a *Minimal (or Near Minimal)* P-Δ Uncertainty Model:

0. **Determine P_{22} and Extract $S_\Delta(\delta)$:**

$$P_{22} = S(p_o) = \begin{bmatrix} A(p_o) & B(p_o) \\ C(p_o) & D(p_o) \end{bmatrix}, \quad S_\Delta(\delta) = \begin{bmatrix} A_\Delta(\delta) & B_\Delta(\delta) \\ C_\Delta(\delta) & D_\Delta(\delta) \end{bmatrix}$$

1. **Define Δ Matrix:** $\Delta(\delta) = \mathrm{diag}(\,\delta_1 I_1, \delta_2 I_2, \ldots, \delta_m I_m\,)$

Repeated Parameters *Only* for nth Order Uncertain Parameters

2. **Determine P_{21} and P_{12} Using Linear Terms (Morton & McAfoos):**

$$P_{21}\Delta(\delta)P_{12} = [S_\Delta(\delta)]_0 = \begin{bmatrix} A_\Delta(\delta) & B_\Delta(\delta) \\ C_\Delta(\delta) & D_\Delta(\delta) \end{bmatrix}_0$$

Known Linear Uncertain Parameter Terms Only (No Uncertain Parameter Crossterms)

Modeling Procedure (cont)

3. Determine P_{11} Using Uncertain Parameter Crossterms:

$$P_{21}\,(\Delta(\delta)\,P_{11})^{1}\,\Delta(\delta)P_{12} \;=\; [S_\Delta(\delta)]_1 \;=\; \begin{bmatrix} A_\Delta(\delta) & B_\Delta(\delta) \\ C_\Delta(\delta) & D_\Delta(\delta) \end{bmatrix}_1 \longleftarrow \text{Known First-Order Crossterms}$$

$$P_{21}\,(\Delta(\delta)\,P_{11})^{2}\,\Delta(\delta)P_{12} \;=\; [S_\Delta(\delta)]_2 \;=\; \begin{bmatrix} A_\Delta(\delta) & B_\Delta(\delta) \\ C_\Delta(\delta) & D_\Delta(\delta) \end{bmatrix}_2 \longleftarrow \text{Known Second-Order Crossterms}$$

$$\vdots$$

$$P_{21}\,(\Delta(\delta)\,P_{11})^{r}\,\Delta(\delta)P_{12} \;=\; [S_\Delta(\delta)]_r \;=\; \begin{bmatrix} A_\Delta(\delta) & B_\Delta(\delta) \\ C_\Delta(\delta) & D_\Delta(\delta) \end{bmatrix}_r \longleftarrow \text{Known rth-Order Crossterms}$$

with Nilpotency Condition Satisfied.

If P_{11} Cannot be Found such that ALL of the above Equations and Condition are Satisfied:

a.) Determine which Parameters Need to be Repeated
b.) Repeat Procedure from Step 1 Augmenting Δ Matrix

Once P_{11} has been Determined,
Minimal (or Near Minimal) P-Δ Model Has Been Found

Example

Given Uncertain System Model :

$$A(p) = \begin{bmatrix} -\frac{V_a}{L_u} & 0 & 0 \\ 0 & -\frac{V_a}{L_w} & 0 \\ 0 & \frac{\sigma_w}{L_w^2}\sqrt{\frac{3V_a}{2\pi}} & -\frac{V_a}{L_w} \end{bmatrix} \qquad B(p) = \begin{bmatrix} \frac{\sigma_u}{L_u^2}\sqrt{\frac{V_a}{\pi}} & 0 \\ 0 & -\frac{V_a}{L_w}(1-\frac{1}{\sqrt{3}}) \\ 0 & \frac{\sigma_w}{L_w^2}\sqrt{\frac{3V_a}{2\pi}} \end{bmatrix}$$

$$C(p) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad D(p) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad A(p) = \begin{bmatrix} -V_a\bar{L}_u & 0 & 0 \\ 0 & -V_a\bar{L}_w & 0 \\ 0 & \sigma_w\bar{L}_w^2\sqrt{\frac{3V_a}{2\pi}} & -V_a\bar{L}_w \end{bmatrix} \quad B(p) = \begin{bmatrix} \sigma_u\bar{L}_u^2\sqrt{\frac{V_a}{\pi}} & 0 \\ 0 & -V_a\bar{L}_w(1-\frac{1}{\sqrt{3}}) \\ 0 & \sigma_w\bar{L}_w^2\sqrt{\frac{3V_a}{2\pi}} \end{bmatrix}$$

where: $\bar{L}_u = \frac{1}{L_u}\ ,\ \bar{L}_w = \frac{1}{L_w}$

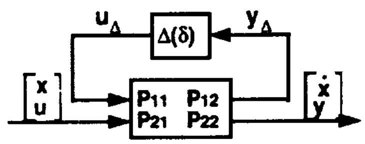
Example (cont)

P-Δ Model Solution:

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = P_{22} \begin{bmatrix} x \\ u \end{bmatrix} + P_{21}\, u_\Delta$$

$$y_\Delta = P_{12} \begin{bmatrix} x \\ u \end{bmatrix} + P_{11}\, u_\Delta$$

$$u_\Delta = \Delta(\delta)\, y_\Delta$$



where:

$$P_{22} = \left[\begin{array}{ccc:cc} -V_a\tilde{L}_{u_o} & 0 & 0 & \sigma_{u_o}\tilde{L}_{u_o}{}^2\sqrt{\frac{V_a}{\pi}} & 0 \\ 0 & -V_a\tilde{L}_{w_o} & 0 & 0 & V_a\tilde{L}_{w_o}(1-\frac{1}{\sqrt{3}}) \\ 0 & \sigma_{w_o}\tilde{L}_{w_o}{}^2\sqrt{\frac{3V_a}{2\pi}} & -V_a\tilde{L}_{w_o} & 0 & \sigma_{w_o}\tilde{L}_{w_o}{}^2\sqrt{\frac{3V_a}{2\pi}} \\ \hdashline 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

Example (cont)

