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A CASE STUDY OF ANALYSIS METHODS FOR LARGE DEFLECTIONS OF A CANTILEVER BEAM

By L.D. Craig

Structures and Dynamics Laboratory
Science and Engineering Directorate

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A load case study of geometric nonlinear large deflections of a cantilever beam is presented. The bending strain must remain elastic. Closed form solution and finite element methods of analysis are illustrated and compared for three common load cases. A nondimensional nomogram for each case is presented in the summary.

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LIST OF SYMBOLS

E	modulus of elasticity
$E(p)$	Legendre's complete elliptic integral of the second kind
$E(p,\beta)$	Legendre's elliptic integral of the second kind
$F(p,\beta)$	Legendre's elliptic integral of the first kind
I	area moment of inertia
$K(p)$	Legendre's complete elliptic integral of the first kind
L	L_1+L_2
L_1	actual beam length
L_2	imaginary beam length
M	bending moment
$M_{\max}$	bending moment at the fixed end
M_o	linear bending moment at the fixed end
P	applied load
P_1	$P \sin \theta_B$, applied load vertical component
P_2	$P \cos \theta_B$, applied load horizontal component
Q	any point along the length of the beam
T	$T = \sum_{n=2}^{\infty} a_n s^n$
T_1	$T_1 = s - \frac{a_3^2 s^7}{14} - \frac{a_3 a_6 s^{10}}{10} - \ldots$
T_2	$T_2 = \frac{a_3 s^4}{4} + \frac{a_6 s^7}{7} + \frac{(a_9 - a_3^3) s^{10}}{10} + \ldots$
$X_{\max}$	deflected horizontal location (from fixed end) of free end
$Y_{\max}$	vertical deflection of free end
Y_o	linear vertical deflection of free end

a_n	series coefficient
c	distance from beam neutral axis to outer surface
h	$\frac{2p}{k}$
k	$\sqrt{\frac{P}{EI}}$
m	$\sin^{-1}\left[-\frac{1}{p}\sin\frac{1}{2}\theta_B\right]$, elliptic integral variable
n	$\sin^{-1}\left[\frac{-\sin\frac{1}{2}(\theta_B+\gamma)}{p}\right]$, elliptic integral variable
p	$-\sin\frac{1}{2}\theta_c$, $-\sin\frac{1}{2}(\theta_B+\gamma)$, elliptic integral variable
q	horizontal deflection component of point Q
q_c	horizontal deflection component of point C (free end)
r	vertical deflection component of point Q
r_c	vertical deflection component of point C (free end)
s	arc length along axis of deflected beam
t	beam thickness
w	applied load per unit length
x	horizontal deflection (from fixed end) at any point along the beam
x_c	deflected horizontal location (from fixed end) at the free end
x_L	deflected horizontal location (from fixed end) at $s = L$ (free end)
y	vertical deflection at any point along the beam
y_c	vertical deflection of free end
y_L	vertical deflection at $x = L$ (free end)
α	beam rotation at the free end

σ_{max}	bending stress at fixed end
β	elliptic integral variable ϕ_B, m, n
ζ	$\frac{wL^3}{EI}$
ψ	elliptic integral variable
γ	beam rotation at free end (point C)
δ_h	horizontal deflection of free end relative to original position
δ_v	vertical deflection of free end
δ_x	deflected horizontal location (from fixed end)
δ_y	vertical deflection
θ	beam rotation
θ_B	angle between load line of action and horizontal axis
θ_C	beam rotation at point C (free end)
ρ	deflected beam curvature
ϕ	beam rotation
ϕ_B	$\sin^{-1}(0.7071/p)$, elliptic integral variable

TECHNICAL MEMORANDUM

A CASE STUDY OF ANALYSIS METHODS FOR LARGE DEFLECTIONS OF A CANTILEVER BEAM

INTRODUCTION

Occasionally the analyst is required to predict the maximum bending deflection and moment of a cantilever beam where the usual linear methods are insufficient. This occurs in slender beams, such as springs, when deflections are large enough that the exact nonlinear Bernoulli-Euler equation must be solved. The basic assumption is that the beam is flexible enough to allow large deflections while the strains remain elastic. Other assumptions are that the beam is initially straight and of uniform cross section and that the bending does not alter the length. This report illustrates analyses via closed-form solution and finite-element methods for three common load cases and presents nomograms for easily obtained approximate solutions.

