# A Toy Model of Sea Ice Growth

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My purpose here is to present a simplified treatment of the growth of sea ice. By ignoring many details, it is possible to obtain several results that help to clarify the ways in which the sea ice cover will respond to climate change. Three models are discussed. The first deals with the growth of sea ice during the cold season. The second describes the cycle of growth and melting for perennial ice. The third model extends the second to account for the possibility that the ice melts away entirely in the summer. In each case, the objective is to understand what physical processes are most important, what ice properties determine the ice behavior, and to which climate variables the system is most sensitive.

## Climate

Divide the year into a cold and a warm season, each of duration Y. During the cold season, the average downwelling longwave radiation is $f_{lwc}$ which equals 180 $W/m^2$ in today's climate. In the warm season, the downwelling radiation is $f_{lww}$ = 270 $W/m^2$, and there is shortwave radiation $f_{sw}$ = 200 $W/m^2$. (These values are from Maykut and Untersteiner, 1971.) A fourth climate variable is the heat supplied to the ice from the ocean, $f_w$. Its value is not known, and it surely varies in space and time. We will regard it as constant throughout the year, and expect it to have a value in the range 0 to 10 $W/m^2$ in the central Arctic. The four parameters $f_{lwc}$, $f_{lww}$, $f_{sw}$, and $f_w$ specify the climate.

To examine the response of the ice to changes in climate, we will have in mind perturbations to the longwave fluxes, such as might accompany tropospheric warming or changes in atmospheric composition:

$$f_{lwc} \rightarrow f_{lwc} + \delta$$

$$f_{lww} \rightarrow f_{lww} + \delta$$

## Sea Ice Parameters

The ice will be described by two variables, its thickness h and surface temperature T. Other ice properties and their values used in this paper are:

| | | |
|---|---|---|
| longwave emissivity | $\varepsilon$ | 1 |
| thermal conductivity | k | 2 W/m/K |
| specific heat | c | $2 \times 10^6$ J/m$^3$/K |
| latent heat | L | $3 \times 10^8$ J/m$^3$ |
| albedo | $\alpha$ | 0.65. |

In taking these parameters to be constant, we ignore the strong dependence some have on temperature and salinity. In effect, we replace the thermal properties of sea ice with those of fresh ice. In so doing, we give up the possibility of resolving the sea ice behavior within a few degrees of the freezing point.

## Ice Growth

The black body radiation from the ice surface can be expressed as a linear approximation to the Stefan-Boltzmann law, in the form A + BT. Here A = 320 W/m$^2$ and B = 4 $\varepsilon$ $\sigma$ $(271.2)^3$ = 4.6 W/m$^2$/K. Here $\sigma$ is the Stefan-Boltzmann constant, $5.7 \times 10^{-8}$ J/m$^2$/s/K$^4$, and 271.2 K is the freezing point of sea water.

In the cold season, and in the absence of upward heat flux from below, the radiation from the ice surface must balance the down-welling longwave radiation. This implies

$$T = -(A - f_{lwc})/B = -D/B = -29°C \tag{1}$$

The quantity D $\equiv$ A – $f_{lwc}$ is a convenient scale for the radiative fluxes. It is the net radiation balance over a surface at the freezing point. Similarly, the quantity –D/B is a natural scale for the surface temperature. It is the minimum temperature the ice surface can attain.

For ice of thickness h, having a linear temperature profile, heat is conducted upward at the rate –kT/h, and this flux must be included in the surface energy balance:

$$A + BT = f_{lwc} - kT/h \tag{2}$$

It follows that the surface temperature and the ice thickness are related as

$$T = -Dh/(k + Bh), \text{ and } h = -kT/(D + BT) \tag{3}$$

These relations hold when the ice is growing, because then the temperature profile through the ice is approximately linear, and the expression for the conductive heat flux is justified.