P-Δ Model Solution (cont):

$$P_{21} = \left[\begin{array}{ccccccc} -1 & \sigma_{u_o}\tilde{L}_{u_o} & 0 & 0 & 0 & \tilde{L}_{w_o} & 0 \\ 0 & 0 & -V_a & V_a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \tilde{L}_{w_o} \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$P_{12} = \left[\begin{array}{ccc:cc} s_{\tilde{L}_u}V_a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2s_{\tilde{L}_u}\sqrt{\frac{V_a}{\pi}} & 0 \\ 0 & s_{\tilde{L}_w} & 0 & 0 & s_{\tilde{L}_w} \\ 0 & 0 & 0 & 0 & s_{\tilde{L}_w}\frac{1}{\sqrt{3}} \\ 0 & 2s_{\tilde{L}_w}\sigma_{w_o}\tilde{L}_{w_o}\sqrt{\frac{3V_a}{2\pi}} & -s_{\tilde{L}_w}V_a & 0 & 2s_{\tilde{L}_w}\sigma_{w_o}\tilde{L}_{w_o}\sqrt{\frac{3V_a}{2\pi}} \\ 0 & 0 & 0 & s_{\sigma_u}\sqrt{\frac{V_a}{\pi}} & 0 \\ 0 & s_{\sigma_w}\sqrt{\frac{3V_a}{2\pi}} & 0 & 0 & s_{\sigma_w}\sqrt{\frac{3V_a}{2\pi}} \end{array}\right]$$

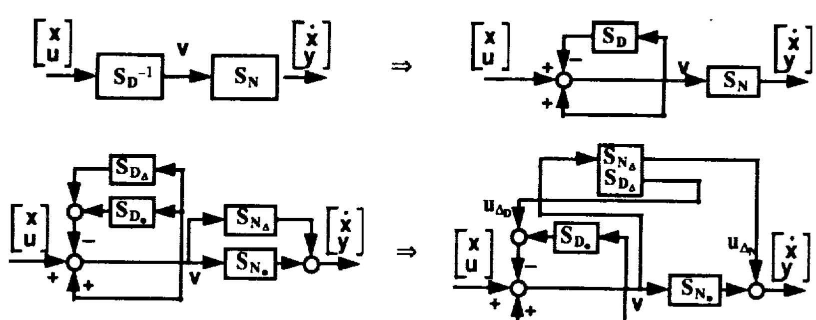
Example (cont)

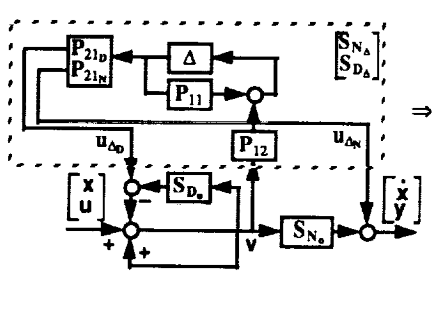
P-Δ Model Solution (cont):

$$
P_{11} = \begin{bmatrix}
0 & -s_{\tilde{L}_u}\frac{\sigma_{u_o}}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{2s_{\tilde{L}_u}}{\sigma_{u_o}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{s_{\tilde{L}_w}}{\sigma_{w_o}}\sqrt{\frac{2\pi}{3V_o}} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{s_{\tilde{L}_w}}{\sigma_{w_o}}\sqrt{\frac{2\pi}{3V_o}} \\
0 & 0 & s_{\tilde{L}_w}\sigma_{w_o}\sqrt{\frac{3V_o}{2\pi}} & 0 & 0 & 0 & 2s_{\tilde{L}_w}\tilde{L}_{w_o} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\Delta(\delta) = \text{diag}\,[\,\delta_{\tilde{L}_u} I_2 \quad \delta_{\tilde{L}_w} I_3 \quad \delta_{\sigma_u} \quad \delta_{\sigma_w}\,]
$$

Extension to Rational Case





$$
\Rightarrow \quad \begin{bmatrix} S_{N_\Delta} \\ S_{D_\Delta} \end{bmatrix} = \begin{bmatrix} P_{21_N} \\ P_{21_D} \end{bmatrix} (I - \Delta P_{11})^{-1} \Delta P_{12}
$$

(Multilinear Problem)

Extension to Rational Case (cont.)

System Equations:

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = S_{N_o} S_{D_o}^{-1} \begin{bmatrix} x \\ u \end{bmatrix} + \left(\begin{bmatrix} P_{21_{N_x}} \\ P_{21_{D_y}} \end{bmatrix} - S_{N_o} S_{D_o}^{-1} \begin{bmatrix} P_{21_{D_x}} \\ P_{21_{D_y}} \end{bmatrix} \right) u_\Delta$$

$$y_\Delta = \begin{bmatrix} P_{12_x} & P_{12_u} \end{bmatrix} S_{D_o}^{-1} \begin{bmatrix} x \\ u \end{bmatrix} + \left(P_{11} - \begin{bmatrix} P_{12_x} & P_{12_u} \end{bmatrix} S_{D_o}^{-1} \begin{bmatrix} P_{21_{D_x}} \\ P_{21_{D_y}} \end{bmatrix} \right) u_\Delta$$

$$u_\Delta = \Delta \, y_\Delta$$

where:

$$S_{N_o} S_{D_o}^{-1} = S_o = \begin{bmatrix} A_o & B_o \\ C_o & D_o \end{bmatrix} = P_{22}$$

$$S_{N_o} = \begin{bmatrix} A_{N_o} & B_{N_o} \\ C_{N_o} & D_{N_o} \end{bmatrix} , \quad S_{D_o}^{-1} = \begin{bmatrix} A_{D_o} & B_{D_o} \\ C_{D_o} & D_{D_o} \end{bmatrix}^{-1}$$

Concluding Remarks

- ***Multilinear Solution Framework***
  - ***Solves Multilinear Parameter Case***
    - ⇒ ***Accomodates nth Order and Inverse Terms***
  - ***Eliminates Symbolic Matrix Inversion in Computation of P_{11}***
    - ⇒ **Computationally Tractable for Symbolic Solution (Symbolic Algebra Tool Required)**
  - ***Can be Extended to Rational Parameter Case***
    - ⇒ ***Preliminary Results***
- ***Systematic Procedure for (Near) Minimal P-Δ Modeling***
  - ***Minimality is Relative to Given State Space Realization***
    - ⇒ **A Lower Dimension P-Δ Model May Exist for Different Realization**
  - ***(Near) Minimality by Construction***
    - ⇒ **Minimality may not Always be Assured**

Further Work

- *Evaluate/Refine/Generalize* **Procedure**
  - ***Wider Class of Problems***
  - ***Multidimensional System Theory***
- *Automate* ***Modeling Procedure***
  - ***Mathematica/Maple***
  - ***Output Files to Matlab***
- *Apply* ***to HSCT Problems***
  - ***Configuration Evaluation***
  - ***Control System Analysis & Design***

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Abstract

Advanced robust control system analysis and design is based on the availability of an uncertainty description which separates the uncertain system elements from the nominal system. Although this modeling structure is relatively straightforward to obtain for multiple unstructured uncertainties modeled throughout the system, it is difficult to formulate for many problems involving real parameter variations. Furthermore, it is difficult to ensure that the uncertainty model is formulated such that the dimension of the resulting model is minimal. This paper presents a procedure for obtaining an uncertainty model for real uncertain parameter problems in which the uncertain parameters can be represented in a multilinear form. Furthermore, the procedure is formulated such that the resulting uncertainty model is minimal (or near minimal) relative to a given state space realization of the system. The approach is demonstrated for a multivariable third-order example problem having four uncertain parameters.