CASE I: Deflection of a Cantilever Beam With Load *P*, Which Remains Vertical, at the Free End[1]

The length of the beam (undeformed or deformed) is given by

$$L_1 = L - L_2 \ ,$$

where L_2 is the imaginary portion of the beam.[1] Now

$$L_2 = \frac{1}{k} F(p, \phi_B) \ ,$$

and

$$L = \frac{1}{k} K(p) \ ,$$

and therefore

$$L_1 = \frac{1}{k} \left\{ K(p) - F\left[p, \sin^{-1}\left(\frac{0.7071}{p} \right) \right] \right\} \ ,$$

where

$$k = \sqrt{\frac{P}{EI}} \ ,$$

and E is the material modulus of elasticity and I is the area moment of inertia. $K(p)$ = Legendre's complete elliptic integral of the first kind and

$$F(p, \phi_B) = \int_0^{\phi_B} [d\psi / (1 - p^2 \sin^2 \psi)^{\frac{1}{2}}] \ ,$$

is Legendre's elliptic integral of the first kind. Where, for this case, since

$$\phi_B = \sin^{-1}(0.7071/p) \ ,$$

and

$$-\sin \frac{1}{2}\theta_B = p(\sin \phi_B) \ ,$$

$$\theta_B = -\frac{1}{2}\pi \ .$$

The value of p must be found by trial and error from the equation for L_1. Having found p, the solution proceeds as follows:

$$h = \frac{2p}{k} \ ,$$

the horizontal deflection of the end of the beam is

$$\delta_h = L_1 - h\cos\phi_B \ ,$$

and the vertical deflection is

$$\delta_v = \frac{2}{k}\left[E(p,\phi_B) - \frac{1}{2}F(p,\phi_B) - E(p) + \frac{1}{2}kL\right] \ .$$

where $E(p)$ is Legendre's complete elliptic integral of the second kind and

$$E(p,\phi_B) = \int_0^{\phi_B} (1-p^2\sin^2\psi)^{\frac{1}{2}}\,d\psi \ ,$$

is Legendre's elliptic integral of the second kind. The geometrical representation of most of the above variables is shown in figure 1.

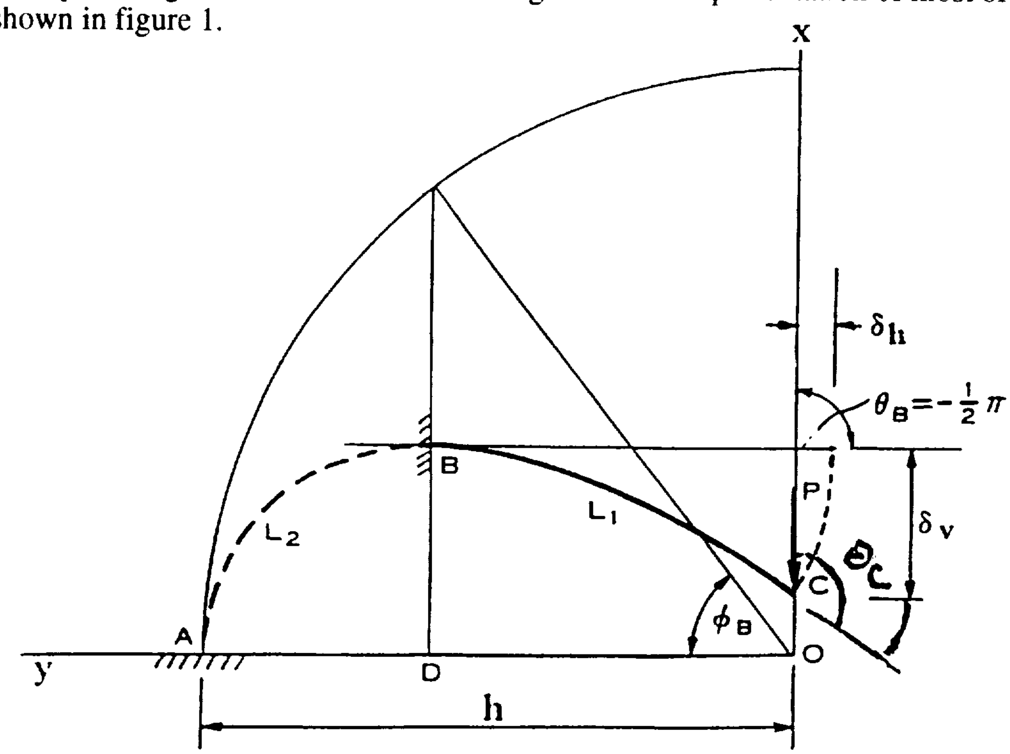

Figure 1. Case I variables and geometry.[1]