At the bottom of the ice, the rate at which heat can be conducted through the ice determines the rate of ice growth:

$$L\, dh/dt = -kT/h - f_w \tag{4}$$

After substituting for T, the equation can be integrated to give the relationship between the thickness h, after time t, and the initial thickness $h_\circ$:

$$tB/L + kDf_w^{-2} \log\left(\frac{k(D - f_w) - f_w Bh}{k(D - f_w) - f_w Bh_\circ}\right) + B(h - h_\circ)f_w^{-1} = 0 \tag{5}$$

In the special case $h_\circ = 0$, the logarithm can be expanded in the form $\log(1 - \beta) \simeq -\beta - \beta^2/2$, provided $f_w \ll D$, leading to

$$L\, dh/dt = -kT/h - f_w \tag{6}$$

This result establishes k/B as a natural scale of ice thickness. B determines how much the surface must cool to maintain radiative equilibrium. Together with k, this fixes the conductive heat flux, and therefore the ice growth. The value of k/B is about 0.4 m. It also appears from this result that kL/BD is a natural time scale, having the value of about 11 days. It is worth noting that the ocean heat flux decreases the thickness through the linear factor $(1 - f_w/D)$. With $D \simeq 140\ \mathrm{W/m^2}$ and $f_w \simeq 1\ \mathrm{W/m^2}$, the ocean heat flux has a small effect on the ice produced in one year, but we will see below that it can have a large effect on the equilibrium thickness.

It can also be noted that a steady state solution to Equations (2) and (4) is possible. The ice must grow to a thickness where the conductive heat flux equals $f_w$. This occurs when $h = (k/B)(D/f_w - 1)$, which is about 50 m. However, the time required to approach this equilibrium is much longer than a single growing season, so the model needs to be extended to account for the seasonal cycle of growing and melting (see "A Model of the Perennial Ice Cover").

For very thin ice, $t \ll kL/BD \approx 11$ days, we have the further approximation that

$$h \simeq (D - f_w)t/L \qquad \text{(small t)} \tag{7}$$

In this case, the conduction through the ice is not important. The ice growth is determined completely by the net heat balance for the slab, and the latent heat, and therefore grows at a constant rate.

For large t, $t \gg kL/BD$, we obtain

$$h \simeq (k/B)(1 - f_w/D)\sqrt{2BDt/kL} \qquad \text{(large t)} \tag{8}$$

which shows that h grows as $t^{1/2}$. The thicker the ice gets, the slower it grows. Here the growth is regulated by the conduction of heat through the slab. With time, the ice both cools and thickens, but the combined effect is that the temperature gradient decreases, which controls the rate at which heat can be removed from the bottom surface, and thus the rate of ice growth.

The ice thickness depends on the climate through the variables $D = A - f_{lwc}$ and $f_w$. It is interesting to note that in the coldest climate possible, $f_{lwc}$ and $f_w = 0$, the ice would only grow to $h = (k/B)(2kAY/BL)^{1/2} \simeq 4$ m in a single growing season, where Y = one-half year.

Now consider the case $h_\circ > 0$, as for ice which has survived the summer. During the subsequent growing season, the ice will reach a thickness h after time t. In the case where the new growth $h - h_\circ$ is small compared to $h_\circ$, we obtain

$$h - h_\circ \simeq \left(\frac{kD}{k + Bh_\circ} - f_w\right)\frac{t}{L} \qquad (h - h_\circ \ll h_\circ) \tag{9}$$

This implies that the ice cannot grow thicker than $(k/B)(D/f_w - 1)$, a quantity that is sensitive to the ocean heat flux $f_w$. Compare this result with Equation (25), below, which accounts for the annual cycle of ice growth and melting.

Equation (5) can be solved for h numerically. Thus we can express

$$h = H(h_\circ t) \tag{10}$$

though we cannot write the function H explicitly.

The assumption of a linear temperature profile through the ice has been essential in obtaining these results. As the ice grows, it also cools, according to Equation (3). There is an internal inconsistency here. An element of ice cannot cool if the temperature profile is linear there. The only way to avoid this kind of inconsistency is to formulate the ice growth as a heat diffusion problem, as Maykut and Untersteiner did. To estimate how large an error the assumption introduces, note that the model properly accounts for the heat released by the new ice growth, Ldh, but it does not account for the heat which must be removed to cool the ice, dQ, where $Q = chT/2$. For ice that grows one meter in the winter, Ldh is $3 \times 10^8$ J/m$^2$. The heat storage term is $(2 \times 10^6 \text{ J/m/K})(1 \text{ m})(-21 \text{ K})/2 = 2 \times 10^7 \text{ J/m}^2$. So we may expect errors of about 5 to 10% to arise because we have not properly accounted for the heat storage (although in the next section it is shown that a large part of the heat storage can easily be accounted for).