1. Introduction

Advanced robust control system analysis and design is based on the availability of an uncertainty description which separates the uncertain system elements from the nominal system. More specifically, the uncertain system components are contained in a block-diagonal Δ matrix, which is connected to the nominal system, P(s), such that the closed-loop uncertain system is described by a linear fractional transformation (LFT). The idea of separating the uncertain part of a system from its nominal part in this manner, for use in robust control system analysis and design, was first posed by John Doyle (see [3] and [4]), and the robust control theory associated with this structured description of uncertainty continues to be an important area of research. A block diagram of this modeling structure can be depicted as follows in Figure 1:

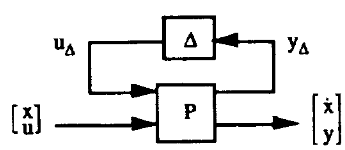

Figure 1. Block Diagram of General Uncertain System

where u contains all external inputs to the system (e.g., disturbances, control inputs, etc.), y contains all outputs from the system (e.g., controlled outputs, measured outputs, etc.) and u_Δ and y_Δ connect the uncertainties represented by Δ to the nominal system, P(s). Although this modeling structure is relatively straightforward to obtain for multiple unstructured uncertainties which occur throughout the system, it is difficult to formulate for many problems involving real parameter variations. Furthermore, it is difficult to ensure that the uncertainty model is formulated such that the dimension of the resulting model is minimal (i.e., the number of repeated parameters in Δ is minimized). Although formulating an uncertainty model is a requirement for utilizing the recently developed robust control analysis and design techniques mentioned above, very little research has been reported in the literature which addresses this problem, particularly for the real parameter uncertainty case. Results to date primarily apply to multiple uncertain parameters which enter the system model in a linear functional form, although some work involving nonlinear special cases have been worked [10]. The results for linear uncertain parameters were first presented in [8] (Morton & McAfoos, 1985) and [9] (Morton, 1985). A later paper [10] (Steinbuch, et. al., 1991) summarizes the general uncertainty modeling problem and the results to date, and presents two simple scalar nonlinear uncertain parameter examples. However, no solution to the general minimal uncertainty modeling problem has been found. The objective of this paper is to present an important extension to these uncertainty modeling results. Specifically, a procedure is presented for obtaining a minimal (or near minimal) uncertainty model (having the form of Figure 1) given the state space realization of an uncertain system with multiple parametric uncertainties entering the model in a multilinear functional form. It should be noted that minimality here is relative to the given state space realization. As discussed in [1] and [2] (Belcastro, et. al., 1989 and 1991), the dimension of the uncertainty model (i.e., the dimension of the Δ matrix) is dependent on the state space realization of the system. Thus, one can consider the minimality of an uncertainty model for a particular state space realization, or one can consider the achievable minimality of the uncertainty model irrespective of the system realization. In this paper, we present a method of obtaining a minimal (or near minimal) uncertainty model relative to the given state space model of the uncertain system for multiple uncertainties entering the model in a multilinear functional form. The multilinear framework significantly reduces the computational complexity involved in obtaining a solution, as compared to solving the problem directly for the rational parameter case. Moreover, it can be shown that the multilinear solution framework can actually be used to solve the rational parameter case, as well. Thus, it provides a means of determining an uncertainty model for many difficult problems of practical interest.

The paper is organized in the following manner. Section 2 presents a formal problem definition for the general uncertain parameter case, briefly summarizes results for the special case of linear parametric uncertainty, and defines the problem to be addressed in this paper. Section 3 summarizes our results for this defined problem, and Section 4 presents an example problem which demonstrates these results. Section 5 briefly discusses the application of the multilinear solution framework to solve the rational uncertain parameter problem, and concluding remarks are given in Section 6.

2. Parametric Uncertainty Modeling: Problem Definition

2.1 General Problem Definition

Consider the state space model of an uncertain system:

$$\dot{x} = A(p)\,x + B(p)\,u\,,\ \ x \in R^{n_x}, u \in R^{n_u} \tag{1a}$$

$$y = C(p)\,x + D(p)\,u\,,\ \ y \in R^{n_y} \tag{1b}$$

where p represents a vector of real uncertain parameters:

$$p = [\,p_1, p_2, \cdots, p_m\,] \in R^m \tag{2}$$

It is assumed that each entry of the model presented in equation (1) is a function of the parameters p. For the general rational case considered in this paper, the uncertain parameters can appear in a rational multivariate functional form within each element of the system model. For example, as given in [10] (Steinbuch et. al., 1991), the $(i,j)^{th}$ entry of the A matrix could have the form:

$$A_{ij}(p) = \frac{p_1 + p_2 a_o + p_2^2 p_3}{p_1 p_3 + a_1 p_4} \tag{3}$$

where a_o and a_1 are constants. It should be noted that n^{th}-order terms are included here because they can be handled within a multilinear framework by defining n-1 additional uncertain parameters which are equal to the parameter being raised to the n^{th} power. For this example, a new uncertain parameter, $p_2' = p_2$, could be defined and p_2^2 would then be replaced by p_2p_2'.

The uncertainty modeling problem consists of three components: scaling of the uncertain parameters, extraction of the uncertainties from the nominal system, and formulation of a linear fractional transformation (LFT) (see [5], Doyle, et. al., 1991 for a review of LFT's). These components are reviewed below.

Uncertainty Scaling:

Each uncertain parameter p_i in p can be bounded by an upper bound, p_{max_i}, and a lower bound, p_{min_i}, as follows:

$$p_{min_i} \le p_i \le p_{max_i} \tag{4}$$

Then the parameter can be written in terms of some nominal value within this range of uncertainty. One way to do this is shown below:

$$p_i = p_{nom_i} + \bar{\delta}_i = p_{nom_i} + s_i\,\delta_i \tag{5}$$

$$p_{nom_i} = \frac{p_{min_i} + p_{max_i}}{2} \tag{6}$$

$$s_i = \frac{p_{max_i} - p_{min_i}}{2} \tag{7}$$

$$|\delta_i| \le 1 \tag{8}$$

Equations (4) - (7) can also be written in vector form by stacking each associated parameter quantity into vectors. The δ_i terms as defined in equations (5) and (8) are the uncertain terms that will be separated into the Δ matrix of Figure 1.

