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Abstract

Usually the theoretical analysis of the Navier-Stokes equations is conducted via the Galerkin method which leads to difficult saddle-point problems. This paper demonstrates that the least-squares method is a useful alternative tool for the theoretical study of partial differential equations since it leads to minimization problems which can often be treated by an elementary technique. The principal part of the Navier-Stokes equations in the first-order velocity-pressure-vorticity formulation consists of two div-curl systems, so the three-dimensional div-curl system is thoroughly studied at first. By introducing a dummy variable and by using the least-squares method, this paper shows that the div-curl system is properly determined and elliptic, and has a unique solution. The same technique then is employed to prove that the Stokes equations are properly determined and elliptic, and that four boundary conditions on a fixed boundary are required for three-dimensional problems. This paper also shows that under four combinations of non-standard boundary conditions the solution of the Stokes equations is unique. This paper emphasizes the application of the least-squares method and the div-curl method to derive a high-order version of differential equations and additional boundary conditions. In this paper an elementary method (integration by parts) is used to prove Friedrichs' inequalities related to the div and curl operators which play an essential role in the analysis.

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Introduction

The incompressible Navier-Stokes(NS) equations which govern the motion of viscous fluids are among the most important partial differential equations in mathematical physics. The Navier-Stokes equations are derived from the mass and momentum conservation laws and the linear local stress-strain relation. While the physical model leading to the Navier-Stokes equations is simple, the situation is quite different from the mathematical point of view, and can be described by quoting the words from a mathematician: “Despite the important work done on these equations, our understanding of them remains fundamentally incomplete.”(see Teman[1]). As pointed out by Teman, the major difficulty in the mathematical study of these equations is related to their nonlinearity.

Even our understanding of the linearized Navier-Stokes equations and the Stokes equations remains incomplete. Here we refer to the fact that there has been no systematic study of permissible boundary conditions(BCs). The boundary conditions applied to the Navier-Stokes equations have been the subject of constant controversy. Quoting next from a recent review on incompressible flow, Gresho[2] states that “To the best of our knowledge, the jury is still out regarding the full story on mathematically permissible BCs for the NS equations”.

Looking into the classic books on the mathematical theory of the Navier-Stokes equations, see for example, Ladyzhenskaya[3], Teman[1,4], and Girault and Raviart[5], one will find that mathematicians study the existence and uniqueness of the solutions mainly for the standard case in which the velocity components are prescribed on the boundary. On the other side, physicists and engineers apply the non-standard boundary conditions often based on physical intuition.

Usually the Galerkin method is employed for numerical solution and theoretical analysis of the incompressible Navier-Stokes equations. The application of the Galerkin method leads to a saddle-point problem, and thus a difficult “Ladyzhenskaya-Babuska-Brezzi(LBB) condition” or “inf-sup condition” is involved[6-10].

In recent years the least-squares finite element method(LSFEM) has been successfully developed and used for the computation of incompressible viscous flows[11-19]. The LSFEM based on the first-order velocity-pressure-vorticity formulation has superior advantages over other methods: together with, for example, a Newton linearization, the resulting discrete equations are symmetric and positive definite, and can be efficiently solved by matrix-free iterative methods; the choice of approximating spaces is not subject to the LBB condition, and thus equal-order interpolation with respect to a single grid for all dependent variables and test functions may be used; only simple algebraic boundary conditions are imposed, no artificial numerical boundary conditions for the vorticity need

be devised at boundaries at which the velocity is specified; accurate vorticity approximations are obtained; neither upwinding nor adjustable parameters are needed; all variables are solved in a fully-coupled manner, operator splitting turns out to be unnecessary. In addition, we found that the least-squares method based on the first-order velocity-pressure-vorticity formulation provides not only a powerful technique for numerical solution but also a useful tool for theoretical analysis of the Navier-Stokes equations.

In our previous work[15], we showed that the original three-dimensional velocity-pressure-vorticity formulation for incompressible Navier-Stokes problems is not elliptic in the ordinary sense, and the compatibility condition, that is, the solenoidality of the vorticity, should be added to make the first-order system elliptic. Based on the first-order elliptic system we showed that for three-dimensional incompressible viscous flows four (three in most cases if the condition for the dummy variable is not counted) boundary conditions are required for a fixed boundary. In this paper, we give a further detailed discussion of the permissible non-standard boundary conditions, and prove the existence and uniqueness of the solution under these boundary conditions by using the least-squares method. The least-squares method leads to a minimization problem which is much easier than a saddle-point problem and often can be dealt with by an elementary technique.

According to the explanation in [15], the non-linear convective terms in the Navier-Stokes equations have no effect on the classification. The ellipticity (in the ordinary sense) of the Navier-Stokes equations is determined only by the principal part of the equations. The principal part of the Navier-Stokes equations is the same as that of the linear Stokes equations. Also according to the theory in [20], error analysis for the nonlinear Navier-Stokes equations is essentially the same as that for the linear Stokes problem, at least away from singular points. Thus, the main goal of this paper is the verification of the well-posedness of the boundary conditions for the Stokes equations.

The Stokes equations written in the first-order velocity-pressure-vorticity formulation consist of two coupled div-curl systems. The three-dimensional div-curl system is traditionally considered as an overdetermined system, and its numerical solution is not trivial, since there are three unknowns and four equations. It is worthwhile to point out a similar situation in electromagnetics. Maxwell's equations also consist of two coupled div-curl systems. For the same reason, some engineering books on electromagnetics claim that Maxwell's equations are overdetermined, and only two curl equations are independent; in some works on computational electromagnetics the divergence-free equation of the electric or magnetic field is just ignored. This paper demonstrates that it is incorrect by simply counting the number of unknowns and equations to judge whether a system of differential equations is overdetermined or not. Because of the importance of this problem, in the first part of this paper we show that the div-curl system is properly determined and elliptic, and analyse the div-curl system by using the least-squares method. In the second part of this paper we use the same technique to deal with the Stokes problem.

In fluid dynamics as well as in electromagnetics there are some good reasons why other higher-order versions of the Navier-Stokes equations and Maxwell's equations are often useful (see the discussion by Gresho[2]). These are all derived from the "primitive" equations by differentiation and often offer additional insights regarding fluid flow and electromagnetic fields. They also serve as the starting point for devising alternative numerical schemes. For example, the pressure Poisson equation is obtained by applying the divergence operator to the momentum equation in the velocity-pressure formulation; the vorticity transport equation is obtained by applying the curl operator to the momentum equation. A key issue is that a derived equation obtained by simple differentiation admits more solutions than do its progenitors, in other words, spurious solutions (or spurious modes in electromagnetic waveguides) may be generated. With careful selection of additional boundary conditions (and additional equations), the solution of the derived equations will also solve the original equations. In this paper, as a by-product, we show that the least-squares method provides a systematic and consistent way to derive higher-order versions of differential equations and corresponding boundary conditions without generating spurious solutions. That is, the Euler-Lagrange equations associated with the least-squares weak formulations are the most appropriate derived equations. We also show an alternative systematic method to derive equivalent higher-order systems. This div-curl method is based on the theorem: if a vector is divergence-free and curl-free in a domain, and its normal component (or tangential components) on the boundary is zero, then this vector is zero. This paper emphasizes that one must apply the div and curl operators together with the boundary condition (either the normal component or the tangential components be zero) to a vector differential equation to derive an equivalent higher-order version of system. We believe that many controversies over the permissibility of the boundary conditions for derived equations can be resolved via this div-curl method.

The paper is organized as follows. In Section 1, we use an elementary method (integration by parts) to prove Friedrichs' inequality related to the div and curl operators. In Section 2, we show that the three-dimensional div-curl system is properly determined and elliptic by introducing a dummy variable, and that the least-squares method for the div-curl system corresponds to solving three independent Poisson equations of three velocity components with three coupled boundary conditions. In Section 2.4, we introduce the div-curl method to change the low-order partial differential equations into an equivalent higher-order form. In Section 3, we study the div-curl system with a different boundary condition. In Section 4, we use the results obtained in Section 1-3 to prove the theorem about the orthogonal decomposition of vectors, and use it to establish Friedrich's second inequality. In Section 5, we show that the Navier-Stokes problem written in the first-order velocity-pressure-vorticity formulation is properly determined and elliptic when the constraint condition (the divergence of the vorticity should be zero) is supplied. In Section 6, we show that the number of permissible boundary conditions is four for three dimensional problems and list all possible combinations of boundary conditions for the Stokes prob-

lem. In Section 7 we prove that under four different nonstandard boundary conditions the Stokes problem has a unique solution. In Section 8, we derive the second-order velocity-pressure-vorticity formulation and the corresponding boundary conditions for the Stokes problem via the least-squares method. In Section 9, we derive the conventional second-order velocity-vorticity formulation by using the div-curl method. Concluding remarks are given in Section 10. In Appendix we list all useful equalities on vector operations, briefly derive some important Green's formulae, and give two Poincare inequalities for scalar functions.

1. Basic Inequalities

1.1 Notations

Let $\Omega \subset \mathfrak{R}^3$ be an open bounded domain with a boundary Γ , $\mathbf{x} = (x, y, z)$ be a point in Ω , $\bar{\mathbf{n}}$ be a unit outward normal vector on the boundary, $\bar{\boldsymbol{\tau}}$ be a tangential vector to Γ , and $\bar{\boldsymbol{\tau}}_1$ and $\bar{\boldsymbol{\tau}}_2$ be two orthogonal tangential vectors to Γ . In order to pay attention to basic ideas and maintain simplicity we often further assume that the domain Ω is simply connected and the boundary Γ is smooth enough, although in many cases these restrictions are not necessary. $L_2(\Omega)$ denotes the space of square-integrable functions defined on Ω equipped with the inner product

$$(u, v) = \int_{\Omega} u v d\Omega \quad u, v \in L_2(\Omega)$$

and the norm

$$\|u\|_{0,\Omega}^2 = (u, u) \quad u \in L_2(\Omega).$$

$H^r(\Omega)$ denotes the Sobolev space of functions with square-integrable derivatives of order up to r . $|\cdot|_{r,\Omega}$ and $\|\cdot\|_{r,\Omega}$ denote the usual seminorm and norm for $H^r(\Omega)$, respectively. For vector-valued functions $\bar{\mathbf{u}}$ with m components, we have the product spaces

$$L_2(\Omega)^m, H^r(\Omega)^m$$

with the inner product

$$(\bar{\mathbf{u}}, \bar{\mathbf{v}}) = \int_{\Omega} \bar{\mathbf{u}} \cdot \bar{\mathbf{v}} d\Omega \quad \bar{\mathbf{u}}, \bar{\mathbf{v}} \in L_2(\Omega)^m$$

and the corresponding norm

$$\|\bar{\mathbf{u}}\|_{0,\Omega}^2 = \sum_{j=1}^m \|u_j\|_{0,\Omega}^2, \quad \|\bar{\mathbf{u}}\|_{r,\Omega}^2 = \sum_{j=1}^m \|u_j\|_{r,\Omega}^2.$$

Further we define

$$\langle u, v \rangle_{\Gamma} = \int_{\Gamma} uv ds \quad u \in H^{1/2}(\Gamma), v \in H^{-1/2}(\Gamma)$$

and the corresponding norm

$$\|u\|_{1/2, \Gamma}^2 = \langle u, u \rangle_{\Gamma}.$$

When there is no chance for confusion, we will often omit the measure Ω or Γ from the inner product and norm designation.

Throughtout the paper C denotes a positive constant dependent on Ω with possibly different values in each appearance.

Lemma 1.1. Let Ω be a bounded open subset of \mathfrak{R}^3 with a piecewise C^1 boundary Γ . Every function \bar{u} of $H^1(\Omega)^3$ with $\bar{n} \times \bar{u} = \bar{0}$ on Γ satisfies

$$|\bar{u}|_1^2 + \int_{\Gamma} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \bar{u}^2 ds = \|\nabla \cdot \bar{u}\|_0^2 + \|\nabla \times \bar{u}\|_0^2, \quad (1.1)$$

where R_1 and R_2 denote the principal radii of curvature for Γ .

Proof. Using Green's formulae (B.2), (B.3) and (B.7), and Equality (A.4), we have

$$\begin{aligned} & \|\nabla \cdot \bar{u}\|_0^2 + \|\nabla \times \bar{u}\|_0^2 \\ &= (\nabla \cdot \bar{u}, \nabla \cdot \bar{u}) + (\nabla \times \bar{u}, \nabla \times \bar{u}) \\ &= (\bar{u}, -\nabla(\nabla \cdot \bar{u})) + \langle \nabla \cdot \bar{u}, \bar{n} \cdot \bar{u} \rangle + (\bar{u}, \nabla(\nabla \cdot \bar{u}) - \Delta \bar{u}) + \langle \nabla \times \bar{u}, \bar{n} \times \bar{u} \rangle \\ &= (\nabla \bar{u}, \nabla \bar{u}) + \langle \nabla \cdot \bar{u}, \bar{n} \cdot \bar{u} \rangle + \langle \nabla \times \bar{u}, \bar{n} \times \bar{u} \rangle - \left\langle \frac{\partial \bar{u}}{\partial n}, \bar{u} \right\rangle. \end{aligned} \quad (1.2)$$

Obviously

$$(\nabla \bar{u}, \nabla \bar{u}) = |\bar{u}|_1^2.$$

Now we turn to the boundary integral terms. $\bar{n} \times \bar{u} = \bar{0}$ on Γ implies that

$$\bar{u} = U \bar{n} \text{ on } \Gamma,$$

where U is a scalar function which depends on the location on the boundary surface. By using (A.1), the boundary integral terms in (1.2) can be written as follows:

$$\int_{\Gamma} \left\{ U \nabla \cdot (U \bar{n}) - U \frac{\partial U}{\partial n} \right\} ds$$

$$\begin{aligned}
&= \int_{\Gamma} \{U(U\nabla \cdot \bar{n} + \bar{n} \cdot \nabla U) - U\nabla U \cdot \bar{n}\} ds \\
&= \int_{\Gamma} U^2 \nabla \cdot \bar{n} ds.
\end{aligned}$$

It can be verified that on a curved surface

$$\nabla \cdot \bar{n} = \frac{1}{R_1} + \frac{1}{R_2} = 2\gamma,$$

where γ is the mean curvature. □

Lemma 1.2. Let Ω be a bounded open subset of \mathfrak{R}^3 with a piecewise C^1 boundary Γ . Every function \bar{u} of $H^1(\Omega)^3$ with $\bar{n} \cdot \bar{u} = 0$ on Γ satisfies

$$|\bar{u}|_1^2 + \int_{\Gamma} \frac{1}{R} \bar{u}^2 ds = \|\nabla \cdot \bar{u}\|_0^2 + \|\nabla \times \bar{u}\|_0^2, \quad (1.3)$$

where $R_1 \leq R \leq R_2$, in which R_1 and R_2 denote the principal radii of curvature for Γ .