## A Model of the Perennial Ice Cover

Since the ice is characterized by its thickness and surface temperature, it is useful to plot its behavior as a trajectory in (h,T) space. Figure 1 displays the results of the Maykut and Untersteiner model this way. We will approximate the annual cycle as follows. Imagine a slab of ice at the end of the melt season, having thickness h and temperature T = 0 throughout. When the weather turns cold, the ice cools until its surface temperature reaches the temperature appropriate for its thickness, Equation (3). Then it grows for the remain-

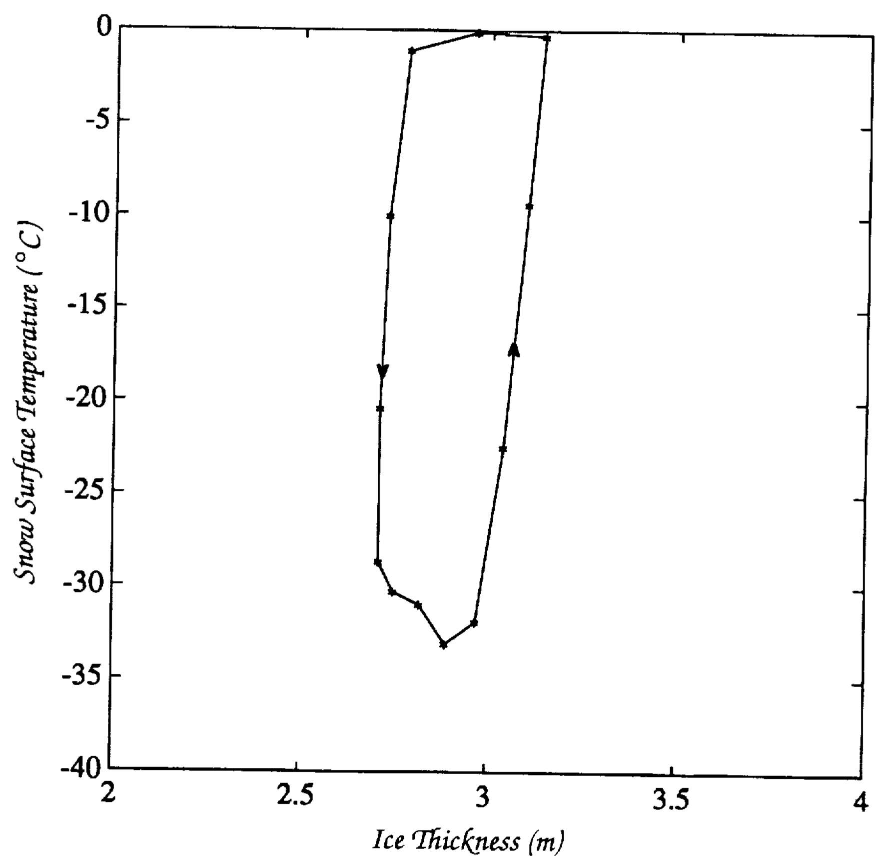


*Figure 1. The equilibrium cycle for the standard case of the Maykut and Untersteiner model, plotted in (h,T) space.*

ing part of the cold season, according to the ice growth model of the previous section. In the warm season, the ice first warms up to the freezing point, and then melts. The cooling and warming processes take place with no change in thickness. They account for the storage of heat by the ice. The equations governing this four-process cycle of cooling, growing, warming, and melting are

Cooling

$$
\begin{aligned}
D &= A - f_{lwc} \\
T &= -Dh/(k + Bh) \\
Q &= -chT/2 \\
\tau_c &= Q/\left(A + BT/2 - f_{lwc} - f_w\right)
\end{aligned}
\tag{11}
$$

Growing

$$
\begin{aligned}
\tau_g &= Y - \tau_c \\
h_g &= H\left(h, \tau_g\right) \qquad \text{(Equations 5 and 10)} \\
T_g &= -Dh_g / \left(k + Bh_g\right)
\end{aligned}
\tag{12}
$$

Warming

$$
\begin{aligned}
Q &= -ch_g T_g / 2 \\
D &= f_{lww} + (1 - \alpha)f_{sw} + f_w - A - BT_g/2 \\
\tau_w &= Q/D
\end{aligned}
\tag{13}
$$

Melting

$$
\begin{aligned}
\tau_m &= Y - \tau_w \\
h &= h_g - \tau_w\left[f_{lww} + (1 - \alpha)f_{sw} + f_w - A\right]/L
\end{aligned}
\tag{14}
$$

where $\tau$ is duration with subscripts c, g, w, and m indicating the four processes.