Uncertainty Extraction:

Using equation (5), the state space model of the uncertain system given in (1) can be rewritten in compact form as follows:

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = S(p)\begin{bmatrix} x \\ u \end{bmatrix} = S(p_{nom})\begin{bmatrix} x \\ u \end{bmatrix} + S_\Delta(\delta)\begin{bmatrix} x \\ u \end{bmatrix} \tag{9}$$

where:

$$\delta = [\,\delta_1, \delta_2, \cdots, \delta_m\,] \in R^m \tag{10}$$

$$S(p) = \begin{bmatrix} A(p) & B(p) \\ C(p) & D(p) \end{bmatrix} = S(p_{nom}) + S_\Delta(\delta) \tag{11}$$

$$S(p_{nom}) = \begin{bmatrix} A(p_{nom}) & B(p_{nom}) \\ C(p_{nom}) & D(p_{nom}) \end{bmatrix} \tag{12a}$$

$$S_\Delta(\delta) = \begin{bmatrix} A_\Delta(\delta) & B_\Delta(\delta) \\ C_\Delta(\delta) & D_\Delta(\delta) \end{bmatrix} \tag{12b}$$

Separation of S(p) into nominal and uncertain parts, $S(p_{nom})$ and $S_\Delta(\delta)$, respectively, results in the extraction of the uncertainties from the nominal system.

Formulation of a Linear Fractional Transformation (LFT):

Equation (9) can be rewritten in the form of an upper (time domain) LFT by defining an input vector, u_Δ, and an output vector, y_Δ, associated with the uncertain part of the system as follows:

$$y_\Delta = P_{11}\,u_\Delta + P_{12}\begin{bmatrix} x \\ u \end{bmatrix} \tag{13}$$

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = P_{21}\,u_\Delta + P_{22}\begin{bmatrix} x \\ u \end{bmatrix} \tag{14}$$

$$u_\Delta = \Delta(\delta)\, y_\Delta \tag{15}$$

$$\Delta(\delta) = \mathrm{diag}(\delta_1 I_1, \delta_2 I_2, \ldots, \delta_m I_m) \tag{16a}$$

$$\Delta(\delta) \in R^{n_\Delta \times n_\Delta} \tag{16b}$$

$$n_\Delta = \sum_{i=1}^{m} r_i \;,\; r_i = \dim(I_i) \tag{17}$$

where P_{11}, P_{12}, P_{21}, and P_{22} are constant matrices with $P_{22} = S(p_{nom})$, and the matrices P_{11}, P_{12}, and P_{21} are related to $S_\Delta(\delta)$. The I_i terms in equation (16a) represent the identity matrix with dimension equal to the repeatedness of parameter δ_i. For example, the squared uncertain parameter of equation (3), i.e. p_2^2, results (after scaling) in the term δ_2^2. Thus, this example would require that both δ_2 and $\delta_2' = \delta_2$ (associated with the uncertain parameter p_2' discussed above) appear in Δ, which means that I_2 in equation (16a) would be a 2-dimensional identity matrix.

The objective of the uncertainty modeling problem is to find the matrices P_{11}, P_{12}, and P_{21} such that the system of equations represented by (13) - (16) is equivalent to the system represented by equation (9). To do this, equations (13) - (15) are combined such that u_Δ and y_Δ are eliminated, as follows:

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = [P_{22} + P_{21}(I - \Delta(\delta)P_{11})^{-1}\Delta(\delta)P_{12}]\begin{bmatrix} x \\ u \end{bmatrix} \tag{18}$$

Thus, the uncertainty modeling problem can be thought of as a multi-dimensional (minimal) realization problem defined by the following equation:

$$S_\Delta(\delta) = P_{21}(I - \Delta(\delta)P_{11})^{-1}\,\Delta(\delta)P_{12} \tag{19}$$

where δ represents the uncertain parameter vector defined in equation (10).

2.2 Summary of Results for Linear Parametric Uncertainties

As indicated previously in this paper, uncertainty modeling results have primarily focused on the special uncertainty case involving multiple uncertain parameters that enter the system model linearly. Results for this case were first presented by [8] (Morton & McAfoos, 1985), and involve solving equation (19) with $P_{11} = 0$. For this case, P_{21} and P_{12} can easily be found by expanding $S_\Delta(\delta)$ as a linear combination of the δ_i terms, and decomposing the resulting coefficient matrices. If any of the coefficient matrices has rank greater than one, then the associated δ_i term must be repeated in Δ a corresponding number of times in order to perform the decomposition. For example, if the coefficient matrix for δ_i is rank 2, then δ_i must appear twice in the Δ matrix. This is also discussed in [9] (Morton, 1985).

2.3 Specific Problem Definition for this Paper: Multilinear Parametric Uncertainties

In this paper, we consider the case of multiple uncertain parameters which enter any element of the system described in equation (1) in a multilinear manner. It should be noted that rational multivariate elements involving only one denominator term can be represented in a multilinear form directly. For example,

$$A_{ij}(p) = \frac{p_1 + p_2 a_o + p_2^2 p_3}{p_1 p_3} \quad (21a)$$

$$= \tilde{p}_3 + \tilde{p}_1 p_2 \tilde{p}_3 a_o + \tilde{p}_1 p_2^2$$

where:

$$\tilde{p}_1 = \frac{1}{p_1} \quad , \quad \tilde{p}_3 = \frac{1}{p_3} \quad (21b)$$

The general multivariate rational uncertainty case containing multiple uncertain terms in the denominator (defined in Section 2.1) could be redefined. For example, an uncertain model element represented by equation (3) could be approximated in a multilinear form as follows:

$$A_{ij}(p) = \frac{p_1 + p_2 a_o + p_2^2 p_3}{p_1 p_3 + a_1 p_4} \quad (20a)$$

$$= \tilde{p}_4 (p_1 + p_2 a_o + p_2^2 p_3)$$

where:

$$\tilde{p}_4 = \frac{1}{p_1 p_3 + a_1 p_4} \quad (20b)$$

Thus, in this formulation the fourth uncertain parameter, $\tilde{p}_4$, is dependent on the uncertain parameters p_1, p_3, and p_4. This approach therefore poses a slight restriction to the general case. However, a brief discussion of a technique for formulating the rational problem in such a way that the multilinear solution framework can be used is presented in Section 5.