Proof. In this case we still have (1.2). Since $\bar{n} \cdot \bar{u} = 0$, we may assume that

$$\bar{u} = U\bar{\tau} \quad \text{on } \Gamma.$$

By virtue of the triple scalar product

$$(\nabla \times \bar{u}) \cdot (\bar{n} \times \bar{u}) = \bar{n} \cdot (\bar{u} \times (\nabla \times \bar{u}))$$

and using (A.5), the boundary integral terms in (1.2) can be written as

$$\begin{aligned}
&\int_{\Gamma} \left\{ \bar{n} \cdot (\bar{u} \times \nabla \times \bar{u}) - \frac{1}{2} \frac{\partial \bar{u}^2}{\partial n} \right\} ds \\
&= \int_{\Gamma} \left\{ \bar{n} \cdot \left(\nabla \left(\frac{1}{2} \bar{u}^2 \right) - (\bar{u} \nabla) \bar{u} \right) - \frac{1}{2} \frac{\partial \bar{u}^2}{\partial n} \right\} ds \\
&= \int_{\Gamma} \left\{ \bar{n} \cdot (-(\bar{u} \nabla) \bar{u}) \right\} ds \\
&= \int_{\Gamma} \left\{ U^2 \bar{n} \cdot (-(\bar{\tau} \nabla) \bar{\tau}) \right\} ds
\end{aligned}$$

$$= \int_{\Gamma} \{U^2 \bar{n} \cdot (\frac{1}{R} \bar{n})\} ds.$$

So the Lemma is proved. Here R is the radius of curvature of the boundary surface in the direction of $\bar{\tau}$. □

Since the curvature $1/R$ is always positive when the boundary surface is convex, we derive immediately the following theorem:

Theorem 1. Let Ω be a bounded and convex open subset of \mathfrak{R}^3 , and (Γ_1, Γ_2) be the pieces of the piecewise C^1 boundary surface Γ . Either Γ_1 or Γ_2 may be empty or with strictly positive measure. Every function \bar{u} of $H^1(\Omega)^3$ with $\bar{n} \cdot \bar{u} = 0$ on Γ_1 and $\bar{n} \times \bar{u} = \bar{0}$ on Γ_2 satisfies:

$$|\bar{u}|_1^2 \leq \|\nabla \cdot \bar{u}\|_0^2 + \|\nabla \times \bar{u}\|_0^2. \quad (1.4)$$

Theorem 2. Suppose that the simply connected Ω and Γ satisfy the same assumption in Theorem 1. If $\bar{u} \in H^1(\Omega)^3$ satisfies $\nabla \cdot \bar{u} = 0$, $\nabla \times \bar{u} = \bar{0}$, $\bar{n} \cdot \bar{u}|_{\Gamma_1} = 0$ and $\bar{n} \times \bar{u}|_{\Gamma_2} = \bar{0}$, then $\bar{u} = \bar{0}$.

Proof. From Theorem 1, we have

$$|\bar{u}|_1^2 \leq 0,$$

that is, \bar{u} must be a constant vector in Ω . From the boundary conditions we know that this constant vector must be zero. □

Theorem 3 (Friedrichs' First Inequality). Suppose that the simply connected Ω and Γ satisfy the same assumption in Theorem 1. Every function \bar{u} of $H^1(\Omega)^3$ with $\bar{n} \cdot \bar{u} = 0$ on Γ_1 and $\bar{n} \times \bar{u} = \bar{0}$ on Γ_2 satisfies:

$$\|\bar{u}\|_1^2 \leq C(\|\nabla \cdot \bar{u}\|_0^2 + \|\nabla \times \bar{u}\|_0^2), \quad (1.5)$$

where the constant $C > 0$ depends only on Ω .

The proof of Theorem 3 can be based on the use of contradiction arguments together with Theorem 1 and 2, see e.g., Saranen[21] and the references therein. In the two-dimensional case, a direct proof is available[22,23].

2. The Div-Curl System (Case 1)

2.1 Determinedness and ellipticity

Let us first consider the following three-dimensional div-curl system:

$$\nabla \cdot \bar{u} = 0 \text{ in } \Omega, \quad (2.1a)$$

$$\nabla \times \bar{u} = \bar{\omega} \text{ in } \Omega, \quad (2.1b)$$

$$\bar{n} \cdot \bar{u} = 0 \text{ on } \Gamma, \quad (2.1c)$$

where the domain Ω is bounded and convex with a piecewise C^1 boundary Γ , the vector function $\bar{\omega} \in L^2(\Omega)^3$ is given and satisfies the compatibility conditions:

$$\nabla \cdot \bar{\omega} = 0 \text{ in } \Omega, \quad (2.2a)$$

$$\int_{\Gamma} \bar{n} \cdot \bar{\omega} ds = 0. \quad (2.2b)$$

The first-order system (2.1) is fundamental for incompressible viscous flow problems in which \bar{u} represents the velocity, and $\bar{\omega}$ the vorticity. The system (2.1) also governs, for example, static magnetic problems in which \bar{u} is the magnetic field, and $\bar{\omega}$ the electric current density (assume that the permeability is one).

At first glance, System (2.1) seems overdetermined, since there are four equations and three unknowns (i.e. three components of the velocity vector). Surely, for this reason solving (2.1) is not easy by conventional finite difference or finite element methods. However, after careful investigation by applying the same trick as used in [24], we shall find that System (2.1) is properly determined and elliptic. By introducing a dummy variable ϕ , System (2.1) can be written as

$$\nabla \cdot \bar{u} = 0 \text{ in } \Omega, \quad (2.3a)$$

$$\nabla \phi + \nabla \times \bar{u} = \bar{\omega} \text{ in } \Omega, \quad (2.3b)$$

$$\bar{u} \cdot \bar{n} = 0 \text{ on } \Gamma, \quad (2.3c)$$

$$\phi = 0 \text{ on } \Gamma. \quad (2.3d)$$

Substituting (2.3b) into (2.2a) and taking into account that $\nabla \cdot \nabla \times \bar{u} = 0$, we obtain $\Delta \phi = 0$. Since $\phi = 0$ on Γ , we must have $\phi \equiv 0$. Therefore System (2.3) is indeed equivalent to System (2.1). In Cartesian coordinates System (2.3) is given as

$$\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0, \\ \frac{\partial \phi}{\partial x} + \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} &= \omega_x, \\ \frac{\partial \phi}{\partial y} + \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} &= \omega_y, \end{aligned} \quad (2.4)$$

$$\frac{\partial \phi}{\partial z} + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \omega_z.$$

Now we write System (2.4) in the standard matrix form:

$$\mathbf{A}_1 \frac{\partial \mathbf{u}}{\partial x} + \mathbf{A}_2 \frac{\partial \mathbf{u}}{\partial y} + \mathbf{A}_3 \frac{\partial \mathbf{u}}{\partial z} + \mathbf{A} \mathbf{u} = \mathbf{f}, \quad (2.5)$$

in which

$$\mathbf{A}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{A}_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{A}_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{f} = \begin{pmatrix} 0 \\ \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} u \\ v \\ w \\ \phi \end{pmatrix}.$$

The characteristic polynomial associated with System (2.4)

$$\det(\mathbf{A}_1 \xi + \mathbf{A}_2 \eta + \mathbf{A}_3 \zeta) = \det \begin{pmatrix} \xi & \eta & \zeta & 0 \\ 0 & -\zeta & \eta & \xi \\ \zeta & 0 & -\xi & \eta \\ -\eta & \xi & 0 & \zeta \end{pmatrix} = (\xi^2 + \eta^2 + \zeta^2)^2 \neq 0$$

for all nonzero triplet (ξ, η, ζ) , System (2.4) and thus System (2.1) is indeed elliptic and properly determined.

The first-order elliptic system (2.3) has four equations and four unknowns, so two boundary conditions are needed to make System (2.3) well-posed. Here $\phi = 0$ serves as one boundary condition, and $\bar{n} \cdot \bar{u} = 0$ as another one.

2.2 The least-squares method

Now let us consider the least-squares method for solving the div-curl equations (2.1), see also a theoretical analysis by Fix and Rose[25] and an application by Hafez and Soliman[26]. We construct the following quadratic functional:

$$I : \mathbf{H} \longrightarrow \Re,$$

$$I(\bar{u}) = \|\nabla \cdot \bar{u}\|_0^2 + \|\nabla \times \bar{u} - \bar{\omega}\|_0^2,$$

where $\mathbf{H} = \{\bar{u} \in H^1(\Omega)^3 \mid \bar{n} \cdot \bar{u} = 0 \text{ on } \Gamma\}$. We note that the introduction of a dummy variable ϕ in § 2.1 is only for the verification of the determinedness, and is not required in the least-squares functional I . Taking the variation of I with respect to \bar{u} , and letting $\delta\bar{u} = \bar{v}$ and $\delta I = 0$, we obtain the least-squares weak formulation: Find $\bar{u} \in \mathbf{H}$ such that

$$B(\bar{u}, \bar{v}) = L(\bar{v}) \quad \forall \bar{v} \in \mathbf{H}, \quad (2.6)$$

where

$$\begin{aligned} B(\bar{u}, \bar{v}) &= (\nabla \cdot \bar{u}, \nabla \cdot \bar{v}) + (\nabla \times \bar{u}, \nabla \times \bar{v}), \\ L(\bar{v}) &= (\bar{\omega}, \nabla \times \bar{v}). \end{aligned}$$

Obviously $B(\bar{u}, \bar{v})$ is symmetric. Due to Theorem 1 we immediately see that

$$|\bar{u}|_1^2 \leq B(\bar{u}, \bar{u}) = L(\bar{u}) \leq |\bar{u}|_1 \|\bar{\omega}\|_0,$$

and

$$|\bar{u}|_1 \leq \|\bar{\omega}\|_0. \quad (2.7)$$

If, in addition, Ω is simply connected, due to Theorem 3 we also see that

$$\frac{1}{C} \|\bar{u}\|_1^2 \leq B(\bar{u}, \bar{u}) = L(\bar{u}) \leq \|\bar{u}\|_1 \|\bar{\omega}\|_0.$$

By virtue of the Lax-Milgram theorem, in fact we have proved the following Lemma.

Lemma 2.1 If Ω is bounded, convex and simply connected, then the solution of (2.1) uniquely exists and satisfies:

$$\|\bar{u}\|_1 \leq C \|\bar{\omega}\|_0. \quad (2.8)$$

2.3 The Euler-Lagrange equation

In order to understand how the least-squares method works, we derive the Euler-Lagrange equations associated with the least-squares weak formulation (2.6) which can be rewritten as: Find $\bar{u} \in \mathbf{H}$ such that

$$(\nabla \cdot \bar{u}, \nabla \cdot \bar{v}) + (\nabla \times \bar{u} - \bar{\omega}, \nabla \times \bar{v}) = 0 \quad \forall \bar{v} \in \mathbf{H}. \quad (2.9)$$

Suppose that \bar{u} and $\bar{\omega}$ are sufficiently smooth. By using Green's formulae (B.3) and (B.5), Equation (2.9) can be written as

$$(-\nabla(\nabla \cdot \bar{u}), \bar{v}) + \langle \nabla \cdot \bar{u}, \bar{n} \cdot \bar{v} \rangle +$$

$$(\nabla \times (\nabla \times \bar{u} - \bar{w}), \bar{v}) - \langle \bar{n} \times (\nabla \times \bar{u} - \bar{w}), \bar{v} \rangle = 0. \quad (2.10)$$

Taking into account (A.4) and that \bar{v} satisfies $\bar{n} \cdot \bar{v} = 0$, from (2.10) we obtain

$$(-\Delta \bar{u} - \nabla \times \bar{w}, \bar{v}) - \langle \bar{n} \times (\nabla \times \bar{u} - \bar{w}), \bar{v} \rangle = 0 \quad (2.11)$$

for all admissible $\bar{v} \in \mathbf{H}$, hence we have the Euler-Lagrange equation and boundary conditions:

$$-\Delta \bar{u} = \nabla \times \bar{w} \text{ in } \Omega, \quad (2.12a)$$

$$\bar{n} \times (\nabla \times \bar{u} - \bar{w}) = \bar{0} \text{ on } \Gamma, \quad (2.12b)$$

$$\bar{n} \cdot \bar{u} = 0 \text{ on } \Gamma. \quad (2.12c)$$

We remark that included in the boundary integral term in (2.11) is a triple vector product of \bar{n} , $(\nabla \times \bar{u} - \bar{w})$ and \bar{v} . Since we already know that \bar{v} is orthogonal to the normal \bar{n} , to make the triple vector product be zero requires only that $(\nabla \times \bar{u} - \bar{w})$ is parallel to \bar{n} on Γ , which is represented algebraically by (2.12b). It is not necessary to require that $(\nabla \times \bar{u} - \bar{w}) = \bar{0}$ on Γ .

The first-order div-curl system (2.1), the least-squares weak formulation (2.9), the second-order Poisson equations (2.12), and the Galerkin weak form (2.11) are all equivalent. Now it turns out that the least-squares method (2.9) for the div-curl equations (2.1) corresponds to using the Galerkin method (2.11) to solve three independent Poisson equations (2.12a) with three coupled boundary conditions (2.12b) and (2.12c). Here (2.12b) serves as two natural boundary conditions, and (2.12c) as an essential boundary condition. The attractions of using (2.12) are obvious. One avoids dealing with the incompressibility constraint (2.1a); instead, one deals with Poisson equations that everyone would rather solve.