A short program can be written to do these calculations. The results (see Table 1) show that for the present climate, the ice is attracted to a periodic orbit, as sketched in Figure 2. The ice is saved from the two alternatives, melting completely or growing without bound, by the negative feedback provided by the ice thickness. If the ice is too thin in the fall, it will grow more in the winter than it melts in the summer. If it is too thick in the fall, it grows less than it melts. In either case, it approaches the equilibrium orbit.

For other climates, the orbit changes. For large enough perturbations to the climate, the equilibrium cycle cannot be maintained. If the climate is too warm, the ice melts completely in the summer; the

*Table 1: Results of model simulations*

Perennial Ice Cover

| $\delta$ (W/m2) | $f_w$ (W/m2) | $\tau_c$ (d) | $h_g$ (m) | $T_g$ (°C) | $\tau_w$ (d) | $h_m$ (m) |
|---|---|---|---|---|---|---|
| -15 | 1 | 43 | 9.4 | -32 | 44 | 9.1 |
| -10 | 1 | 21 | 5.3 | -30 | 23 | 4.8 |
| 0 | 1 | 7 | 2.9 | -26 | 11 | 1.9 |
| 5 | 1 | 3 | 2.5 | -25 | 8 | 1.1 |
| 10 | 1 | 1 | 2.2 | -24 | 7 | 0.6 |
| 15 | 1 | 0.1 | 2.0 | -22 | 6 | 0.2 |
| 0 | 2 | 5 | 2.7 | -26 | 10 | 1.6 |
| 0 | 5 | 3 | 2.3 | -26 | 8 | 1.0 |
| 0 | 10 | 0.5 | 1.8 | -25 | 6 | 0.3 |

Seasonal Ice Cover

| $\delta$ (W/m2) | $f_w$ (W/m2) | $T_{ml}$ (°C) | $\tau_{ml}$ (d) | $h_g$ (m) | $T_g$ (°C) | $\tau_w$ (d) | $\tau_h$ (d) |
|---|---|---|---|---|---|---|---|
| 20 | 1 | 2 | 37 | 1.7 | -21 | 5 | 36 |
| 30 | 1 | 6 | 117 | 1.0 | -16 | 2 | 115 |
| 37 | 1 | 9 | 170 | 0.3 | -9 | 0.4 | 165 |
| 20 | 5 | 4 | 82 | 1.3 | -19 | 3 | 82 |
| 20 | 10 | 6 | 119 | 0.9 | -18 | 2 | 119 |

$h_g$ and $T_g$ are the ice thickness and temperature at the end of the growing season. $h_m$ is the thickness at the end of the melting season. $\tau_c$, $\tau_w$, $\tau_{ml}$, and $\tau_h$ are the durations of the processes which cool and warm the ice, and cool and heat the mixed layer. $\delta$ is the perturbation to the longwave radiation fluxes. $f_w$ is the upward flux of heat from the ocean. $T_{ml}$ is the temperature of model's upper ocean layer.

model is patched up to treat this case below. If the climate is too cold, the ice grows without bound. The behavior of the model as a function of the perturbation $\delta$ is sketched in Figure 3. The longwave fluxes can vary up to about $\pm$ 20 W/m$^2$ from the present climate and still support an annual equilibrium cycle.

## A Seasonal Ice Model

If the ice melts away before the end of the summer, the positive heat balance at the surface must cause the upper ocean to warm. The model of the preceding section can be modified to account for this heat by including an upper ocean layer. The thickness of the layer $h_{ml}$ is assumed to be fixed at 50 m, and its temperature is allowed to change. The layer is assumed to be well mixed, so that a single temperature describes its state. Begin the annual cycle at the end of summer with the mixed layer having temperature $T_{ml}$. We calculate how long it takes for the mixed layer to cool, and then how much ice will grow during the remainder of the cold season. During the warm season, the ice must warm up to the freezing point, and