2.4 Formal Problem Statement

A formal problem statement based on the above discussion can be summarized as follows:

Given: An uncertain system in state space form as in equation (1), i.e.:

$$\dot{x} = A(p)\,x + B(p)\,u\,, \quad x \in R^{n_x},\ u \in R^{n_u}$$
$$y = C(p)\,x + D(p)\,u\,, \quad y \in R^{n_y}$$

which can be rewritten as in equation (9), i.e.:

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = S(p)\begin{bmatrix} x \\ u \end{bmatrix} = S(p_{nom})\begin{bmatrix} x \\ u \end{bmatrix} + S_\Delta(\delta)\begin{bmatrix} x \\ u \end{bmatrix}$$

Find: The matrices P_{21}, P_{12}, and P_{11} such that the above system can be expressed as in equations (13-16), i.e.:

$$y_\Delta = P_{11}\,u_\Delta + P_{12}\begin{bmatrix} x \\ u \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = P_{21}\,u_\Delta + P_{22}\begin{bmatrix} x \\ u \end{bmatrix}$$

$$u_\Delta = \Delta(\delta)\,y_\Delta$$

$$\Delta(\delta) = \mathrm{diag}(\,\delta_1 I_1, \delta_2 I_2, \ldots, \delta_m I_m\,)$$

A detailed discussion of a solution to this problem for uncertainties which are represented within a multilinear framework, as discussed above, will be presented in the next section.

3. Parametric Uncertainty Modeling: A Multilinear Problem Solution

3.1 Multilinear Solution Framework

As indicated in Section 2, the solution to the uncertainty modeling problem posed above involves finding the matrices P_{21}, P_{12}, and P_{11} such that the $S_\Delta(\delta)$ matrices given by (12) and (19) are equal, i.e.:

$$S_\Delta(\delta) = \begin{bmatrix} A_\Delta(\delta) & B_\Delta(\delta) \\ C_\Delta(\delta) & D_\Delta(\delta) \end{bmatrix}$$

$$= P_{21}(I - \Delta(\delta)P_{11})^{-1}\,\Delta(\delta)P_{12} \quad (22)$$

$$= \begin{bmatrix} P_{21_x} \\ P_{21_y} \end{bmatrix}(I - \Delta(\delta)P_{11})^{-1}\,\Delta(\delta)\,[\,P_{12_x}\ \ P_{12_u}\,]$$

where the $A_\Delta(\delta)$, $B_\Delta(\delta)$, $C_\Delta(\delta)$, and $D_\Delta(\delta)$ terms in equation (22) are formed by scaling the uncertain parameters p and extracting the uncertain δ terms from the nominal system, as discussed in Section 2, and P_{21} and P_{12} are partitioned appropriately. Thus, the matrices $A_\Delta(\delta)$, $B_\Delta(\delta)$, $C_\Delta(\delta)$, and $D_\Delta(\delta)$ are known matrix functions of the δ parameters, and the matrices P_{21}, P_{12}, and P_{11} are the unknown matrix variables for which equation (22) is solved. This section presents the main result of the paper - namely a solution to the above problem for uncertainties that are represented within the multilinear framework described in Section 2.3.

As stated above, the solution to this problem involves solving equation (22) for P_{21}, P_{12}, and P_{11}. However, the inversion of the quantity ($I - \Delta(\delta)\,P_{11}$) in equation (22) for multiple parameter problems can become very cumbersome because P_{11} is of the same dimension as $\Delta(\delta)$, and the inversion has to be performed symbolically. Moreover, each element of P_{11} must be determined such that equation (22) is satisfied. Within the multilinear framework, however, this quantity can be replaced by a finite series. To see this, consider the matrix equation:

$$I - A^{n+1} = (I - A)\,(I + A + A^2 + A^3 + \ldots + A^n)$$

which can be written for any matrix A. Assuming that the matrix ($I - A$) is invertible, this equation can be rewritten as:

$$(I - A)^{-1}\,(I - A^{n+1}) = I + A + A^2 + A^3 + \ldots + A^n$$

If matrix A is structured such that $A^{n+1} = 0$ (i.e., A is nilpotent), then:

$$(I - A)^{-1} = I + A + A^2 + A^3 + \ldots + A^n$$

This development is similar to the Neuman series expansion developed in [6] (Halmos, 1974) for a matrix A such that $\| A \| < 1$. For our problem, however, $A = \Delta(\delta)\,P_{11}$, where $\Delta(\delta)$ is a diagonal matrix and P_{11} is unknown. Although $\Delta(\delta)$ is norm-bounded by unity, P_{11} is not norm-bounded. However, since P_{11} is to be determined, requiring P_{11} to be structured such that:

$$(\Delta(\delta)\,P_{11})^{r+1} = 0 \quad (23)$$

yields:

$$(I - \Delta(\delta)\,P_{11})^{-1} = I + (\Delta(\delta)\,P_{11}) + (\Delta(\delta)\,P_{11})^2 + \ldots + (\Delta(\delta)\,P_{11})^r \quad (24)$$

Substituting this into equation (22) results in:

$$S_\Delta(\delta) = \begin{bmatrix} A_\Delta(\delta) & B_\Delta(\delta) \\ C_\Delta(\delta) & D_\Delta(\delta) \end{bmatrix} \quad (25)$$

$$= P_{21}[I + \Delta(\delta)P_{11} + (\Delta(\delta)P_{11})^2 + \ldots + (\Delta(\delta)P_{11})^r]\Delta(\delta)P_{12}$$

which can be rewritten as:

$$S_\Delta(\delta) = P_{21}\Delta(\delta)P_{12} + P_{21}[\Delta(\delta)P_{11} + (\Delta(\delta)\,P_{11})^2 + \ldots \ldots(\Delta(\delta)P_{11})^r]\Delta(\delta)P_{12} \quad (26)$$

The first term on the right side of equation (26) represents the linear uncertain components of $S_\Delta(\delta)$, and the second term adds in the nonlinear terms. Furthermore, since the nonlinear terms of $S_\Delta(\delta)$ consist of cross terms and n^{th}-order terms (which can be represented as cross terms), the order, r, of the highest term in the series of equation (26) is defined by the highest cross term order required to realize $S_\Delta(\delta)$. Thus, r is defined by the order of the highest cross-term occurring in $A_\Delta(\delta)$, $B_\Delta(\delta)$, $C_\Delta(\delta)$, and $D_\Delta(\delta)$, i.e.:

$$r = \max(O_A, O_B, O_C, O_D) \tag{27a}$$

and O_A, O_B, O_C, and O_D represent the order of the highest-order cross-product term in $A_\Delta(\delta)$, $B_\Delta(\delta)$, $C_\Delta(\delta)$, and $D_\Delta(\delta)$, respectively. That is, for a general uncertain mxn matrix M:

$$O_M = \max[\, \text{order}(m_{ij})\, ;\ \text{for all } i = 1,2,\ldots,m \text{ and } j = 1,2,\ldots,n\,] \tag{27b}$$

where the order of each m_{ij} is the order of its highest-order cross-product term, and cross-product term order is defined as:

$$\text{order}(\delta_1 \delta_2 \delta_3 \ldots \delta_i) = i - 1 \quad \text{for } i = 1, 2, \ldots, n_\Delta. \tag{27c}$$