It is well known that the Galerkin method is a perfect method for Poisson equations. Here “perfect” means that the corresponding finite element method has an optimal rate of convergence and leads to a symmetric positive definite matrix. This fact explains why the least-squares method is a perfect method for the first-order system (2.1).

Unfortunately, this perfect least-squares method is often misunderstood. Some people think that “the Euler-Lagrange equation derived from the least-squares weak formulation is some derivative of the original equation, hence this introduces the possibility of spurious solutions if incorrect boundary conditions are used.” However, our derivation clearly shows that the least-squares method does not require any additional boundary condition. Only if someone would like to use, for example, the finite difference method to solve the associated Euler-Lagrange equation (2.12a), additional natural boundary conditions (2.12b) (which are simply taken from the original first-order system) are needed in order to obtain the solution.

2.4 The div-curl method

The derivation in §2.3 shows that the least-squares method converts the difficult first-order div-curl system into an easy second-order system and reveals that each component of the velocity satisfies a Poisson equation. Here we show how to derive a high-order version of the differential equations without any risk of generating spurious solutions by another systematic way. By virtue of Theorem 2, System (2.1) is equivalent to

$$\nabla \cdot \bar{u} = 0 \text{ in } \Omega, \quad (2.13a)$$

$$\nabla \cdot (\nabla \times \bar{u} - \bar{w}) = 0 \text{ in } \Omega, \quad (2.13b)$$

$$\nabla \times (\nabla \times \bar{u} - \bar{w}) = \bar{0} \text{ in } \Omega, \quad (2.13c)$$

$$\bar{n} \times (\nabla \times \bar{u} - \bar{w}) = \bar{0} \text{ on } \Gamma, \quad (2.13d)$$

$$\bar{n} \cdot \bar{u} = 0 \text{ on } \Gamma. \quad (2.13e)$$

Due to the compatibility constraint (2.2a) and $\nabla \cdot \nabla \times \bar{u} = 0$, Equation (2.13b) is always satisfied. After simplification System (2.13) becomes

$$-\Delta \bar{u} = \nabla \times \bar{w} \text{ in } \Omega, \quad (2.14a)$$

$$\bar{n} \times (\nabla \times \bar{u} - \bar{w}) = \bar{0} \text{ on } \Gamma, \quad (2.14b)$$

$$\bar{n} \cdot \bar{u} = 0 \text{ on } \Gamma, \quad (2.14c)$$

which is the same as Equation (2.12) obtained by the least-squares method. Here we remark that by this div-curl method itself it is not clear whether the divergence-free equation of the velocity, i.e. Equation (2.13a), can be eliminated or not. It is only through the least-squares method that we are able to make sure that the satisfaction of Equation (2.14a) and boundary condition (2.14b) and (2.14c) can guarantee that the solution of \bar{u} is divergence-free.

Yet from Theorem 2 we may choose another boundary condition to replace (2.14b), so that we solve the following system:

$$\nabla \cdot \bar{u} = 0 \text{ in } \Omega, \quad (2.15a)$$

$$-\Delta \bar{u} = \nabla \times \bar{w} \text{ in } \Omega, \quad (2.15b)$$

$$\bar{n} \cdot (\nabla \times \bar{u} - \bar{w}) = 0 \text{ on } \Gamma, \quad (2.15c)$$

$$\bar{n} \cdot \bar{u} = 0 \text{ on } \Gamma. \quad (2.15d)$$

In this case the divergence-free equation (2.15a) cannot be eliminated.

3. The Div-Curl System (Case 2)

Let us consider the following three-dimensional div-curl equations with another important boundary condition:

$$\nabla \cdot \bar{u} = \rho \text{ in } \Omega, \quad (3.1a)$$

$$\nabla \times \bar{u} = \bar{0} \text{ in } \Omega, \quad (3.1b)$$

$$\bar{n} \times \bar{u} = \bar{0} \text{ on } \Gamma, \quad (3.1c)$$

where ρ is given. The first-order system (3.1) governs, for example, static electric problems in which \bar{u} is the electric field, and ρ the density of the electric charge. System (2.1) has only one algebraic boundary condition, while System (3.1) includes two independent algebraic boundary conditions. We shall explain why these two boundary conditions are correct.

At first we show that System (3.1) is also properly determined and elliptic. By introducing a dummy variable ϕ , System (3.1) can be written as

$$\nabla \cdot \bar{u} = \rho \text{ in } \Omega, \quad (3.3a)$$

$$\nabla \phi + \nabla \times \bar{u} = \bar{0} \text{ in } \Omega, \quad (3.3b)$$

$$\bar{n} \times \bar{u} = \bar{0} \text{ on } \Gamma. \quad (3.3c)$$

Notice that in this case we don't impose any boundary condition for the dummy variable ϕ . By virtue of Theorem 2, (3.3b) is equivalent to the following equations and boundary condition:

$$\nabla \cdot (\nabla \phi + \nabla \times \bar{u}) = 0 \text{ in } \Omega, \quad (3.4a)$$

$$\nabla \times (\nabla \phi + \nabla \times \bar{u}) = \bar{0} \text{ in } \Omega, \quad (3.4b)$$

$$\bar{n} \cdot (\nabla \phi + \nabla \times \bar{u}) = 0 \text{ on } \Gamma. \quad (3.4c)$$

Since $\bar{n} \times \bar{u} = \bar{0}$ on Γ , that is, \bar{u} is parallel with \bar{n} on Γ , or \bar{u} is perpendicular to Γ , we necessarily have

$$\bar{n} \cdot (\nabla \times \bar{u}) = 0 \text{ on } \Gamma. \quad (3.5)$$

The proof of (3.5) is straightforward. Assume the contrary, say, $\bar{n} \cdot (\nabla \times \bar{u}) > 0$ at a point P on Γ , then in a neighbourhood $\partial\Gamma$ of P we have

$$\bar{n} \cdot (\nabla \times \bar{u}) > \varepsilon > 0,$$

in which ε is a small positive constant. From the Stokes theorem we have a contradiction:

$$0 = \oint_c \bar{u} \cdot d\bar{l} = \int_{\partial\Gamma} (\nabla \times \bar{u}) \cdot \bar{n} ds > 0,$$

where c is the boundary contour of $\partial\Gamma$.

Taking into account (3.5), (3.4a) and (3.4c) lead to

$$\Delta\phi = 0 \text{ in } \Omega, \quad (3.6)$$

$$\frac{\partial\phi}{\partial n} = 0 \text{ on } \Gamma. \quad (3.7)$$

That is, ϕ is a constant, or $\nabla\phi \equiv \bar{0}$ in Ω . Therefore System (3.3) with four equations and four unknowns is equivalent to System (3.1). System (3.3) is elliptic and properly determined, so is System (3.1). The boundary condition (3.1c) requires that two tangential components of \bar{u} must be zero on the surface Γ . In this case no boundary condition on the dummy variable ϕ should be specified, so altogether there are only two boundary conditions which are consistent with a 2×2 first-order elliptic system.

4. Orthogonal Decomposition of Vectors

Theorem 4. Every vector $\bar{u} \in H^1(\Omega)^3$ has the orthogonal decomposition:

$$\bar{u} = \nabla q + \nabla \times \bar{\psi}, \quad (4.1)$$

where $q \in H^2(\Omega)/\mathfrak{R}$ and $\bar{\psi} \in H^2(\Omega)^3$.

Proof. we note that for the purpose of investigation in this paper, there is no need to find q and $\bar{\psi}$. This theorem shall be proved by directly seeking for ∇q and $\nabla \times \bar{\psi}$. By virtue of Theorem 2, Equation (4.1) is equivalent to the following equations and boundary condition:

$$\nabla \cdot (\nabla q + \nabla \times \bar{\psi} - \bar{u}) = 0 \text{ in } \Omega, \quad (4.2a)$$

$$\nabla \times (\nabla q + \nabla \times \bar{\psi} - \bar{u}) = \bar{0} \text{ in } \Omega, \quad (4.2b)$$

$$\bar{n} \cdot (\nabla q + \nabla \times \bar{\psi} - \bar{u}) = 0 \text{ on } \Gamma. \quad (4.2c)$$

Taking into account $\nabla \cdot \nabla \times \bar{\psi} = \bar{0}$ and $\nabla \times \nabla q = 0$, System (4.2) can be written as follows:

$$\Delta q = \nabla \cdot \bar{u} \text{ in } \Omega, \quad (4.3a)$$

$$\nabla \times (\nabla \times \bar{\psi}) = \nabla \times \bar{u} \text{ in } \Omega, \quad (4.3b)$$

$$\bar{n} \cdot (\nabla q + \nabla \times \bar{\psi}) = \bar{n} \cdot \bar{u} \text{ on } \Gamma. \quad (4.3c)$$

We may solve the following Neuman problem of Poisson equation to obtain q :

$$\Delta q = \nabla \cdot \bar{u} \text{ in } \Omega, \quad (4.4a)$$

$$\bar{n} \cdot \nabla q = \bar{n} \cdot \bar{u} \text{ on } \Gamma, \quad (4.4b)$$

where the boundary condition (4.4b) is additionally supplied. Although q is not unique, i.e., an arbitrary constant can be added into q , ∇q is uniquely determined.

Now $\bar{\psi}$ should satisfy

$$\nabla \cdot (\nabla \times \bar{\psi}) = 0 \text{ in } \Omega, \quad (4.5a)$$

$$\nabla \times (\nabla \times \bar{\psi}) = \nabla \times \bar{u} \text{ in } \Omega, \quad (4.5b)$$

$$\bar{n} \cdot (\nabla \times \bar{\psi}) = 0 \text{ on } \Gamma. \quad (4.5c)$$

$\nabla \times \bar{\psi}$ in System (4.5) may be considered as an unknown vector and can be uniquely determined by the least-squares method described in Section 2. Therefore the validation of the decomposition (4.1) is proved.

Using (B.4) and (4.5c) we have

$$(\nabla q, \nabla \times \bar{\psi}) = \langle \bar{n} \cdot (\nabla \times \bar{\psi}), q \rangle = 0,$$

that is, ∇q and $\nabla \times \bar{\psi}$ are orthogonal. □

Since q is the solution of the Neuman problem of Poisson equation (4.4), we have the following regularity result[27,28]:

$$|q|_2 \leq C \{ \|\nabla \cdot \bar{u}\|_0 + \|\bar{n} \cdot \bar{u}\|_{1/2, \Gamma} \}. \quad (4.6)$$

Since $\nabla \times \bar{\psi}$ is the least-squares solution of (4.5), by virtue of (2.7) we have

$$|\nabla \times \bar{\psi}|_1 \leq \|\nabla \times \bar{u}\|_0. \quad (4.7)$$

Hence by using (4.1), (4.6) and (4.7) we obtain

$$\begin{aligned} \|\bar{u}\|_1 &\leq \|\nabla q\|_1 + \|\nabla \times \bar{\psi}\|_0 + |\nabla \times \bar{\psi}|_1 \\ &\leq \|\nabla q\|_1 + \|\bar{u}\|_0 + \|\nabla q\|_0 + |\nabla \times \bar{\psi}|_1 \\ &\leq \|\bar{u}\|_0 + |q|_2 + |\nabla \times \bar{\psi}|_1. \end{aligned}$$

$$\leq C\{\|\bar{u}\|_{0,\Omega} + \|\nabla \cdot \bar{u}\|_{0,\Omega} + \|\bar{n} \cdot \bar{u}\|_{1/2,\Gamma} + \|\nabla \times \bar{u}\|_{0,\Omega}\}.$$

In fact we have proved the following Theorem.

Theorem 5 (Friedrich's Second Inequality). Let Ω be a bounded and convex open region of \mathfrak{R}^3 with a piecewise C^1 boundary Γ . Every $\bar{u} \in H^1(\Omega)^3$ satisfies

$$\|\bar{u}\|_{1,\Omega} \leq C\{\|\bar{u}\|_{0,\Omega} + \|\nabla \cdot \bar{u}\|_{0,\Omega} + \|\nabla \times \bar{u}\|_{0,\Omega} + \|\bar{n} \cdot \bar{u}\|_{1/2,\Gamma}\}. \quad (4.8)$$

5. The Velocity-Pressure-Vorticity Formulation of the Navier-Stokes Equations

Usually the Navier-Stokes equations governing the steady-state incompressible viscous flows are written in the following velocity-pressure formulation

$$\bar{u} \cdot \nabla \bar{u} + \nabla p - \frac{1}{Re} \Delta \bar{u} = \bar{f} \quad \text{in } \Omega, \quad (5.1a)$$

$$\nabla \cdot \bar{u} = 0 \quad \text{in } \Omega. \quad (5.1b)$$

We assume that the domain Ω is bounded, convex and simply connected with a piecewise C^1 boundary Γ . All variables in (5.1) are nondimensionalized, $\bar{u} = (u, v, w)$ denotes the velocity, p the pressure, $\bar{f} = (f_x, f_y, f_z) \in L^2(\Omega)^3$ the body force, and Re the Reynolds number, defined as

$$Re = \frac{UL}{\nu},$$

where L is a reference length, U a reference velocity and ν the kinematic viscosity.

All mathematical analyses of the existence of the solution of (5.1) are conducted by using the Galerkin method mainly under the standard velocity boundary condition

$$\bar{u} = \bar{u}_\Gamma \quad \text{on } \Gamma, \quad (5.1c)$$

where \bar{u}_Γ denotes a given function defined on the boundary Γ .

It is well known that the Galerkin mixed method leads to a saddle-point problem, thus the sophisticated LBB condition is invoked to guarantee the existence of the solution. It is notoriously difficult to verify and satisfy the LBB condition. From a numerical point of view, the most difficult problem associated with the Galerkin mixed method is that the resulting discretized algebraic equations are nonsymmetric and nonpositive-definite which are hard to deal with for large problems. All these difficulties motivated us to apply the least-squares method.