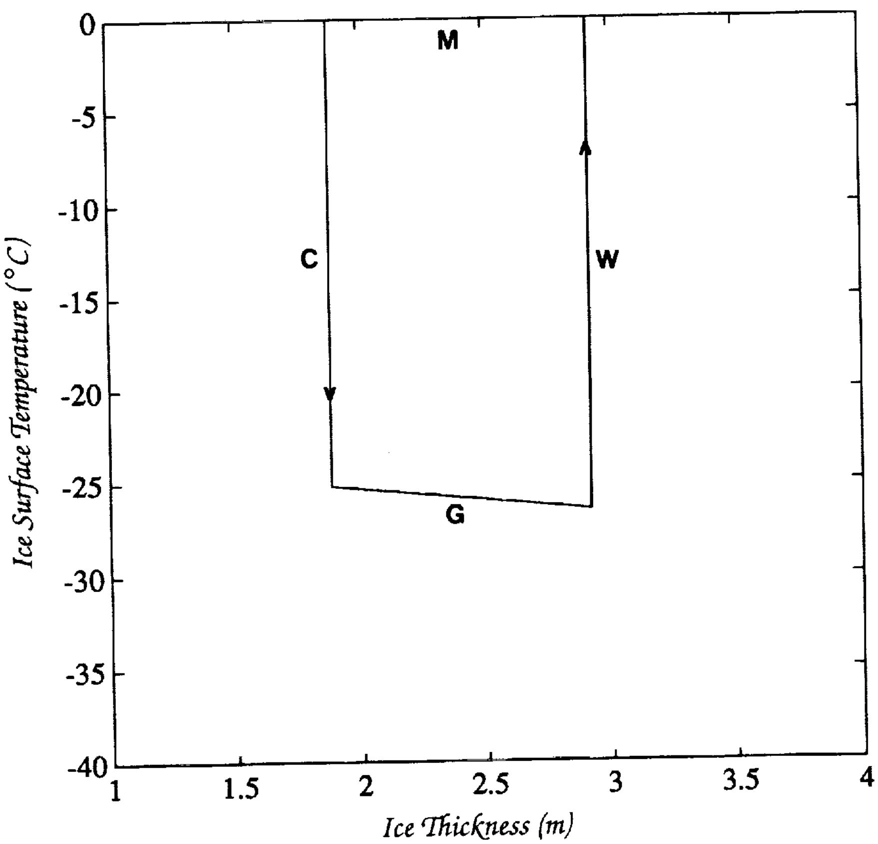


*Figure 2. The equilibrium cycle for the perennial ice model, under the present climate; C is cooling, G is growing, W is warming, M is melting.*

then melt, requiring times $\tau_w$ and $\tau_m$. The remainder of the warm season is spent heating the mixed layer. The model equations are:

Cooling the mixed layer:

$$Q = c_w h_{ml} T_{ml} \qquad (15)$$

$$\tau_{ml} = Q/(A + BT_{ml}/2 - f_{lwc} - f_w)$$

Ice growth:

$$\tau_g = Y - \tau_{ml} \qquad (16)$$

$$h = H(0, \tau_g)$$

$$T = -(A - f_{lwc})h/(k + Bh)$$

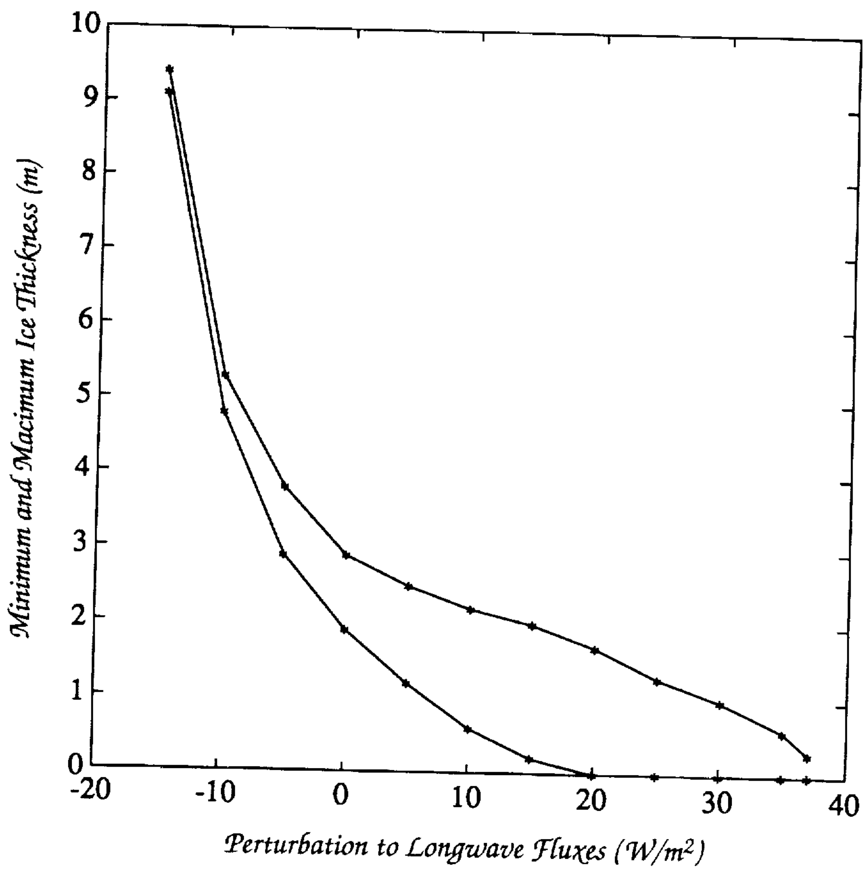


*Figure 3. The maximum and minimum thicknesses in the annual cycle for the perennial and seasonal ice models, as functions of the perturbation δ to the longwave fluxes.*

Warming:

$$
\begin{aligned}
&Q = -chT/2 \\
&D = f_{lww} + (1-\alpha)f_{sw} + f_w - A - BT/2 \\
&\tau_w = Q/D
\end{aligned}
\tag{17}
$$

Melting:

$$\tau_m = hL/\left[f_{lww} + (1-\alpha)f_{sw} + f_w - A\right] \tag{18}$$

Heating the mixed layer:

$$
\begin{aligned}
&\tau_h = Y - \tau_w - \tau_m \\
&D = f_{lww} + (1-\alpha_w)f_{sw} + f_w - A \\
&T_{ml} = (D/B)\left[1 - \exp(-B\tau_h/c_w h_{ml})\right]
\end{aligned}
\tag{19}
$$

For sea water the specific heat is $c_w = 4 \times 10^6$ J/m$^3$/K and the albedo is taken to be 0.2. The last equation is the result of integrating the heat balance for the mixed layer:

$$d(c_w h_{ml} T_{ml})/dt = D - BT_{ml} \tag{20}$$

For positive perturbations to the longwave fluxes in the range 17 W/m$^2$ < $\delta$ < 40 W/m$^2$, a seasonal ice pack is possible (see Figure 3). For example, with $\delta$ = 30 W/m$^2$, the mixed layer reaches 6°C. It requires 117 days to cool off in the fall. In the remaining 65 days of cold weather, the ice grows to 1 m, reaching a surface temperature of −16°C. In the warm season, it takes 2 days to warm the ice to the freezing point, 65 days to melt it, and 115 days to warm the mixed layer.

For climates warmer than $\delta$ = 40 W/m$^2$, the mixed layer gets so warm that it cannot cool completely in half a year, so ice never forms.

## The Annual Energy Balance

The coldest surface temperature and the thickest ice occur at the end of the freezing season. A simple energy argument leads to estimates of these quantities. Since the temperature varies between T = 0 and T = $T_g$ during the cold season and during the warm season, the radiated energy for the entire year must be approximately

$$2Y\left(A + BT_g/2\right)$$

In an equilibrium cycle, this must equal the energy reaching the ice, which is

$$Y(f_{lwc} + f_w) + Y\left[f_{lww} + (1-\alpha)f_{sw} + f_w\right]$$

Equating these, and neglecting $f_w$ for the moment, leads to

$$T_g = \left(D_m - D_g\right)/B \tag{21}$$

in which $D_g$ and $D_m$ are the net radiation balances over an ice surface at the freezing point during the growing and melting seasons. Using this expression in the surface heat balance, Equation (3), gives the maximum thickness as

$$h_g = (k/B)\left(D_g - D_m\right)/D_m \tag{22}$$

With $D_m = f_{lww} + (1 - \alpha)f_{sw} - A$ = 20 W/m$^2$, and $D_g = A - f_{lwc}$ = 140 W/m$^2$, we obtain $T_g$ = −26°C, and $h_g$ = 2.6 m. The expression for $h_g$ underlines the sensitivity to the heat balance in the melting season. As $D_m$ approaches zero, the ice grows without bound.

The simulations in "A Model of the Perennial Ice Cover" showed that the perennial ice cover gives way to a seasonal ice cover for a positive perturbation of 18 $W/m^2$ to the longwave fluxes. To see why this is, we develop a condition on $D_g$ and $D_m$ which just causes the ice to melt completely at the end of the melting season. If h = 0 at the beginning of the cold season, the thickness at the end of the season will be, Equation (6),

$$h = (k/B)\left[\left(1 + 2BYD_g/kL\right)^{1/2} - 1\right] \tag{23}$$

In an equilibrium cycle, this must equal the ice melted in the warm season, which is approximately $YD_m/L$. Combining these results gives the constraint

$$\left(1 + BYD_m/L\right)^2 = 1 + 2BYD_g/kL \tag{24}$$

which is plotted in Figure 4. For climates having ($D_g$, $D_m$) below this line, the ice will melt away completely during the summer. If we set $D_m = 20 + \delta$ and $D_g = 140 + \delta$ the equation requires $\delta = 17$ $W/m^2$, in good agreement with the simulations.