Thus, the maximum value of r is $r_{max} = n_\Delta - 1$, where n_Δ is the dimension of the Δ matrix and is given by equation (17). The nilpotent requirement of equation (23) for $(\Delta(\delta)\, P_{11})$ can be satisfied if the elements of P_{11}, p_{ij}, satisfy the following structure:

$$\begin{array}{ll} 1.) & p_{ii} = 0\, ; \qquad i = 1, 2, \ldots, n_\Delta \\ 2.) & \text{If } \ p_{ij} \neq 0\, , \ \text{then for} \\ & i = 1, 2, \ldots, n_\Delta \ \text{and} \ j = 1, 2, \ldots, n_\Delta : \\ & a.)\ p_{ji} = 0\, ; \\ & b.)\ p_{i\oplus 1, j\oplus 1} = 0 \ \text{or} \ p_{i\oplus 2, j\oplus 2} = 0 \ \text{or} \\ & \quad \ldots \ \text{or} \ p_{i\oplus(n_\Delta - 1), j\oplus(n_\Delta - 1)} = 0 \end{array} \tag{28}$$

where the symbol "$\oplus$" represents "modulo n_Δ" addition [7] (Horowitz and Sahni, 1978) over the set $\{1, 2, \ldots, n_\Delta\}$, i.e.:

$$a \oplus b = \left\{ \begin{array}{lll} a + b & \text{if} & a + b \leq n_\Delta \\ a + b - n_\Delta & \text{if} & a + b > n_\Delta \end{array} \right\}$$

$$1 \leq a \leq n_\Delta \ , \ 1 \leq b \leq n_\Delta$$

and n_Δ is the dimension of Δ (and, hence, P_{11}) as defined in equation (17). It should be noted that requiring P_{11} to satisfy the conditions of (28) does not impose a restriction in solving the uncertainty modeling problem, but rather it is a means of removing unnecessary freedom in determining P_{11} based on the uncertain system being modeled. Thus, (28) assists in the process of solving for P_{11}.

Using this multilinear framework, P_{21} and P_{12} can be found using the linear uncertain terms of $S_\Delta(\delta)$, and P_{11} can be found using the nonlinear terms of $S_\Delta(\delta)$ such that the conditions of (28) are satisfied. Thus, the procedure presented in [8] (Morton & McAfoos, 1985) (and briefly described in Section 2.2) for obtaining an uncertainty model for multiple linear uncertain parameters can be used to obtain P_{21} and P_{12}, and these matrices can be used in the second right-hand term of equation (26) so that P_{11} can be determined directly using equations (26) and (28). Details of the procedure for doing this are presented in [1] and [2] (Belcastro, et. al., 1989 and 1991), and an example problem is presented in Section 4 which demonstrates these results.

3.2 Uncertainty Modeling Procedure

Obviously, in order to reduce computational complexity in robust control system analysis and design, it is desired to obtain an uncertainty model of minimal dimension. As discussed in [1] and [2] (Belcastro et. al., 1989 and 1991), the dimension of the uncertainty model is dependent on the system state space realization. These papers address the problem of obtaining a state space realization of an uncertain single-input single-output (SISO) system (given its transfer function) such that an uncertainty model of minimal dimension can be determined. For practical multivariable applications, however, it is usually desired to retain physical relevance to the problem being considered in assigning the states of the system, so that a particular state space realization may be preferred. Therefore, given a desired state space model of an uncertain system, one would like to be able to determine a minimal uncertainty model for this particular realization - which may or may not be an overall minimal uncertainty model for the system. A procedure to obtain a minimal (or near minimal) uncertainty model relative to a particular state space realization (based on the multilinear framework presented in Section 3.2) is therefore given in this section.

Given a state space realization of an uncertain system whose matrix elements are multilinear functions of the uncertain parameters of the system, it is desired to obtain an uncertainty model of the form of Figure 1, which has a minimal (or near minimal) number of repeated parameters in Δ. This can be done using the following approach:

1. Define a Δ matrix of the form of equation (16) which has only those repeated uncertain parameters necessary to realize the nth-order uncertain terms in the model, as discussed in Section 2.1.

2. Follow the procedure given in [8] (Morton & McAfoos, 1985) and [9] (Morton, 1985) for the linear uncertain parameter case to obtain P_{21} and P_{12} using equations (22) and (26). If problems with rank occur in defining P_{21} and P_{12}, go back to step 1 and add a repeated parameter to Δ, as described in Section 2.2.

3. Once P_{21} and P_{12} have been obtained, use the nonlinear uncertain terms in equations (22) and (26) to obtain P_{11} such that the conditions of (28) and, hence, equation (23) are satisfied. If P_{11} cannot be determined such that all of these equations and conditions are satisfied, the dimension of Δ is not large enough. If this occurs, it must be determined which parameter must be repeated (based on the specific problem encountered in trying to satisfy the above equations), and the process begins again at step 1 with the repeated parameter being added to the Δ matrix. Once P_{11} has been successfully determined such that all equations and conditions are satisfied, the minimal (or near minimal) uncertainty model for the given state space realization of the system has been determined, and equations (13) - (16) can be used to model the uncertain system as depicted in Figure 1.

It should be noted that the above procedure yields a minimal (or near minimal) uncertainty model by construction, since the initial Δ matrix defined in step 1 is of the smallest possible dimension required to model the given system, and additional parameters are added to this Δ matrix in steps 2 and 3 only if required. An example problem illustrating the above procedure is presented in Section 4.