Since the momentum equation (5.1a) involves the second-order derivatives of velocity, the application of the least-squares method to (5.1) requires the use of impractical continuously differentiable functions. In order to avoid this trouble, one has to consider the governing equations of incompressible flow in the form of a first-order system. The velocity-pressure-stress formulation is one of the choices. However, this formulation has too many unknowns. Moreover, as pointed out in [15], the three-dimensional velocity-pressure-stress formulation has nine independent unknowns and nine independent equations, and thus cannot be elliptic in the ordinary sense. Consequently, the selection of proper boundary conditions becomes a difficult task. Instead, we introduce the vorticity $\bar{\omega} = (\omega_x, \omega_y, \omega_z) = \nabla \times \bar{u}$ as an additional independent unknown vector, and rewrite the incompressible Navier-Stokes equations in the following first-order quasi-linear velocity-pressure-vorticity formulation:

$$\bar{u} \cdot \nabla \bar{u} + \nabla p + \frac{1}{Re} \nabla \times \bar{\omega} = \bar{f} \text{ in } \Omega, \quad (5.2a)$$

$$\nabla \cdot \bar{\omega} = 0 \quad \text{in } \Omega, \quad (5.2b)$$

$$\bar{\omega} - \nabla \times \bar{u} = \bar{0} \text{ in } \Omega, \quad (5.2c)$$

$$\nabla \cdot \bar{u} = 0 \text{ in } \Omega. \quad (5.2d)$$

Here we have included the compatibility constraint condition (5.2b), i.e., the divergence of the vorticity vector equals zero, to make System (5.2) elliptic in the ordinary sense. The determinedness and ellipticity of (5.2) have been proved in [15]. For completeness we briefly repeat the proof in the following.

At first glance, one may think that System (5.2) is overdetermined, since there are seven known variables, i.e., three velocity components u, v, w , one pressure p and three vorticity components $\omega_x, \omega_y, \omega_z$, and eight equations. We shall show that System (5.2) is really properly determined and elliptic by using the same technique as discussed in Section 2. As explained in [15], the nonlinear convective terms and the Reynolds number have no effect on the classification, so we may just consider the following first-order system of the Stokes problem:

$$\nabla p + \nabla \times \bar{\omega} = \bar{f} \text{ in } \Omega, \quad (5.3a)$$

$$\nabla \cdot \bar{\omega} = 0 \quad \text{in } \Omega, \quad (5.3b)$$

$$-\bar{\omega} + \nabla \phi + \nabla \times \bar{u} = \bar{0} \text{ in } \Omega, \quad (5.3c)$$

$$\nabla \cdot \bar{u} = 0 \text{ in } \Omega. \quad (5.3d)$$

Here we have already introduced a dummy variable ϕ in (5.3c), which satisfies the boundary condition $\phi = 0$ on Γ . Substituting (5.3c) into (5.3b) yields $\Delta \phi = 0$, thus $\phi \equiv 0$ in Ω . That is, the introduction of ϕ does not change anything. However, now there are eight

unknowns and eight equations in (5.3), and System (5.3) and hence System (5.2) is indeed properly determined.

We note that in some cases the specification of the boundary condition for the dummy variable ϕ is unnecessary, and $\nabla\phi \equiv \bar{0}$ can also be guaranteed, see Section 6.

Now let us classify System (5.3). In Cartesian co-ordinates, System (5.3) is given as

$$\begin{aligned}
\frac{\partial p}{\partial x} + \frac{\partial \omega_z}{\partial y} - \frac{\partial \omega_y}{\partial z} &= f_x, \\
\frac{\partial p}{\partial y} + \frac{\partial \omega_x}{\partial z} - \frac{\partial \omega_z}{\partial x} &= f_y, \\
\frac{\partial p}{\partial z} + \frac{\partial \omega_y}{\partial x} - \frac{\partial \omega_x}{\partial y} &= f_z, \\
\frac{\partial \omega_x}{\partial x} + \frac{\partial \omega_y}{\partial y} + \frac{\partial \omega_z}{\partial z} &= 0, \\
-\omega_x + \frac{\partial \phi}{\partial x} + \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} &= 0, \\
-\omega_y + \frac{\partial \phi}{\partial y} + \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} &= 0, \\
-\omega_z + \frac{\partial \phi}{\partial z} + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} &= 0, \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0.
\end{aligned} \tag{5.4}$$

We may write (5.4) in the standard matrix form:

$$\mathbf{A}_1 \frac{\partial \mathbf{u}}{\partial x} + \mathbf{A}_2 \frac{\partial \mathbf{u}}{\partial y} + \mathbf{A}_3 \frac{\partial \mathbf{u}}{\partial z} + \mathbf{A} \mathbf{u} = \mathbf{f}, \tag{5.5}$$

in which

$$\mathbf{A}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{A}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{A}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{f} = \begin{pmatrix} f_x \\ f_y \\ f_z \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} u \\ v \\ w \\ \phi \\ \omega_x \\ \omega_y \\ \omega_z \\ p_1 \end{pmatrix}.$$

The characteristic polynomial associated with System (5.5)

$$\det(\mathbf{A}_1\xi + \mathbf{A}_2\eta + \mathbf{A}_3\zeta) = \det \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -\zeta & \eta & \xi \\ 0 & 0 & 0 & 0 & \zeta & 0 & -\xi & \eta \\ 0 & 0 & 0 & 0 & -\eta & \xi & 0 & \zeta \\ 0 & 0 & 0 & 0 & \xi & \eta & \zeta & 0 \\ 0 & -\zeta & \eta & \xi & 0 & 0 & 0 & 0 \\ \zeta & 0 & -\xi & \eta & 0 & 0 & 0 & 0 \\ -\eta & \xi & 0 & \zeta & 0 & 0 & 0 & 0 \\ \xi & \eta & \zeta & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = (\xi^2 + \eta^2 + \zeta^2)^4 \neq 0$$

for all nonzero triplet (ξ, η, ζ) , System (5.5) or (5.4) and thus System (5.3) is indeed elliptic. The ellipticity can be easily understood in the following way. Equations (5.3a) and (5.3b) constitute a div-curl system of the vorticity, and Equations (5.3c) and (5.3d) constitute a div-curl system of the velocity. These two div-curl systems are coupled through the vorticity in Equation (5.3c) and form the Stokes equations. In other words, the principal part of the Stokes operator consists of two identical elliptic div-curl operators.

The classification in this paper is based on an ordinary method, so Equation (5.2b) is needed to guarantee ellipticity. System (5.2) is also elliptic in the Agmon-Douglis-Nirenberg(ADN) sense[29]. In fact, System (5.2) without (5.2b) is also elliptic in the ADN sense. In the ADN theory the non-principal part of the operator must be considered. The corresponding least-squares method can be developed based on minimizing a weighted (mesh-dependent) L_2 norm of the residuals following the idea proposed in [30] for general

elliptic systems and the Stokes equations in the velocity-pressure formulation, and in [31] for the Stokes equations in the velocity-pressure-vorticity formulation.

6. Boundary Conditions for the Navier-Stokes Equations

The boundary conditions for the Navier-Stokes equations have been extensively discussed in the literature. The most complete and thorough examination of this topic may be found in Gresho's state-of-the-art reports[2,32,33] and the references cited therein. For boundary conditions other than the specification of the velocity one may also consult Gunzburger[9], Girault[34], Pironneau[10] and Verfurth[35]. Based on different choices for the formulation of the viscous term in the velocity-pressure formulation and the Galerkin method, Gunzburger correctly gives many possible combinations of nonstandard boundary conditions. Here we should mention that for nonstandard boundary conditions there were very few rigorous analyses available. In this section we shall list all possible combinations of boundary conditions based on the first-order velocity-pressure-vorticity formulation.

Before investigation of boundary conditions we first discuss the equivalence between the velocity-pressure formulation (5.1) and the velocity-pressure-vorticity formulation (5.2). System (5.2) is obtained from (5.1) by introducing the vorticity $\bar{\omega}$ which is some kind of derivative of the velocity \bar{u} and thus reducing the order of differential operator. Reducing the order of differential equations in this way does not generate spurious solutions. It means that System (5.2) is equivalent to System (5.1), that is, the solution of (5.2) is the solution of (5.1). Conversely, we may think that System (5.1) is deduced from System (5.2) when the definition of the vorticity (5.2c) is substituted into (5.2a) and (5.2b). This type of substitution and combination does not generate spurious solutions either. Therefore, the velocity-pressure formulation (5.1) and the velocity-pressure-vorticity (5.2) formulation are mutually equivalent. This equivalence implies that the permissible boundary conditions for System (5.2) must be the permissible boundary conditions for System (5.1). The reverse is also true. Consequently, it is not possible that some boundary conditions that have been shown to be legitimate for System (5.1) might not be so for System (5.2), and vice versa.

As pointed out in the previous section that the nonlinear convective term has no effect on the classification of the Navier-Stokes equations, hence the boundary conditions for the Stokes equations are valid for the Navier-Stokes equations. So we need only to analyse the Stokes problem (5.3). Since the system (5.3) is of first-order, the boundary conditions do not involve the derivatives of unknowns. In other words, there are only essential boundary conditions for the solution of first-order partial differential equations. This fact precludes the mathematical legitimacy of taking the derivative of pressure as a boundary condition.

For convenience we rewrite the Stokes problem (5.3) as the following two coupled

Table 1. Combination of Boundary Conditions

| <i>Boundary Conditions</i> | <i>3D</i> | <i>2D</i> | <i>Remarks</i> |
|--|--|---|--------------------------|
| (1) Normal velocity Tangential vorticity | $\bar{n} \cdot \bar{u}$ $\bar{n} \times \bar{\omega}$ | $\bar{n} \cdot \bar{u}$ ω | |
| (2) Pressure Normal vorticity Normal velocity | p $\bar{n} \cdot \bar{\omega}$ $\bar{n} \cdot \bar{u}$ | p $\bar{n} \cdot \bar{u}$ | |
| (3) Pressure Normal vorticity Tangential velocity | p $\bar{n} \cdot \bar{\omega}$ $\bar{n} \times \bar{u}$ | p $\bar{n} \times \bar{u}$ | |
| (4) Tangential velocity Tangential vorticity | $\bar{n} \times \bar{u}$ $\bar{n} \times \bar{\omega}$ | $\bar{n} \times \bar{u}$ ω | |
| (5) Velocity | $\bar{n} \cdot \bar{u}$ $\bar{n} \times \bar{u}$ | $\bar{n} \cdot \bar{u}$ $\bar{n} \times \bar{u}$ | ADN |
| (6) Pressure Normal vorticity Tangential vorticity | p $\bar{n} \cdot \bar{\omega}$ $\bar{n} \times \bar{\omega}$ | p ω | on a part of Γ |

div-curl systems:

$$\nabla p + \nabla \times \bar{\omega} = \bar{f} \text{ in } \Omega, \quad (6.1a)$$

$$\nabla \cdot \bar{\omega} = 0 \quad \text{in } \Omega, \quad (6.1b)$$

and

$$\nabla \phi + \nabla \times \bar{u} = \bar{\omega} \text{ in } \Omega, \quad (6.2a)$$

$$\nabla \cdot \bar{u} = 0 \quad \text{in } \Omega. \quad (6.2b)$$

If the vorticity $\bar{\omega}$ in (6.2a) is known, the structure of these two div-curl systems would be identical. In Section 2 we have thoroughly investigated this type of div-curl system. In order to solve (6.1) to obtain p and $\bar{\omega}$, on the boundary we should specify

$$(p \text{ and } \bar{n} \cdot \bar{\omega}) \quad \text{or} \quad \bar{n} \times \bar{\omega}.$$

When $\bar{\omega}$ is obtained, to obtain ϕ and \bar{u} we may solve (6.2) with the boundary condition of

$$(\phi \text{ and } \bar{n} \cdot \bar{u}) \quad \text{or} \quad \bar{n} \times \bar{u}.$$

From the above consideration we can immediately list four permissible combinations of boundary conditions (1)-(4) in Table 1 for the Stokes problem. In Table 1 for three-dimensional problems we don't explicitly include the boundary condition $\phi = 0$ on Γ . If we understand that $\phi = 0$ always comes with the condition of $\bar{n} \cdot \bar{u}$ and count this condition, then the total number of boundary conditions is four for 3D problems. Since there are eight equations and eight unknowns in the first-order elliptic system (5.3) or (6.1) and (6.2), we need four boundary conditions on each fixed boundary. In (5) and (6) of Table 1 we list other possible choices which are not from the above consideration but still satisfy the requirement of four conditions on the boundary. For two-dimensional problems in the first-order velocity-pressure-vorticity formulation there are four unknowns, i.e., u, v, p, ω , and four equations, i.e., two momentum equations, one definition of the vorticity and the incompressibility; and no dummy variable is involved, see [12]. Therefore, two boundary conditions are needed for two-dimensional problems. In Table 1 we also list corresponding boundary conditions for two-dimensional problems.

7. Permissibility of the boundary conditions

In this section we shall rigorously prove the well-posedness of the boundary conditions (1)-(4) in Table 1. The boundary conditions (1)-(5) in Table 1 can be used on the entire boundary or on a part of boundary Γ . For simplicity we consider only one kind of homogeneous boundary conditions on the entire boundary. The results can be extended to mixed and inhomogeneous cases without difficulty.

Given the elliptic differential operator, the question of well-posedness reduces to verification of the permissibility of the boundary conditions. For general elliptic systems one may use the ADN theory to accomplish this task. For first-order systems in the plane, one may also try to answer this question by applying the theory developed in [36]. However, both techniques invoke the modern theory of elliptic partial differential equations and are quite difficult to understand by engineers. In this paper we try to identify the permissible boundary conditions by using an elementary treatment. The mathematical tools used are the least-squares method, Green's formulae (integration by parts) and Friedrich's inequalities established for div and curl operators. An elliptic system with supplied boundary conditions is considered to be well-posed, if one can prove that the corresponding least-squares method leads to a coercive bilinear form. In the following, we do this case by case.