## Sensitivity to the Ocean Heat Flux

One result of the Maykut and Untersteiner simulations is a strong dependence on the ocean heat flux. They showed that increasing $f_w$ from 0 to 7 $W/m^2$ caused the equilibrium thickness to decrease from about 6 m to zero. When one considers how poorly the ocean heat flux is known, and the uncertainties in the other heat fluxes, this sensitivity is unsettling.

The present models have been run using different values of $f_w$, with results that support the Maykut and Untersteiner results. In particular, holding other climate variables fixed, $f_w = 1$ $W/m^2$ produced ice of thickness 1.9 m at the end of summer. With $f_w = 12$ $W/m^2$ that thickness is reduced to 0.1 m.

We can appreciate how this works by restoring $f_w$ to the expressions for $T_g$ and $h_g$, Equations (21) and (22):

$$T_g = \left(D_m - D_g + 2f_w\right)/B \tag{25}$$

$$h_g = \frac{k}{B}\frac{\left(D_g - D_m + 2f_w\right)}{\left(D_m + 2f_w\right)}$$

From the first expression, we see that the surface temperature responds to the ocean heat flux exactly as it responds to the radiative fluxes. If an increment of heat $\Delta$ is added to the ice, in any way, the ice must respond by warming its surface $\Delta/B$ to maintain the overall energy balance. However, the response of the ice thickness is

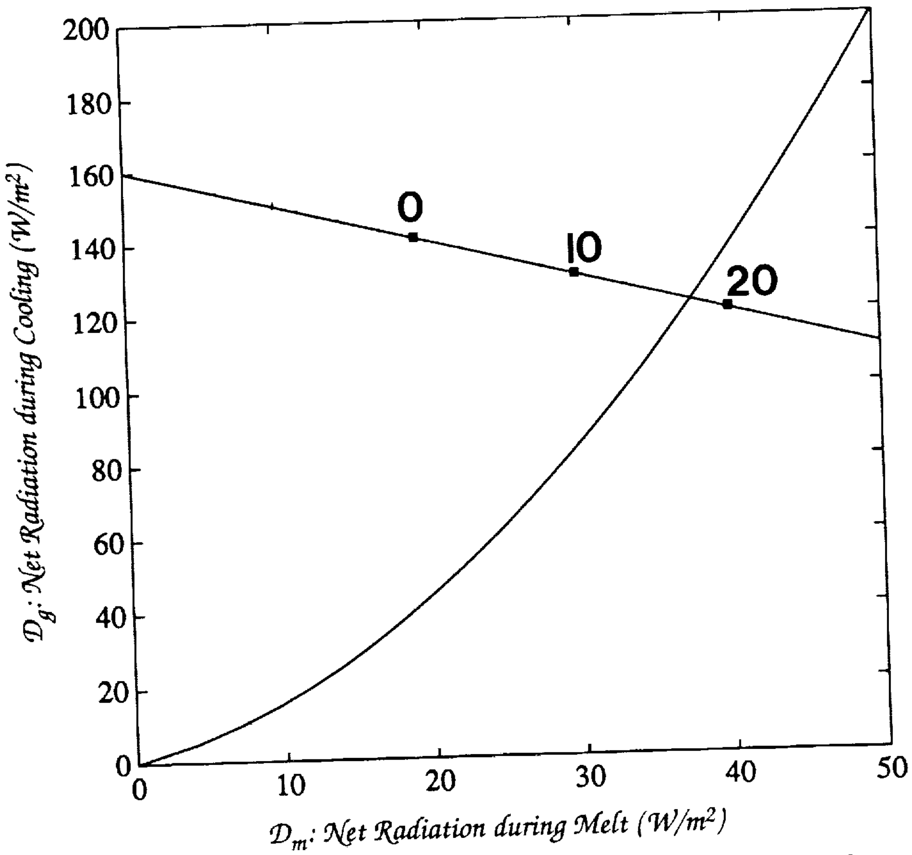


*Figure 4. The condition on the net radiation balances in the cooling and melting seasons, $D_g$ and $D_m$, which causes the ice just to vanish at the end of the summer. The straight line represents climates in which the longwave fluxes differ by $\delta$ from the present climate.*

more subtle. An increase to $f_w$ decreases the numerator and increases the denominator, and therefore has a larger effect on the thickness than one might expect.