4. Example

Consider the third-order multivariable system described in state space form as in equation (1) by the following realization:

$$A(p) = \begin{bmatrix} -\frac{V_a}{L_u} & 0 & 0 \\ 0 & -\frac{V_a}{L_w} & 0 \\ 0 & \frac{\sigma_w}{L_w{}^2}\sqrt{\frac{3V_a}{2\pi}} & -\frac{V_a}{L_w} \end{bmatrix} \tag{29a}$$

$$B(p) = \begin{bmatrix} \frac{\sigma_u}{L_u^2}\sqrt{\frac{V_a}{\pi}} & 0 \\ 0 & -\frac{V_a}{L_w}\left(1-\frac{1}{\sqrt{3}}\right) \\ 0 & \frac{\sigma_w}{L_w^2}\sqrt{\frac{3V_a}{2\pi}} \end{bmatrix} \tag{29b}$$

$$C(p) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D(p) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{29c}$$

where the uncertain parameters L_u, L_w, σ_u, and σ_w vary over the following ranges:

$$105.7 \le L_u \le 841.1 \tag{30a}$$

$$10.4 \le L_w \le 795.5 \tag{30b}$$

$$5.74 \le \sigma_u \le 9.69 \tag{30c}$$

$$3.95 \le \sigma_w \le 13.4 \tag{30d}$$

The elements of equation (29) can be expressed as multilinear functions of the uncertain parameters as follows:

$$A(p) = \begin{bmatrix} -V_a\tilde{L}_u & 0 & 0 \\ 0 & -V_a\tilde{L}_w & 0 \\ 0 & \sigma_w\tilde{L}_w^2\sqrt{\frac{3V_a}{2\pi}} & -V_a\tilde{L}_w \end{bmatrix} \tag{31a}$$

$$B(p) = \begin{bmatrix} \sigma_u\tilde{L}_u^2\sqrt{\frac{V_a}{\pi}} & 0 \\ 0 & -V_a\tilde{L}_w\left(1-\frac{1}{\sqrt{3}}\right) \\ 0 & \sigma_w\tilde{L}_w^2\sqrt{\frac{3V_a}{2\pi}} \end{bmatrix} \tag{31b}$$

$$C(p) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D(p) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{31c}$$

where:

$$\tilde{L}_u = \frac{1}{L_u}, \quad \tilde{L}_w = \frac{1}{L_w} \tag{32}$$

$$.001189 \le \tilde{L}_u \le .009461, \; .0013 \le \tilde{L}_w \le .00962$$

The first step is to extract the uncertain δ terms from the nominal system by scaling the uncertain parameters as in equation (5), as follows:

$$\begin{aligned} \tilde{L}_u &= \tilde{L}_{u_o} + s_{\tilde{L}_u}\delta_{\tilde{L}_u} = \tilde{L}_{u_o} + \bar{\delta}_{\tilde{L}_u} \\ \tilde{L}_w &= \tilde{L}_{w_o} + s_{\tilde{L}_w}\delta_{\tilde{L}_w} = \tilde{L}_{w_o} + \bar{\delta}_{\tilde{L}_w} \\ \sigma_u &= \sigma_{u_o} + s_{\sigma_u}\delta_{\sigma_u} = \sigma_{u_o} + \bar{\delta}_{\sigma_u} \\ \sigma_w &= \sigma_{w_o} + s_{\sigma_w}\delta_{\sigma_w} = \sigma_{w_o} + \bar{\delta}_{\sigma_w} \end{aligned} \tag{33}$$

so that, as in equation (12):

$$S(p_{nom}) = \begin{bmatrix} A(p_o) B(p_o) \\ C(p_o) D(p_o) \end{bmatrix}; \quad S_\Delta(\delta) = \begin{bmatrix} A_\Delta(\delta) & B_\Delta(\delta) \\ C_\Delta(\delta) & D_\Delta(\delta) \end{bmatrix} \tag{34}$$

where:

$$A(p_o) = \begin{bmatrix} -V_a\tilde{L}_{u_o} & 0 & 0 \\ 0 & -V_a\tilde{L}_{w_o} & 0 \\ 0 & \sigma_{w_o}\tilde{L}_{w_o}^2\sqrt{\frac{3V_a}{2\pi}} & -V_a\tilde{L}_{w_o} \end{bmatrix} \tag{35a}$$

$$B(p_o) = \begin{bmatrix} \sigma_{u_o}\tilde{L}_{u_o}^2\sqrt{\frac{V_a}{\pi}} & 0 \\ 0 & -V_a\tilde{L}_{w_o}\left(1-\frac{1}{\sqrt{3}}\right) \\ 0 & \sigma_{w_o}\tilde{L}_{w_o}^2\sqrt{\frac{3V_a}{2\pi}} \end{bmatrix} \tag{35b}$$

$$C(p_o) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D(p_o) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{35c}$$

$$A_\Delta(\delta) = \begin{bmatrix} -V_a\bar{\delta}_{\tilde{L}_u} & 0 & 0 \\ 0 & -V_a\bar{\delta}_{\tilde{L}_w} & 0 \\ 0 & a_\Delta\sqrt{\frac{3V_a}{2\pi}} & -V_a\bar{\delta}_{\tilde{L}_w} \end{bmatrix} \tag{36a}$$

$$B_\Delta(\delta) = \begin{bmatrix} b_{\Delta_1}\sqrt{\frac{V_a}{\pi}} & 0 \\ 0 & -V_a\left(1-\frac{1}{\sqrt{3}}\right)\bar{\delta}_{\tilde{L}_w} \\ 0 & b_{\Delta_2}\sqrt{\frac{3V_a}{2\pi}} \end{bmatrix} \tag{36b}$$

$$C_\Delta(\delta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad D_\Delta(\delta) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{36c}$$

where:

$$\begin{aligned} a_{\Delta_1} = \; & 2\sigma_{w_o}\tilde{L}_{w_o}\bar{\delta}_{\tilde{L}_w} + \tilde{L}_{w_o}\bar{\delta}_{\sigma_w} + 2\tilde{L}_{w_o}\bar{\delta}_{\tilde{L}_w}\bar{\delta}_{\sigma_w} \\ & + \sigma_{w_o}\bar{\delta}_{\tilde{L}_w}^2 + \bar{\delta}_{\sigma_w}\bar{\delta}_{\tilde{L}_w}^2 \end{aligned} \tag{37a}$$

$$\begin{aligned} b_{\Delta_1} = \; & 2\sigma_{u_o}\tilde{L}_{u_o}\bar{\delta}_{\tilde{L}_u} + \tilde{L}_{u_o}\bar{\delta}_{\sigma_u} + 2\tilde{L}_{u_o}\bar{\delta}_{\tilde{L}_u}\bar{\delta}_{\sigma_u} \\ & + \sigma_{u_o}\bar{\delta}_{\tilde{L}_u}^2 + \bar{\delta}_{\sigma_u}\bar{\delta}_{\tilde{L}_u}^2 \end{aligned} \tag{37b}$$

$$\begin{aligned} b_{\Delta_2} = \; & 2\sigma_{w_o}\tilde{L}_{w_o}\bar{\delta}_{\tilde{L}_w} + \tilde{L}_{w_o}\bar{\delta}_{\sigma_w} + 2\tilde{L}_{w_o}\bar{\delta}_{\tilde{L}_w}\bar{\delta}_{\sigma_w} \\ & + \sigma_{w_o}\bar{\delta}_{\tilde{L}_w}^2 + \bar{\delta}_{\sigma_w}\bar{\delta}_{\tilde{L}_w}^2 \end{aligned} \tag{37c}$$