7.1 $u_n = 0, \omega_{\tau 1} = 0, \omega_{\tau 2} = 0$ ($\bar{n} \cdot \bar{u} = 0, \bar{n} \times \bar{\omega} = \bar{0}$) on Γ

These boundary conditions may be used for the symmetric plane. The inhomogeneous version may be used for the uniform inflow boundary in which the normal components of the velocity and two tangential components of the vorticity are prescribed. These conditions correspond to those in the velocity-pressure formulation, i.e., the normal velocity and the tangential stresses are given. For example, let us consider a piece of boundary with $\bar{n} = (1, 0, 0)$. We have that

$$\begin{aligned} u &= 0, \\ \omega_y &= \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0, \\ \omega_z &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0. \end{aligned}$$

From $u = 0$ on this boundary we deduce that

$$\begin{aligned} \frac{\partial u}{\partial z} &= 0, \\ \frac{\partial u}{\partial y} &= 0. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\partial w}{\partial x} &= 0, \\ \frac{\partial v}{\partial x} &= 0. \end{aligned}$$

This implies that

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0,$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0.$$

That is, the tangential strains and thus the tangential stresses are zero.

In order to guarantee the uniqueness of the solution of pressure p , we require that the pressure has zero mean over Ω :

$$\int_{\Omega} p dx = 0. \quad (7.1)$$

The least-squares method minimizes the following functional

$$J(\mathbf{U}) = \|\nabla p + \nabla \times \bar{\omega} - \bar{f}\|_0^2 + \|\nabla \cdot \bar{\omega}\|_0^2 + \|\bar{\omega} - \nabla \times \bar{u}\|_0^2 + \|\nabla \cdot \bar{u}\|_0^2, \quad (7.2)$$

where $\mathbf{U} = (\bar{u}, p, \bar{\omega}) \in \mathbf{H} = H^1(\Omega)^7$, and \mathbf{U} satisfies the corresponding homogeneous boundary conditions on Γ . Furthermore, $\bar{f} \in L_2(\Omega)^3$. Following standard arguments of variational calculus, we deduce that the least-squares weak solution \mathbf{U} necessarily satisfies

$$B(\mathbf{U}, \mathbf{V}) = L(\mathbf{V}) \quad \forall \mathbf{V} = (\bar{v}, q, \bar{\tau}) \in \mathbf{H}, \quad (7.3)$$

in which

$$B(\mathbf{U}, \mathbf{V}) = (\nabla p + \nabla \times \bar{\omega}, \nabla q + \nabla \times \bar{\tau}) + (\nabla \cdot \bar{\omega}, \nabla \cdot \bar{\tau}) + (\bar{\omega} - \nabla \times \bar{u}, \bar{\tau} - \nabla \times \bar{v}) + (\nabla \cdot \bar{u}, \nabla \cdot \bar{v}), \quad (7.4)$$

$$L(\mathbf{V}) = (\bar{f}, \nabla q + \nabla \times \bar{\tau}), \quad (7.5)$$

and \mathbf{V} satisfies the same homogeneous boundary conditions as \mathbf{U} .

Clearly $B(\mathbf{U}, \mathbf{V})$ is symmetric. If we can prove $B(\mathbf{V}, \mathbf{V})$ is coersive, then the existence and the uniqueness of the weak solution follow from the Lax-Milgram theorem in a standard manner. Consequently, the corresponding finite element method has an optimal rate of convergence for all unknowns.

Now we examine the coersivity of $B(\mathbf{V}, \mathbf{V})$. We have

$$B(\mathbf{V}, \mathbf{V}) = \|\nabla q + \nabla \times \bar{\tau}\|_0^2 + \|\nabla \cdot \bar{\tau}\|_0^2 + \|\bar{\tau} - \nabla \times \bar{v}\|_0^2 + \|\nabla \cdot \bar{v}\|_0^2. \quad (7.6)$$

Let us expand the first term in (7.6). Since $\bar{n} \times \bar{\tau} = \bar{0}$ on Γ , using Green's formula (B.6), we have

$$(\nabla q, \nabla \times \bar{\tau}) = \langle \nabla q, \bar{n} \times \bar{\tau} \rangle = 0, \quad (7.7)$$

and thus

$$\|\nabla q + \nabla \times \bar{\tau}\|_0^2 = \|\nabla q\|_0^2 + \|\nabla \times \bar{\tau}\|_0^2 + 2(\nabla q, \nabla \times \bar{\tau}) = \|\nabla q\|_0^2 + \|\nabla \times \bar{\tau}\|_0^2. \quad (7.8)$$

By virtue of (7.8) we have

$$B(\mathbf{V}, \mathbf{V}) = \|\nabla q\|_0^2 + \|\nabla \times \bar{\tau}\|_0^2 + \|\nabla \cdot \bar{\tau}\|_0^2 + \|\bar{\tau} - \nabla \times \bar{v}\|_0^2 + \|\nabla \cdot \bar{v}\|_0^2. \quad (7.9)$$

From (7.9) we have

$$B(\mathbf{V}, \mathbf{V}) \geq \|\nabla q\|_0^2 = |q|_1^2. \quad (7.10)$$

Since q satisfies the zero mean constraint (7.1), from the Poincare inequality (C.1) we have

$$C|q|_1^2 \geq \|q\|_1^2. \quad (7.11)$$

Combining (7.10) with (7.11) yields

$$CB(\mathbf{V}, \mathbf{V}) \geq \|q\|_1^2. \quad (7.12)$$

From (7.9) we also have

$$B(\mathbf{V}, \mathbf{V}) \geq \|\nabla \times \bar{\tau}\|_0^2 + \|\nabla \cdot \bar{\tau}\|_0^2, \quad (7.13)$$

$$B(\mathbf{V}, \mathbf{V}) \geq \|\bar{\tau} - \nabla \times \bar{v}\|_0^2, \quad (7.14)$$

$$B(\mathbf{V}, \mathbf{V}) \geq \|\nabla \cdot \bar{v}\|_0^2. \quad (7.15)$$

Since $\bar{n} \times \bar{\tau} = \bar{0}$ on Γ , from Theorem 3 (Friedrichs' inequality) we have the inequality:

$$C(\|\nabla \times \bar{\tau}\|_0^2 + \|\nabla \cdot \bar{\tau}\|_0^2) \geq \|\bar{\tau}\|_1^2 \geq \|\bar{\tau}\|_0^2. \quad (7.16)$$

Combining (7.13) with (7.16) yields

$$CB(\mathbf{V}, \mathbf{V}) \geq \|\bar{\tau}\|_1^2 \quad (7.17)$$

and

$$CB(\mathbf{V}, \mathbf{V}) \geq \|\bar{\tau}\|_0^2$$

or

$$C(B(\mathbf{V}, \mathbf{V}))^{\frac{1}{2}} \geq \|\bar{\tau}\|_0. \quad (7.18)$$

From (7.14) we have

$$(B(\mathbf{V}, \mathbf{V}))^{\frac{1}{2}} \geq \|\bar{\tau} - \nabla \times \bar{v}\|_0. \quad (7.19)$$

Combining (7.18) with (7.19) and using the triangle inequality we have

$$C(B(\mathbf{V}, \mathbf{V}))^{\frac{1}{2}} \geq \|\bar{\tau}\|_0 + \|\bar{\tau} - \nabla \times \bar{v}\|_0 \geq \|\nabla \times \bar{v}\|_0,$$

that is

$$C(B(\mathbf{V}, \mathbf{V})) \geq \|\nabla \times \bar{v}\|_0^2. \quad (7.20)$$

Combining (7.15) with (7.20) leads to

$$CB(\mathbf{V}, \mathbf{V}) \geq \|\nabla \times \bar{v}\|_0^2 + \|\nabla \cdot \bar{v}\|_0^2. \quad (7.21)$$

Since $\bar{n} \cdot \bar{v} = 0$ on Γ , again from Theorem 3 we have the inequality

$$C(\|\nabla \times \bar{v}\|_0^2 + \|\nabla \cdot \bar{v}\|_0^2) \geq \|\bar{v}\|_1^2. \quad (7.22)$$

Combining (7.21) with (7.22) yields

$$CB(\mathbf{V}, \mathbf{V}) \geq \|\bar{v}\|_1^2. \quad (7.23)$$

Combining (7.12), (7.17) and (7.23) together we finally obtain that

$$CB(\mathbf{V}, \mathbf{V}) \geq \|\bar{v}\|_1^2 + \|q\|_1^2 + \|\bar{\tau}\|_1^2. \quad (7.24)$$

This shows that $B(\mathbf{V}, \mathbf{V})$ is indeed bounded below in H^1 norm and thus coersive. Consequently, it is trivial to prove that this problem has a unique solution that satisfies the following bound:

$$\|\bar{u}\|_1 + \|p\|_1 + \|\bar{\omega}\|_1 \leq C\|\bar{f}\|_0.$$

7.2 $p = 0, u_n = 0, \omega_n = 0$ ($p = 0, \bar{n} \cdot \bar{u} = 0, \bar{n} \cdot \bar{\omega} = 0$) on Γ

The related inhomogeneous case represents, for example, the well developed inflow boundary, in which the normal velocity is given, and the normal vorticity and the pressure are prescribed be zero. In two-dimensional cases, only u_n and p are prescribed. These boundary conditions seem difficult to justify by the Galerkin method. The numerical results can be found in Bochev and Gunzburger[19,31].

The corresponding least-squares method minimizes the same functional (7.2). The proof of the coersivity of $B(\mathbf{V}, \mathbf{V})$ follows the steps of the previous case.

We have

$$B(\mathbf{V}, \mathbf{V}) = \|\nabla q + \nabla \times \bar{\tau}\|_0^2 + \|\nabla \cdot \bar{\tau}\|_0^2 + \|\bar{\tau} - \nabla \times \bar{v}\|_0^2 + \|\nabla \cdot \bar{v}\|_0^2. \quad (7.25)$$

We expand the first term in (7.25). Since $q = 0$ on Γ , using Green's formula (B.4), we have

$$(\nabla q, \nabla \times \bar{\tau}) = \langle q, \bar{n} \cdot (\nabla \times \bar{\tau}) \rangle = 0, \quad (7.26)$$

and thus

$$\|\nabla q + \nabla \times \bar{\tau}\|_0^2 = \|\nabla q\|_0^2 + \|\nabla \times \bar{\tau}\|_0^2 + 2(\nabla q, \nabla \times \bar{\tau}) = \|\nabla q\|_0^2 + \|\nabla \times \bar{\tau}\|_0^2. \quad (7.27)$$

By virtue of (7.27) we have

$$B(\mathbf{V}, \mathbf{V}) = \|\nabla q\|_0^2 + \|\nabla \times \bar{\tau}\|_0^2 + \|\nabla \cdot \bar{\tau}\|_0^2 + \|\bar{\tau} - \nabla \times \bar{v}\|_0^2 + \|\nabla \cdot \bar{v}\|_0^2. \quad (7.28)$$

From (7.28) we have

$$B(\mathbf{V}, \mathbf{V}) \geq \|\nabla q\|_0^2 = |q|_1^2. \quad (7.29)$$

Since $q = 0$ on Γ , from Poincare inequality (C.2) we have

$$C|q|_1^2 \geq \|q\|_1^2. \quad (7.30)$$

Combining (7.29) with (7.30) yields

$$CB(\mathbf{V}, \mathbf{V}) \geq \|q\|_1^2. \quad (7.31)$$

From (7.28) we also have

$$B(\mathbf{V}, \mathbf{V}) \geq \|\nabla \times \bar{\tau}\|_0^2 + \|\nabla \cdot \bar{\tau}\|_0^2, \quad (7.32)$$

$$B(\mathbf{V}, \mathbf{V}) \geq \|\bar{\tau} - \nabla \times \bar{v}\|_0^2, \quad (7.33)$$

$$B(\mathbf{V}, \mathbf{V}) \geq \|\nabla \cdot \bar{v}\|_0^2. \quad (7.34)$$

Since $\bar{n} \cdot \bar{\tau} = 0$ on Γ , from Theorem 3 we have the inequality:

$$C(\|\nabla \times \bar{\tau}\|_0^2 + \|\nabla \cdot \bar{\tau}\|_0^2) \geq \|\bar{\tau}\|_1^2 \geq \|\bar{\tau}\|_0^2. \quad (7.35)$$

Combining (7.32) with (7.35) we have

$$CB(\mathbf{V}, \mathbf{V}) \geq \|\bar{\tau}\|_1^2 \quad (7.36)$$

and

$$CB(\mathbf{V}, \mathbf{V}) \geq \|\bar{\tau}\|_0^2$$

or

$$C(B(\mathbf{V}, \mathbf{V}))^{\frac{1}{2}} \geq \|\bar{\tau}\|_0. \quad (7.37)$$

From (7.33) we have

$$(B(\mathbf{V}, \mathbf{V}))^{\frac{1}{2}} \geq \| -\bar{\tau} + \nabla \times \bar{v} \|_0. \quad (7.38)$$

Combining (7.37) with (7.38) and using the triangle inequality lead to

$$C(B(\mathbf{V}, \mathbf{V}))^{\frac{1}{2}} \geq \|\bar{\tau}\|_0 + \| -\bar{\tau} + \nabla \times \bar{v} \|_0 \geq \|\nabla \times \bar{v}\|_0,$$

that is

$$C(B(\mathbf{V}, \mathbf{V})) \geq \|\nabla \times \bar{v}\|_0^2. \quad (7.39)$$

Combining (7.34) with (7.39) leads to

$$CB(\mathbf{V}, \mathbf{V}) \geq \|\nabla \times \bar{v}\|_0^2 + \|\nabla \cdot \bar{v}\|_0^2. \quad (7.40)$$

Since $\bar{n} \cdot \bar{v} = 0$ on Γ , from Theorem 3 we have the inequality

$$C(\|\nabla \times \bar{v}\|_0^2 + \|\nabla \cdot \bar{v}\|_0^2) \geq \|\bar{v}\|_1^2. \quad (7.41)$$

Combining (7.40) with (7.41) yields

$$CB(\mathbf{V}, \mathbf{V}) \geq \|\bar{v}\|_1^2. \quad (7.42)$$

Combining (7.31), (7.36) and (7.42) together we finally obtain

$$CB(\mathbf{V}, \mathbf{V}) \geq \|\bar{v}\|_1^2 + \|q\|_1^2 + \|\bar{\tau}\|_1^2. \quad (7.43)$$

Therefore, the coersivity of $B(\mathbf{V}, \mathbf{V})$ is proved.