## The Time Scale for the Ice Response

By combining the ice grown in the cold season and the ice melted in the warm, we obtain the ice thickness after one year, beginning with initial thickness $h_n$.

$$h_{n+1} = F(h_n) = h_n + YL^{-1}\left(\frac{kD}{k + Bh_n} - 2f_w - D_m\right) \qquad (26)$$

Now the time scale for the ice to reach its equilibrium cycle is

$$\Gamma = h_n - h_{eq}/(h_n - h_{n+1}) \qquad \text{(years)} \qquad (27)$$

If we also write

$$h_{n+1} = h_{eq} + \left(h_n - h_{eq}\right) dF/dt \tag{28}$$

and substitute for $h_{n+1}$ in (27), we find

$$\Gamma = \left(1 - dF/dh_n\right)^{-1} \tag{29}$$

where the derivative is evaluated at the equilibrium thickness

$$h_{eq} = (k/B)\left(\frac{D}{D_m + 2f_w} - 1\right) \tag{30}$$

This gives the time scale

$$\Gamma = (kL/YB)\frac{D}{(D_m + 2f_w)^2} \cong 3 \text{ yr} \tag{31}$$

## Discussion

The essential mechanism at work is that the ice adjusts its surface temperature to maintain the energy balance at the surface. In the present climate, the annual energy balance over a surface at the melting point is $D_m - D_g = -120$ W/m$^2$, meaning that a surface at T = 0°C would lose 120 W/m$^2$ more than it receives. The surface adjusts by cooling to about –26°C.

The second important idea is that during ice growth, the ice thickness is related to its surface temperature; the colder the temperature, the thicker the ice. Therefore we can assess the response of the ice thickness to a climate perturbation, in the form of a change dE in the energy reaching the surface, by evaluating

$$dT/dE \quad \text{and}$$

$$dh/dE = dh/dT \bullet dT/dE = -\left[(k + Bh)^2/DBk\right] \tag{32}$$

For h = 3 m, the sensitivity is about –0.2 m for 1 W/m$^2$ increase in incoming energy. Note that the sensitivity is a strong function of thickness.

Thus the thickness is rather sensitive to the energy fluxes. Perturbations of 10 W/m$^2$ can mean several meters of ice thickness. Nevertheless, the system can adjust to perturbations of this magnitude.

However, for larger fluctuations, the system cannot adjust. A cooler climate in which the longwave fluxes were 20 W/m$^2$ less than present (about 5°C of tropospheric cooling) would allow the ice to grow without bound. A positive perturbation of the same magnitude would cause the ice to melt away completely in the summer, so that the entire Arctic would support only a seasonal ice cover. With a positive perturbation of about 40 W/m$^2$, enough heat is stored in the upper ocean during the warm season to prevent ice from forming during the winter.

Of course, with a model that ignores many physical processes and treats others badly, one cannot have great confidence in the numerical results. Still, I expect the relationships between quantities in these models to have a qualitative validity, in the sense that in a more careful model, the signs and powers in the expressions would survive, while different numerical factors, of order unity, might appear.

These ideas have implications for climate models that attempt to simulate the Arctic ice cover. First, the level of detail in the perennial and seasonal models discussed here may be about right for use in a climate model. The models surely allow an interactive ice cover without a heavy computational burden. A second point is that we have found limits on errors that can be tolerated in the energy fluxes supplied by the climate model to the ice. Fluxes in error by $\pm$ 10 $W/m^2$ will not destroy the ice, but errors of $\pm$ 20 $W/m^2$ will lead to a qualitatively wrong ice pack. Finally, given the sensitivity of the ice thickness and temperature to perturbations on the order of 10 $W/m^2$, and the likelihood that climate models will have this much uncertainty for some time to come, it is not realistic to expect a climate model to reproduce the observed ice thickness, or to predict how much the thickness will change. It is enough to ask that a climate model distinguish between a perennial ice cover and a seasonal ice cover, or between an ice cover and no ice at all.

Of course, the actual Arctic ice cover is too complicated for a toy model. There is in fact a great range of thicknesses, all growing and melting at different rates, and all responding to mechanical as well as thermal forcing. I do not want to leave the impression that these processes are unimportant.

## Acknowledgments

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