As can be seen by the last term in equation (37) (for either a_{Δ_1}, b_{Δ_1}, or b_{Δ_2}), $r = 2$ for this example problem (as defined by equation (27)). Since $S_\Delta(\delta)$ contains 2nd-order terms associated with L_u and L_w, the δ terms associated with these variables will have to appear twice in Δ. Thus, the dimension of Δ going into Step 1 of Section 3.2 is six. For a six-dimensional Δ, the matrices P_{21} and P_{12} can be determined, as described in Step 2 of Section 3.2. However, it is impossible to obtain a P_{11} matrix which satisfies all of the equations discussed in Step 3 of Section 3.2. Moreover, it is determined in that step that the δ term associated with L_w must be repeated a third time. Therefore, when steps 1 - 3 of Section 3.2 are repeated, the resulting uncertainty model can be expressed as in equations (13) - (16) and (22), where:

$$\begin{aligned} \Delta &= \mathrm{diag}[\, \delta_{\tilde{L}_u} \;\; \delta_{\tilde{L}_u} \;\; \delta_{\tilde{L}_w} \;\; \delta_{\tilde{L}_w} \;\; \delta_{\tilde{L}_w} \;\; \delta_{\sigma_u} \;\; \delta_{\sigma_w} \,] \\ &= \mathrm{diag}[\, \delta_{\tilde{L}_u} I_2 \;\; \delta_{\tilde{L}_w} I_3 \;\; \delta_{\sigma_u} \;\; \delta_{\sigma_w} \,] \end{aligned} \tag{38}$$

$$P_{21_x} = \begin{bmatrix} -1 & \sigma_{u_o}\tilde{L}_{u_o} & 0 & 0 & 0 & \tilde{L}_{u_o} & 0 \\ 0 & 0 & -V_a & V_a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \tilde{L}_{w_o} \end{bmatrix} \tag{39a}$$

$$P_{21_y} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{39b}$$

$$P_{12_x} = \begin{bmatrix} s_{\tilde{L}_u} V_a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & s_{\tilde{L}_w} & 0 \\ 0 & 0 & 0 \\ 0 & 2 s_{\tilde{L}_w} \sigma_{w_o} \tilde{L}_{w_o} \sqrt{\frac{3V_a}{2\pi}} & -s_{\tilde{L}_w} V_a \\ 0 & 0 & 0 \\ 0 & s_{\sigma_w} \sqrt{\frac{3V_a}{2\pi}} & 0 \end{bmatrix} \tag{39c}$$

$$P_{12_u} = \begin{bmatrix} 0 & 0 \\ 2 s_{\tilde{L}_u} \sqrt{\frac{V_a}{\pi}} & 0 \\ 0 & s_{\tilde{L}_w} \\ 0 & s_{\tilde{L}_w} \frac{1}{\sqrt{3}} \\ 0 & 2 s_{\tilde{L}_w} \sigma_{w_o} \tilde{L}_{w_o} \sqrt{\frac{3V_a}{2\pi}} \\ s_{\sigma_u} \sqrt{\frac{V_a}{\pi}} & 0 \\ 0 & s_{\sigma_w} \sqrt{\frac{3V_a}{2\pi}} \end{bmatrix} \tag{39d}$$

$$P_{11} = \begin{bmatrix} 0 & p_{12} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & p_{26} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & p_{37} \\ 0 & 0 & 0 & 0 & 0 & 0 & p_{47} \\ 0 & 0 & p_{53} & 0 & 0 & 0 & p_{57} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{40a}$$

$$\begin{aligned} p_{12} &= -s_{\tilde{L}_u} \frac{\sigma_{u_o}}{2} \\ p_{26} &= \frac{2 s_{\tilde{L}_u}}{\sigma_{u_o}} \\ p_{37} &= \frac{s_{\tilde{L}_w}}{\sigma_{w_o}} \sqrt{\frac{2\pi}{3V_a}} \\ p_{47} &= \frac{s_{\tilde{L}_w}}{\sigma_{w_o}} \sqrt{\frac{2\pi}{3V_a}} \\ p_{53} &= s_{\tilde{L}_w} \sigma_{w_o} \sqrt{\frac{3V_a}{2\pi}} \\ p_{57} &= 2 s_{\tilde{L}_w} \tilde{L}_{w_o} \end{aligned} \tag{40b}$$

and the nominal system matrices are given above in equation (35). It should be noted that a certain amount of freedom exists in determining the above matrices, so that an uncertainty model obtained for a given uncertain system is not unique. It should also be noted that in the above uncertainty model development, the scaling terms s_{p_i} were incorporated into the model at the end so as to reduce the number of symbolic terms involved in the determination of the P_{21}, P_{12}, and P_{11} matrices.

5. Extension to Rational Case

The above procedure for solving the multilinear uncertainty modeling problem can in fact also be used to solve the more general rational uncertainty modeling problem. This is done by obtaining a matrix fraction description of the uncertain system, and representing the denominator matrix in a feedback loop so as to remove the inverse. The numerator and denominator matrices are then multivariate polynomial matrices which can be concatenated together and modeled using the multilinear techniques discussed above. Details of this approach will be presented in a subsequent paper.

6. Conclusions

This paper has summarized previous results in parametric uncertainty modeling, and has presented and demonstrated an important extension to these results. The extension consists of a framework for modeling multiple parametric uncertainties which can be represented in a multilinear functional form, and includes a procedure for obtaining a minimal (or near minimal) uncertainty model relative to a given state space realization of the uncertain system. As discussed in the paper, the multilinear framework can also be used to solve the more general rational uncertain parameter case, and provides a mechanism for significantly simplifying the computational complexity involved in determining an uncertainty model for a given uncertain system. Thus, many practical problems of interest can be solved within this framework. To demonstrate the results of the paper, an example problem was presented which consisted of a multivariable third-order uncertain system with four uncertain parameters. A minimal (or near minimal) uncertainty model was determined for the given state space realization of this system, and the resulting model had a dimension of seven. Although two of the uncertain parameters entered into the given model as squared terms and as fractions, they were easily modeled within the multilinear framework.

Further work being addressed in this area includes evaluating/refining/generaliziing this modeling procedure for a wider class of problems, automating the generalized modeling procedure, and applying the procedure to practical application problems.

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