7.3 $p = 0, u_{\tau_1} = 0, u_{\tau_2} = 0, \omega_n = 0$ ($p = 0, \bar{n} \times \bar{u} = \bar{0}, \bar{n} \cdot \bar{\omega} = 0$) on Γ

This boundary condition may be used for the well developed exit boundary. Here four boundary conditions are prescribed. As mentioned in §3.3 $\bar{n} \times \bar{u} = \bar{0}$ on Γ analytically implies that $\bar{n} \cdot (\nabla \times \bar{u}) = 0$ on Γ . It seems that there are too many boundary conditions. In the previous cases we have specified the boundary condition $\phi = 0$ on Γ for the dummy variable ϕ in advance, so only three boundary conditions are needed. In the following we show that in the present case no boundary condition is needed for the dummy variable ϕ , so it is all right to specify four conditions.

By virtue of Theorem 2, Equation (5.3c) is equivalent to the following equations and boundary condition:

$$\nabla \times (-\bar{\omega} + \nabla \phi + \nabla \times \bar{u}) = \bar{0} \text{ in } \Omega, \quad (7.44a)$$

$$\nabla \cdot (-\bar{\omega} + \nabla \phi + \nabla \times \bar{u}) = 0 \text{ in } \Omega, \quad (7.44b)$$

$$\bar{n} \cdot (-\bar{\omega} + \nabla \phi + \nabla \times \bar{u}) = 0 \text{ on } \Gamma. \quad (7.44c)$$

Taking into account (5.3b) and $\nabla \cdot \nabla \times \bar{u} = 0$, Equation (7.44b) becomes

$$\Delta \phi = 0 \text{ in } \Omega. \quad (7.45a)$$

Taking into account $\omega_n = 0$ on Γ and (3.5), the boundary condition (7.44c) becomes

$$\frac{\partial \phi}{\partial n} = 0 \text{ on } \Gamma. \quad (7.45b)$$

Equation (7.45a) and (7.45b) imply that ϕ is a constant or $\nabla \phi \equiv \bar{0}$ in Ω . Therefore, four conditions in the present case automatically guarantee that the dummy variable ϕ can be eliminated in Equation (5.3c).

These boundary conditions correspond to those in the velocity-pressure formulation, i.e., the tangential velocity components and the normal stress are prescribed. To show this let's consider, for example, the surface with $\bar{n} = (1, 0, 0)$. Since $v = w = 0$, we have

$$\frac{\partial v}{\partial y} = 0,$$

$$\frac{\partial w}{\partial z} = 0.$$

Hence from the continuity of velocity we know that

$$\frac{\partial u}{\partial x} = 0.$$

Therefore,

$$\sigma_x = p + 2\nu \frac{\partial u}{\partial x} = 0,$$

that is, the normal stress is zero.

The least-squares method minimizes the same functional (7.2). We shall now prove the coersivity of $B(\mathbf{V}, \mathbf{V})$. We have

$$B(\mathbf{V}, \mathbf{V}) = \|\nabla q + \nabla \times \bar{\tau}\|_0^2 + \|\nabla \cdot \bar{\tau}\|_0^2 + \|\bar{\tau} - \nabla \times \bar{v}\|_0^2 + \|\nabla \cdot \bar{v}\|_0^2. \quad (7.46)$$

Since $q = 0$ on Γ , by virtue of (7.26) we have

$$\|\nabla q + \nabla \times \bar{\tau}\|_0^2 = \|\nabla q\|_0^2 + \|\nabla \times \bar{\tau}\|_0^2. \quad (7.47)$$

Thus we obtain

$$B(\mathbf{V}, \mathbf{V}) = \|\nabla q\|_0^2 + \|\nabla \times \bar{\tau}\|_0^2 + \|\nabla \cdot \bar{\tau}\|_0^2 + \|\bar{\tau} - \nabla \times \bar{v}\|_0^2 + \|\nabla \cdot \bar{v}\|_0^2. \quad (7.48)$$

Therefore

$$B(\mathbf{V}, \mathbf{V}) \geq \|\nabla q\|_0^2 = |q|_1^2. \quad (7.49)$$

Since $q = 0$ on Γ , from Poincare inequality (C.2) we have

$$C|q|_1 \geq \|q\|_1. \quad (7.50)$$

Combining (7.49) with (7.50) yields

$$CB(\mathbf{V}, \mathbf{V}) \geq \|q\|_1^2. \quad (7.51)$$

From (7.48) we know that

$$B(\mathbf{V}, \mathbf{V}) \geq \|\nabla \times \bar{\tau}\|_0^2 + \|\nabla \cdot \bar{\tau}\|_0^2. \quad (7.52)$$

Since $\bar{n} \cdot \bar{\tau} = 0$ on Γ , from Theorem 3 we have the inequality

$$C(\|\nabla \times \bar{\tau}\|_0^2 + \|\nabla \cdot \bar{\tau}\|_0^2) \geq \|\bar{\tau}\|_1^2. \quad (7.53)$$

Combining (7.52) with (7.53) yields

$$CB(\mathbf{V}, \mathbf{V}) \geq \|\bar{\tau}\|_1^2. \quad (7.54)$$

From (7.48) we also know that

$$B(\mathbf{V}, \mathbf{V}) \geq \|\bar{\tau} - \nabla \times \bar{v}\|_0^2, \quad (7.55)$$

$$B(\mathbf{V}, \mathbf{V}) \geq \|\nabla \cdot \bar{v}\|_0^2. \quad (7.56)$$

From (7.54) we have

$$CB(\mathbf{V}, \mathbf{V})^{\frac{1}{2}} \geq \|\bar{\tau}\|_0. \quad (7.57)$$

From (7.55) we know

$$B(\mathbf{V}, \mathbf{V})^{\frac{1}{2}} \geq \|-\bar{\tau} + \nabla \times \bar{v}\|_0. \quad (7.58)$$

Combining (7.57) with (7.58) and using the triangle inequality lead to

$$CB(\mathbf{V}, \mathbf{V})^{\frac{1}{2}} \geq \|\bar{\tau}\|_0 + \|-\bar{\tau} + \nabla \times \bar{v}\|_0 \geq \|\nabla \times \bar{v}\|_0,$$

that is

$$CB(\mathbf{V}, \mathbf{V}) \geq \|\nabla \times \bar{v}\|_0^2. \quad (7.59)$$

Combining (7.59) with (7.56) yields

$$CB(\mathbf{V}, \mathbf{V}) \geq \|\nabla \times \bar{v}\|_0^2 + \|\nabla \cdot \bar{v}\|_0^2. \quad (7.60)$$

Since $\bar{n} \times \bar{v} = \bar{0}$ on Γ , from Theorem 3 we have the inequality

$$C(\|\nabla \times \bar{v}\|_0^2 + \|\nabla \cdot \bar{v}\|_0^2) \geq \|\bar{v}\|_1^2. \quad (7.61)$$

Combining (7.60) and (7.61) leads to

$$CB(\mathbf{V}, \mathbf{V}) \geq \|\bar{v}\|_1^2. \quad (7.62)$$

Combining (7.51), (7.54) and (7.62) together we finally obtain the coersivity

$$CB(\mathbf{V}, \mathbf{V}) \geq \|q\|_1^2 + \|\bar{v}\|_1^2 + \|\bar{\tau}\|_1^2. \quad (7.63)$$

If we really specify the dummy variable ϕ be zero on Γ in advance, then only three boundary conditions are needed, and that $\omega_n = 0$ can be imposed in a weak sense. In this case, the least-squares method minimizes the following functional:

$$J(\mathbf{U}) = \|\nabla p + \nabla \times \bar{\omega} - \bar{f}\|_0^2 + \|\nabla \cdot \bar{\omega}\|_0^2 + \|\bar{\omega} - \nabla \times \bar{u}\|_0^2 + \|\nabla \cdot \bar{u}\|_0^2 + \|\bar{n} \cdot \bar{\omega}\|_{1/2, \Gamma}^2. \quad (7.64)$$

We have

$$\begin{aligned} B(\mathbf{U}, \mathbf{V}) &= (\nabla p + \nabla \times \bar{\omega}, \nabla q + \nabla \times \bar{\tau}) + (\nabla \cdot \bar{\omega}, \nabla \cdot \bar{\tau}) \\ &+ (\bar{\omega} - \nabla \times \bar{u}, \bar{\tau} - \nabla \times \bar{v}) + (\nabla \cdot \bar{u}, \nabla \cdot \bar{v}) + \langle \bar{\omega} \cdot \bar{n}, \bar{\tau} \cdot \bar{n} \rangle, \end{aligned} \quad (7.65)$$

$$L(\mathbf{V}) = (\bar{f}, \nabla q + \nabla \times \bar{\tau}), \quad (7.66)$$

and

$$B(\mathbf{V}, \mathbf{V}) = \|\nabla q + \nabla \times \bar{\tau}\|_0^2 + \|\nabla \cdot \bar{\tau}\|_0^2 + \|\bar{\tau} - \nabla \times \bar{v}\|_0^2 + \|\nabla \cdot \bar{v}\|_0^2 + \|\bar{n} \cdot \bar{\tau}\|_{1/2, \Gamma}^2. \quad (7.67)$$

Since $q = 0$ on Γ , by virtue of (7.26) we have

$$\|\nabla q + \nabla \times \bar{\tau}\|_0^2 = \|\nabla q\|_0^2 + \|\nabla \times \bar{\tau}\|_0^2. \quad (7.68)$$

Thus we obtain

$$B(\mathbf{V}, \mathbf{V}) = \|\nabla q\|_0^2 + \|\nabla \times \bar{\tau}\|_0^2 + \|\nabla \cdot \bar{\tau}\|_0^2 + \|\bar{\tau} - \nabla \times \bar{v}\|_0^2 + \|\nabla \cdot \bar{v}\|_0^2 + \|\bar{n} \cdot \bar{\tau}\|_{1/2, \Gamma}^2. \quad (7.69)$$

Therefore

$$B(\mathbf{V}, \mathbf{V}) \geq \|\nabla q\|_0^2 = |q|_1^2. \quad (7.70)$$

Since $q = 0$ on Γ , from the Poincare inequality (C.2) we have

$$C|q|_1 \geq \|q\|_1. \quad (7.71)$$

Combining (7.70) with (7.71) yields

$$CB(\mathbf{V}, \mathbf{V}) \geq \|q\|_1^2. \quad (7.72)$$

From (7.69) we also know that

$$B(\mathbf{V}, \mathbf{V}) \geq \|\nabla \times \bar{\tau}\|_0^2 + \|\nabla \cdot \bar{\tau}\|_0^2 + \|\bar{n} \cdot \bar{\tau}\|_{1/2, \Gamma}^2, \quad (7.73)$$

$$B(\mathbf{V}, \mathbf{V}) \geq \|\bar{\tau} - \nabla \times \bar{v}\|_0^2, \quad (7.74)$$

$$B(\mathbf{V}, \mathbf{V}) \geq \|\nabla \cdot \bar{v}\|_0^2. \quad (7.75)$$

Now let us expand the fourth term in (7.69). Since $\bar{n} \times \bar{v} = \bar{0}$, using Green's formula (B.5) we have

$$\|\bar{\tau} - \nabla \times \bar{v}\|_0^2 = \|\bar{\tau}\|_0^2 + \|\nabla \times \bar{v}\|_0^2 - 2(\bar{\tau}, \nabla \times \bar{v}) = \|\bar{\tau}\|_0^2 + \|\nabla \times \bar{v}\|_0^2 - 2(\nabla \times \bar{\tau}, \bar{v}). \quad (7.76)$$

Therefore (7.69) becomes

$$\begin{aligned} B(\mathbf{V}, \mathbf{V}) &= \|\nabla q\|_0^2 + \|\nabla \times \bar{\tau}\|_0^2 + \|\nabla \cdot \bar{\tau}\|_0^2 + \|\bar{\tau}\|_0^2 + \|\nabla \times \bar{v}\|_0^2 \\ &\quad - 2(\nabla \times \bar{\tau}, \bar{v}) + \|\nabla \cdot \bar{v}\|_0^2 + \|\bar{n} \cdot \bar{\tau}\|_{1/2, \Gamma}^2. \end{aligned} \quad (7.77)$$

Since $\bar{n} \times \bar{v} = \bar{0}$ on Γ , from Theorem 3 we have the inequality:

$$C(\|\nabla \times \bar{v}\|_0^2 + \|\nabla \cdot \bar{v}\|_0^2) \geq \|\bar{v}\|_1^2 \geq \|\bar{v}\|_0^2. \quad (7.78)$$

Multiplying (7.77) by C^2 and taking into account (7.78) we have

$$C^2 B(\mathbf{V}, \mathbf{V}) \geq C^2 \|\bar{\tau}\|_0^2 + C^2 (\|\nabla \times \bar{v}\|_0^2 + \|\nabla \cdot \bar{v}\|_0^2) - 2C^2 (\nabla \times \bar{\tau}, \bar{v})$$

or

$$C^2 B(\mathbf{V}, \mathbf{V}) \geq C^2 \|\bar{\tau}\|_0^2 + C \|\bar{v}\|_0^2 - 2C^2 (\nabla \times \bar{\tau}, \bar{v}). \quad (7.79)$$

From (7.73) we have

$$C^3 B(\mathbf{V}, \mathbf{V}) \geq C^3 \|\nabla \times \bar{\tau}\|_0^2. \quad (7.80)$$

Adding (7.79) and (7.80) yields

$$(C^3 + C^2) B(\mathbf{V}, \mathbf{V}) \geq C^2 \|\bar{\tau}\|_0^2 + C \|\nabla \times \bar{\tau} - \bar{v}\|_0^2. \quad (7.81)$$

From (7.81) obviously we have

$$CB(\mathbf{V}, \mathbf{V}) \geq \|\bar{\tau}\|_0^2. \quad (7.82)$$

Combining (7.73) with (7.82) leads to

$$CB(\mathbf{V}, \mathbf{V}) \geq \|\nabla \times \bar{\tau}\|_0^2 + \|\nabla \cdot \bar{\tau}\|_0^2 + \|\bar{\tau}\|_0^2 + \|\bar{\tau} \cdot \bar{n}\|_{1/2, \Gamma}^2. \quad (7.83)$$

From Theorem 5 we know that $\bar{\tau}$ satisfies

$$C\{\|\bar{\tau}\|_0 + \|\nabla \cdot \bar{\tau}\|_0 + \|\nabla \times \bar{\tau}\|_0 + \|\bar{\tau} \cdot \bar{n}\|_{1/2, \Gamma}\} \geq \|\bar{\tau}\|_1. \quad (7.84)$$

Combining (7.83) with (7.84) leads to

$$CB(\mathbf{V}, \mathbf{V}) \geq \|\bar{\tau}\|_1^2. \quad (7.85)$$

From (7.82) we have

$$CB(\mathbf{V}, \mathbf{V})^{\frac{1}{2}} \geq \|\bar{\tau}\|_0. \quad (7.86)$$

From (7.74) we know

$$CB(\mathbf{V}, \mathbf{V})^{\frac{1}{2}} \geq \|\bar{\tau} + \nabla \times \bar{v}\|_0. \quad (7.87)$$

Combining (7.86) with (7.87) and using the triangle inequality lead to

$$CB(\mathbf{V}, \mathbf{V})^{\frac{1}{2}} \geq \|\bar{\tau}\|_0 + \|\bar{\tau} + \nabla \times \bar{v}\|_0 \geq \|\nabla \times \bar{v}\|_0,$$

that is

$$CB(\mathbf{V}, \mathbf{V}) \geq \|\nabla \times \bar{v}\|_0^2. \quad (7.88)$$

Combining (7.75) with (7.88) and considering (7.78) yield

$$CB(\mathbf{V}, \mathbf{V}) \geq \|\bar{v}\|_1^2. \quad (7.89)$$

Combining (7.72), (7.85) and (7.89) together we finally obtain the coersivity

$$CB(\mathbf{V}, \mathbf{V}) \geq \|q\|_1^2 + \|\bar{v}\|_1^2 + \|\bar{\tau}\|_1^2. \quad (7.90)$$

7.4 $u_{\tau_1} = 0, u_{\tau_2} = 0, \omega_{\tau_1} = 0, \omega_{\tau_2} = 0$ ($\bar{n} \times \bar{u} = \bar{0}, \bar{n} \times \bar{\omega} = \bar{0}$) on Γ

For the same reason as explained in §7.3, in this case $\phi \equiv 0$ is guaranteed even no boundary condition of ϕ is specified. The coersivity of \mathbf{B} can be proved by just following the steps in §7.1.

7.5 $u_n = 0, u_{\tau_1} = 0, u_{\tau_2} = 0$ ($\bar{n} \cdot \bar{u} = 0, \bar{n} \times \bar{u} = \bar{0}$) on Γ

Obviously this is a standard permissible boundary condition. However, it seems that the permissibility cannot be proved by the elementary method presented in this paper.

Fortunately, one may rely on the ADN theory to fulfil the task, see Bochev and Gunzburger[31].

7.6 $p = 0, \omega_n = 0, \omega_{\tau_1} = 0, \omega_{\tau_2} = 0$ ($p = 0, \bar{n} \cdot \bar{\omega} = 0, \bar{n} \times \bar{\omega} = \bar{0}$) on Γ

Using the boundary conditions $p = 0, \omega_n = 0$ one can solve Equation (6.1) to obtain p and $\bar{\omega}$. However, \bar{u} cannot be uniquely determined by solving (8.3f) with the natural boundary conditions in (8.3g) and (8.3f) (see the discussion in the next section). Therefore, this combination can only be used on a part of boundary.

8. Euler-Lagrange Equations Associated with the Least-Squares Method for the Stokes Equations

In order to understand how the least-squares method works, we derive the Euler-Lagrange equations associated with the least-squares weak formulation (7.3) for the Stokes problems which can be rewritten as

$$\begin{aligned} & (\nabla p + \nabla \times \bar{\omega} - \bar{f}, \nabla q + \nabla \times \bar{\tau}) + (\nabla \cdot \bar{\omega}, \nabla \cdot \bar{\tau}) + \\ & (\bar{\omega} - \nabla \times \bar{u}, \bar{\tau} - \nabla \times \bar{v}) + (\nabla \cdot \bar{u}, \nabla \cdot \bar{v}) = 0, \end{aligned}$$

or

$$\begin{aligned} & (\nabla p + \nabla \times \bar{\omega} - \bar{f}, \nabla q) \\ & + (\nabla p + \nabla \times \bar{\omega} - \bar{f}, \nabla \times \bar{\tau}) \\ & + (\nabla \cdot \bar{\omega}, \nabla \cdot \bar{\tau}) \\ & + (\bar{\omega} - \nabla \times \bar{u}, \bar{\tau}) \\ & - (\bar{\omega} - \nabla \times \bar{u}, \nabla \times \bar{v}) \\ & + (\nabla \cdot \bar{u}, \nabla \cdot \bar{v}) = 0. \end{aligned} \tag{8.1}$$

Using Green's formulae (B.1), (B.3) and (B.5) from (8.1) we have that

$$\begin{aligned} & -(\nabla \cdot (\nabla p + \nabla \times \bar{\omega} - \bar{f}), q) + \langle \bar{n} \cdot (\nabla p + \nabla \times \bar{\omega} - \bar{f}), q \rangle \\ & + (\nabla \times (\nabla p + \nabla \times \bar{\omega} - \bar{f}), \bar{\tau}) - \langle \bar{n} \times (\nabla p + \nabla \times \bar{\omega} - \bar{f}), \bar{\tau} \rangle \\ & - (\nabla(\nabla \cdot \bar{\omega}), \bar{\tau}) + \langle \nabla \cdot \bar{\omega}, \bar{n} \cdot \bar{\tau} \rangle \\ & + (\bar{\omega} - \nabla \times \bar{u}, \bar{\tau}) \\ & - (\nabla \times (\bar{\omega} - \nabla \times \bar{u}), \bar{v}) + \langle \bar{n} \times (\bar{\omega} - \nabla \times \bar{u}), \bar{v} \rangle \end{aligned}$$

$$-(\nabla(\nabla \cdot \bar{u}), \bar{v}) + \langle \nabla \cdot \bar{u}, \bar{n} \cdot \bar{v} \rangle = 0. \quad (8.2)$$

From (8.2) after simplification we obtain the following Euler-Lagrange equations and boundary conditions:

$$\Delta p = \nabla \cdot \bar{f} \text{ in } \Omega, \quad (8.3a)$$

$$q = 0 \text{ or } \bar{n} \cdot (\nabla p + \nabla \times \bar{\omega} - \bar{f}) = 0 \text{ on } \Gamma, \quad (8.3b)$$

$$\Delta \bar{\omega} - \bar{\omega} + \nabla \times \bar{u} = -\nabla \times \bar{f} \text{ in } \Omega, \quad (8.3c)$$

$$\bar{n} \times \bar{\tau} = \bar{0} \text{ or } \bar{n} \times (\nabla p + \nabla \times \bar{\omega} - \bar{f}) = \bar{0} \text{ on } \Gamma, \quad (8.3d)$$

$$\bar{n} \cdot \bar{\tau} = 0 \text{ or } \nabla \cdot \bar{\omega} = 0 \text{ on } \Gamma, \quad (8.3e)$$

$$\Delta \bar{u} + \nabla \times \bar{\omega} = \bar{0} \text{ in } \Omega, \quad (8.3f)$$

$$\bar{n} \times \bar{v} = \bar{0} \text{ or } \bar{n} \times (\bar{\omega} - \nabla \times \bar{u}) = \bar{0} \text{ on } \Gamma, \quad (8.3g)$$

$$\bar{n} \cdot \bar{v} = 0 \text{ or } \nabla \cdot \bar{u} = 0 \text{ on } \Gamma, \quad (8.3h)$$

Equation (8.3) reveals that the least-squares weak formulation corresponds to seven second-order elliptic equations and seven boundary conditions in which the original boundary conditions serve as the essential boundary conditions and some first-order equations serve as the natural boundary conditions. In the following we list the combinations of boundary conditions for different cases:

(1) $\bar{n} \cdot \bar{u}, \bar{n} \times \bar{\omega}$ given

$$\bar{n} \cdot (\nabla p + \nabla \times \bar{\omega} - \bar{f}) = 0,$$

$$\nabla \cdot \bar{\omega} = 0,$$

$$\bar{n} \times (\bar{\omega} - \nabla \times \bar{u}) = \bar{0}.$$

(2) $p, \bar{n} \cdot \bar{u}, \bar{n} \cdot \bar{\omega}$ given

$$\bar{n} \times (\nabla p + \nabla \times \bar{\omega} - \bar{f}) = \bar{0},$$

$$\bar{n} \times (\bar{\omega} - \nabla \times \bar{u}) = \bar{0}.$$

(3) $p, \bar{n} \times \bar{u}, \bar{n} \cdot \bar{\omega}$ given

$$\bar{n} \times (\nabla p + \nabla \times \bar{\omega} - \bar{f}) = \bar{0},$$

$$\nabla \cdot \bar{u} = 0.$$

(4) $\bar{n} \times \bar{u}, \bar{n} \times \bar{\omega}$ given

$$\bar{n} \cdot (\nabla p + \nabla \times \bar{\omega} - \bar{f}) = 0,$$

$$\nabla \cdot \bar{\omega} = 0,$$

$$\nabla \cdot \bar{u} = 0.$$

(5) \bar{u} given

$$\nabla p + \nabla \times \bar{\omega} - \bar{f} = \bar{0},$$

$$\nabla \cdot \bar{\omega} = 0.$$

(6) $p, \bar{n} \cdot \bar{\omega}, \bar{n} \times \bar{\omega}$ given

$$\bar{n} \times (\bar{\omega} - \nabla \times \bar{u}) = \bar{0},$$

$$\nabla \cdot \bar{u} = 0.$$

We emphasize again that the least-squares method based on the first-order velocity-pressure-vorticity formulation (5.5) does not need any additional boundary conditions. Only if someone would like to use, for example, the finite difference method to solve the second-order velocity-pressure-vorticity formulation (8.3a), (8.3c) and (8.3f), should the additional natural boundary conditions be included.

We notice that in the second-order velocity-pressure-vorticity formulation in general the solution of the pressure Poisson equation (8.3a) is coupled with the solution of the velocity and the vorticity through the boundary conditions. The significant advantages of the present second-order formulation are that (1) it guarantees the satisfaction of the continuity of velocity and the solenoidality constraint on the vorticity without explicitly including these two divergence-free equations; (2) it is suitable not only for the standard boundary condition but also for non-standard boundary conditions; (3) the differential operator is self-adjoint (symmetrical).

9. The Div-Curl Method for the Navier-Stokes Equations

Let us first consider the following first-order system of the Stokes problem with the standard boundary condition:

$$\nabla p + \nabla \times \bar{\omega} = \bar{f} \text{ in } \Omega, \tag{9.1a}$$

$$\nabla \cdot \bar{\omega} = 0 \text{ in } \Omega, \tag{9.1b}$$

$$-\bar{\omega} + \nabla \times \bar{u} = \bar{0} \text{ in } \Omega, \tag{9.1c}$$

$$\nabla \cdot \bar{u} = 0 \text{ in } \Omega, \tag{9.1d}$$

$$\bar{u} = \bar{u}_\Gamma \text{ on } \Gamma. \tag{9.1e}$$

Of course the boundary data \bar{u}_Γ should satisfy the global mass conservation:

$$\int_{\Gamma} \bar{n} \cdot \bar{u}_\Gamma = 0. \quad (9.1f)$$

From Theorem 2 we know that System (9.1) is equivalent to the following system:

$$\nabla \times (\nabla p + \nabla \times \bar{\omega} - \bar{f}) = \bar{0} \text{ in } \Omega, \quad (9.2a)$$

$$\nabla \cdot (\nabla p + \nabla \times \bar{\omega} - \bar{f}) = 0 \text{ in } \Omega, \quad (9.2b)$$

$$\bar{n} \cdot (\nabla p + \nabla \times \bar{\omega} - \bar{f}) = 0 \text{ on } \Gamma, \quad (9.2c)$$

$$\nabla \times (\bar{\omega} - \nabla \times \bar{u}) = \bar{0} \text{ in } \Omega, \quad (9.2d)$$

$$\nabla \cdot (\bar{\omega} - \nabla \times \bar{u}) = 0 \text{ in } \Omega, \quad (9.2e)$$

$$\bar{n} \times (\bar{\omega} - \nabla \times \bar{u}) = \bar{0} \text{ on } \Gamma, \quad (9.2f)$$

$$\nabla \cdot \bar{\omega} = 0 \text{ in } \Omega, \quad (9.2g)$$

$$\nabla \cdot \bar{u} = 0 \text{ in } \Omega, \quad (9.2h)$$

$$\bar{u} = \bar{u}_\Gamma \text{ on } \Gamma. \quad (9.2i)$$

Taking into account $\nabla \times \nabla p = \bar{0}$, $\nabla \cdot \nabla \times \bar{\omega} = 0$, $\nabla \cdot \nabla \times \bar{u} = 0$ and Equality (A.4), System (9.2) can be simplified as:

$$\Delta \bar{\omega} = -\nabla \times \bar{f} \text{ in } \Omega, \quad (9.3a)$$

$$\Delta p = \nabla \cdot \bar{f} \text{ in } \Omega, \quad (9.3b)$$

$$\bar{n} \cdot (\nabla p + \nabla \times \bar{\omega} - \bar{f}) = 0 \text{ on } \Gamma, \quad (9.3c)$$

$$\Delta \bar{u} + \nabla \times \bar{\omega} = \bar{0} \text{ in } \Omega, \quad (9.3d)$$

$$\bar{n} \times (\bar{\omega} - \nabla \times \bar{u}) = \bar{0} \text{ on } \Gamma, \quad (9.3e)$$

$$\nabla \cdot \bar{\omega} = 0 \text{ in } \Omega, \quad (9.3f)$$

$$\nabla \cdot \bar{u} = 0 \text{ in } \Omega, \quad (9.3g)$$

$$\bar{u} = \bar{u}_\Gamma \text{ on } \Gamma, \quad (9.3h)$$

As explained in §2.3, Equation (9.3g) can be eliminated, since Equation (9.3d) and (9.3f) and the boundary conditions (9.3e) and (9.3h) guarantee the divergence-free of the velocity.

Since Equation (9.3a) implies that

$$\Delta(\nabla \cdot \bar{\omega}) = 0 \text{ in } \Omega, \quad (9.4a)$$

if we specify that

$$\nabla \cdot \bar{\omega} = 0 \text{ on } \Gamma, \quad (9.4b)$$

then $\nabla \cdot \bar{\omega} \equiv 0$ in Ω , that is, the solenoidality of the vorticity vector is guaranteed. Therefore we can replace Equation (9.3f) by the boundary condition (9.4b). Furthermore, if we replace (9.3e) by that

$$\bar{\omega} - \nabla \times \bar{u} = \bar{0} \text{ on } \Gamma, \quad (9.4e)$$

then (9.3e) and (9.4b) all are satisfied. Finally we obtain the conventional second-order velocity-pressure-vorticity formulation for the Stokes problems:

$$\Delta \bar{\omega} = -\nabla \times \bar{f} \text{ in } \Omega, \quad (9.5a)$$

$$\Delta \bar{u} + \nabla \times \bar{\omega} = \bar{0} \text{ in } \Omega, \quad (9.5b)$$

$$\bar{\omega} - \nabla \times \bar{u} = \bar{0} \text{ on } \Gamma, \quad (9.5c)$$

$$\bar{u} = \bar{u}_\Gamma \text{ on } \Gamma, \quad (9.5d)$$

$$\Delta p = \nabla \cdot \bar{f} \text{ in } \Omega, \quad (9.5e)$$

$$\bar{n} \cdot (\nabla p + \nabla \times \bar{\omega} - \bar{f}) = 0 \text{ on } \Gamma. \quad (9.5f)$$

From (9.5) we understand that the calculation of the velocity and vorticity is decoupled from that of the pressure, for this reason in the literature Equation (9.5a) and (9.5b) are called the velocity-vorticity formulation. We note that this decoupling does not hold if on a part of boundary the pressure is prescribed.

For the Navier-Stokes equations, Equation (9.1a) in system (9.1) is replaced by the following non-linear momentum equation[2,15]:

$$\bar{\omega} \times \bar{u} + \nabla p + \frac{1}{Re} \nabla \times \bar{\omega} = \bar{f} \text{ in } \Omega, \quad (9.6)$$

where p should be understood as the total pressure. Following the same steps as those for the Stokes equations we obtain the second-order velocity-vorticity formulation:

$$\frac{1}{Re} \Delta \bar{\omega} - \nabla \times (\bar{\omega} \times \bar{u}) = -\nabla \times \bar{f} \text{ in } \Omega, \quad (9.7a)$$

$$\Delta \bar{u} + \nabla \times \bar{\omega} = \bar{0} \text{ in } \Omega, \quad (9.7b)$$

$$\bar{\omega} - \nabla \times \bar{u} = \bar{0} \text{ on } \Gamma, \quad (9.7c)$$

$$\bar{u} = \bar{u}_\Gamma \text{ on } \Gamma, \quad (9.7d)$$

$$\Delta p + \nabla \cdot (\bar{\omega} \times \bar{u}) = \nabla \cdot \bar{f} \text{ in } \Omega, \quad (9.7e)$$

$$\bar{n} \cdot (\bar{\omega} \times \bar{u} + \nabla p + \frac{1}{Re} \nabla \times \bar{\omega} - \bar{f}) = 0 \text{ on } \Gamma. \quad (9.7f)$$

10. Conclusions

The least-squares method based on the first-order differential equations is not only a powerful technique for numerical solution, but also a useful tool for theoretical study of the div-curl equations and the Navier-Stokes equations. The div-curl equations and the Navier-Stokes equations in the first-order velocity-pressure-vorticity formulation are not overdetermined. The three-dimensional div-curl equations should have two boundary conditions. Since the principal part of the Navier-Stokes equations consists of two div-curl systems, four boundary conditions on a fixed boundary are needed (if three conditions are given, the dummy $\phi = 0$ on Γ should be counted as the fourth one) for three-dimensional problems, and two for two-dimensional problems. Four different combinations of non-standard boundary conditions are rigorously proved to be permissible for the Navier-Stokes problems by using the least-squares method. Consequently, the corresponding least-squares finite element method with equal-order interpolations has an optimal rate of convergence for all unknowns. The least-squares method and the div-curl method are systematic and consistent methods to obtain a high-order derived version of the differential equations without generating spurious solutions. Specially, the self-adjoint second-order differential equations obtained by the least-squares method automatically satisfy the divergence-free equations and are suitable for any boundary conditions.

Appendix

A. Operations on Vectors

$$\nabla \cdot (q\bar{v}) = q\nabla \cdot \bar{v} + \nabla q \cdot \bar{v}, \quad (A.1)$$

$$\nabla \times (q\bar{v}) = q\nabla \times \bar{v} + \nabla q \times \bar{v}, \quad (A.2)$$

$$\nabla \cdot (\bar{u} \times \bar{v}) = (\nabla \times \bar{u}) \cdot \bar{v} - \bar{u} \cdot (\nabla \times \bar{v}), \quad (A.3)$$

$$\nabla \times \nabla \times \bar{v} = \nabla(\nabla \cdot \bar{v}) - \Delta \bar{v}, \quad (A.4)$$

$$\bar{v} \times (\nabla \times \bar{v}) = \frac{1}{2} \nabla(\bar{v}^2) - (\bar{v} \nabla) \bar{v}. \quad (A.5)$$

B. Green's Formula

Assume that \bar{u} , \bar{v} and q are smooth enough. Integrating (A.1) and using the Gauss divergence theorem lead to

$$(\nabla \cdot \bar{v}, q) + (\bar{v}, \nabla q) = \langle \bar{n} \cdot \bar{v}, q \rangle. \quad (B.1)$$

Substituting $\bar{v} = \nabla p$ into (B.1) yields

$$(\Delta p, q) + (\nabla p, \nabla q) = \langle \bar{n} \cdot \nabla p, q \rangle. \quad (B.2)$$

Substituting $q = \nabla \cdot \bar{u}$ into (B.1) yields

$$(\nabla \cdot \bar{v}, \nabla \cdot \bar{u}) + (\bar{v}, \nabla(\nabla \cdot \bar{u})) = \langle \bar{n} \cdot \bar{v}, \nabla \cdot \bar{u} \rangle. \quad (B.3)$$

Replacing \bar{v} by $\nabla \times \bar{v}$ in (B.1) leads to

$$(\nabla \times \bar{v}, \nabla q) = \langle \bar{n} \cdot (\nabla \times \bar{v}), q \rangle. \quad (B.4)$$

Integrating (A.3) and using the Gauss divergence theorem lead to

$$(\nabla \times \bar{u}, \bar{v}) - (\bar{u}, \nabla \times \bar{v}) = \langle \bar{n} \times \bar{u}, \bar{v} \rangle. \quad (B.5)$$

Substituting $\bar{u} = \nabla q$ into (B.5) yields

$$(\nabla \times \bar{v}, \nabla q) = - \langle \bar{n} \times \nabla q, \bar{v} \rangle = \langle \nabla q, \bar{n} \times \bar{v} \rangle. \quad (B.6)$$

Replacing \bar{v} by $\nabla \times \bar{v}$ in (B.5) yields

$$(\nabla \times \bar{u}, \nabla \times \bar{v}) - (\bar{u}, \nabla \times \nabla \times \bar{v}) = \langle \bar{n} \times \bar{u}, \nabla \times \bar{v} \rangle. \quad (B.7)$$

C. Poincare Inequality

Let Ω be a bounded domain with a piecewise C^1 boundary Γ , then

$$\|p\|_1^2 \leq C\{\|\nabla p\|_0^2 + (\int_{\Omega} p dx)^2\} \quad \forall p \in H^1(\Omega), \quad (C.1)$$

$$\|p\|_1^2 \leq C\{\|\nabla p\|_0^2 + (\int_{\Gamma} p dx)^2\} \quad \forall p \in H^1(\Omega). \quad (C.2)$$

REFERENCES

- [1] R. Temam, Navier-Stokes Equations and Nonlinear Functional Analysis (Society for Industrial and Applied Mathematics, Philadelphia, 1983).
- [2] P.M. Gresho, Some interesting issues in incompressible fluid dynamics, both in the continuum and in numerical simulation, *Advances in Applied Mechanics* 28 (1992) 45-137.
- [3] O.A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow* (Goldon and Breach, New York, 1969).
- [4] R. Temam, *Navier-Stokes Equations* (North- Holland, Amsterdam, 1979).
- [5] V. Girault and P.-A. Raviart, *Finite Element Methods for Navier-Stokes Equations* (Springer-Verlag, Berlin, 1986).
- [6] I. Babuska, The finite element method with Lagrange multipliers, *Numer. Math.* 20 (1973) 179-192.
- [7] F. Brezzi, On the existence, uniqueness and approximation of saddle-point problems arising from Lagrange multipliers, *R.A.I.R.O., Anal. Numer.* R2 (1974) 129-151.
- [8] J.T. Oden and G.F. Carey, *Finite Elements: Mathematical Theory*, Vol.VI (Prentice-Hall, Englewood Cliffs, NJ, 1986).
- [9] M.D. Gunzburger, *Finite Element Methods for Viscous Incompressible Flows* (Academic Press, 1989).
- [10] O. Pironneau, *Finite Element Methods for Fluids* (John Wiley and Sons, Chichester, 1989).
- [11] B.N. Jiang and L.A. Povinelli, Least-squares finite element method for fluid dynamics, *Comput.Meth.Appl. Mech.Engrg.* 81 (1990) 13-37.
- [12] B.N. Jiang, A least-squares finite element method for incompressible Navier-Stokes problems, *Inter.J.Numer.Meth.Fluids* 14 (1992) 843-859.

- [13] B.N. Jiang and V. Sonnad, Least-squares solution of incompressible Navier-Stokes equations with the P-version of finite Element, NASA TM 105203, ICOMP-91-14.
- [14] B.N. Jiang and C.L.Chang, Least-squares finite element for Stokes problem, *Comput.Methods Appl.Mech.Engrg.* 78 (1990) 297-311.
- [15] B.N. Jiang, T.L. Lin and L.A. Povinelli, Large-Scale Computation of Incompressible Viscous Flow by Least-Squares Finite Element Method, *Comput.Methods Appl.Mech.Engrg.*, to appear.
- [16] B.N. Jiang, L.T. Hou and T.L. Lin, Least-Squares Finite Element Solutions for Three-Dimensional Backward-Facing Step Flow, NASA TM 106353, ICOMP-93-31.
- [17] D. Lefebvre, J. Peraire and K. Morgan, Least squares finite element solution of compressible and incompressible flows, *Int.J.Num.Methods Heat Trans.Fluid Flow* 2 (1992) 99-113.
- [18] L.Q. Tang and T.T.H. Tsang, A least-squares finite element method for time-dependent incompressible flows with thermal convection, *Inter.J.Numer.Meth.Fluids* 17 (1993) 271-289.
- [19] P.B. Bochev and M.D. Gunzburger, Accuracy of least-squares methods for the Navier-Stokes equations, *Comput. Fluids* 22 (1993) 549-563.
- [20] F. Brezzi, J. Rappaz and P.-A. Raviart, Finite-dimensional approximation of nonlinear problems, Part I: Branches of nonsingular solutions, *Numer. Math.* 36 (1980) 1-25.
- [21] J. Saranen, On an inequality of Friedrichs, *Math. Scand.* 51 (1982) 310-322.
- [22] M. Krizek and P. Neittaanmaki, On the validity of Friedrich's inequalities, *Math. Scand.* 54 (1984) 17-26.
- [23] M. Krizek and P. Neittaanmaki, *Finite Element Approximation of Variational Problems and Applications* (Pitman Scientific and Technical, Hawlow, England, 1990).
- [24] B.N. Jiang and L.A. Povinelli, Optimal least-squares finite element method for elliptic problems, *Comput.Methods Appl.Mech.Engrg.* 102 (1993) 199-212.
- [25] G.F. Fix and M.E. Rose, A comparative study of finite element and finite difference methods for Cauchy-Reimann-type equations, *SIAM J. Num. Analysis* 22 (1985) 250-260.
- [26] M. Hafez, and J. Soliman, A vector/vorticity method for viscous incompressible flow calculations, *Lecture Notes in Physics*, Vol.323 (1989).
- [27] J.L. Lions and E. Magenes, *Non-homogeneous Boundary-Value Problems and Applications*, Vol.I (Springer-Verlag, New York, 1972).
- [28] J.T. Oden and J.N. Reddy, *An Introduction to the Mathematical Theory of Finite Elements* (John Wiley and Sons, New York, 1976).
- [29] S. Agmon, A. Douglis and L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions II, *Comm.Pure.Appl.Math.* 17 (1964) 35-92.
- [30] A.K. Aziz, R.B. Kellogg and A.B. Stephens, Least squares methods for elliptic systems, *Math.Comp.* 44 (1985) 53-70.

- [31] P.B. Bochev and M.D. Gunzburger, Analysis of least-squares finite element methods for the Stokes equations, *Math. Comp.*, to appear.
- [32] P.M. Gresho, Incompressible fluid dynamics: some fundamental formulation issues, *Ann. Rev. Fl. Mech. Vol. 23* (Annual Reviews, Inc., Palo alto, 1991).
- [33] P.M. Gresho, Some current CFD issues relevant to the incompressible Navier-Stokes equations, *Comput. Methods Appl.Mech. Engrg.* 87 (1991) 201-252.
- [34] V. Girault, Incompressible finite element methods for Navier-Stokes equations with nonstandard boundary conditions in R^3 , *Math.Comp.* 51 (1988) 55-74.
- [35] R. Verfurth, Mixed finite element approximation of the vector potential, *Numer.Math.* 50 (1987) 685-695.
- [36] W. Wendland, *Elliptic Systems in the Plane* (Pitman, London, 1979).

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