An example problem (example 2, ref. 1) will now be solved to demonstrate the use of these equations. Find the horizontal and vertical deflection of the end of a 150-inch long horizontal cantilever of $2^1/2$ by $^1/4$ inch cross section with a vertical load $P = 5$ lb applied at the end ($E = 30\times10^6$ lb/in^2).

$$I = bt^3/12 = (2.5)(0.25)^3/12 = 3.2552\times10^{-3}\ \text{in}^4 \ ,$$

$$EI = 9.7656\times10^4\ \text{lb-in}^2 \ ,$$

$$P/EI = k^2 = 5.12\times10^{-6}\ \text{in}^{-2} \Rightarrow k = 1/139.754\ \text{in}^{-1} = 7.1554\times10^{-3}\ \text{in}^{-1} \ ,$$

$$L_1 = 139.754\{K(p)-F[p, \sin^{-1}(0.7071/p)]\} = 150\ \text{in} \ .$$

This equation can be solved for p by trial and error in the following manner:

$$K(p)-F(p,\phi_B) = \frac{150}{139.754} = 1.0733 \ ,$$

where

$$\phi_B = \sin^{-1}(0.7071/p) \ ,$$

and introducing the expression

$$p = \sin\frac{1}{2}\,\theta_C\ \text{(ref. 2)} \ ,$$

where θ_C is the rotation at the end of the beam relative to the vertical axis. Initially let $\theta_C = 100^\circ$ which is a 10° rotation of the end with respect to the horizontal axis.

$$p = 0.766 \Rightarrow \phi_B = 67.3766^\circ \ .$$

Use elliptic integral tables to find $K(0.766)$ and $F(0.766, 67.3766^\circ)$.

$$K(p) = 1.9356 \text{ at } \sin^{-1}(0.766) = 50^\circ$$

and

$$F(p,\phi_B) = 1.3414 \text{ at } \sin^{-1}(0.766) = 50^\circ$$

(linearly interpolated values between 67° and 68°).

$$K(p)-F(p,\phi_B) = 0.5942 \neq 1.0733 \ ,$$

therefore try another value of θ_C. Let

$$\theta_C = 140^\circ \Rightarrow p = 0.93969 \Rightarrow \phi_B = 48.8057^\circ \ .$$

Using elliptic integral tables again,

$$K(p) = 2.5046 \quad \text{and} \quad F(p,\phi_B) = 0.95792 \ ,$$

$$K(p)-F(p\phi_B) = 1.54668 \neq 1.0733 \ ,$$

therefore try another value of θ_C. Let

$$\theta_C = 120^\circ \Rightarrow p = 0.86603 \Rightarrow \phi_B = 54.7348^\circ \ .$$

Then using elliptic integral tables yields

$$K(p) = 2.1565 \quad \text{and} \quad F(p,\phi_B) = 1.0783 \ ,$$

$$K(p) - F(p,\phi_B) = 1.0782 \pm 1.0733 \ .$$

Now, using these values, $h = 242.06$ in and $L = 139.754(2.1565) = 301.38$ in and the horizontal deflection is

$$\delta_h = 150 - 242.06 \cos(54.7348) = 10.244 \text{ in.}$$

Two additional terms are evaluated from the elliptic integral tables for the vertical deflection

$$E(p) = E(0.86603) = 1.2111 \ ,$$

$$E(p,\phi_B) = E(0.86603, 54.7348^\circ) = 0.8555 \ ,$$

therefore

$$\delta_v = 2(139.754)\{0.8555 - 0.5(1.0782) - 1.2111 + [301.38/(2)(139.754)]\} = 51.3 \text{ in.}$$

According to linear theory, the vertical deflection would be

$$\delta_v = \frac{PL_1^3}{3EI} = 57.6 \ in \ ,$$

and, of course, the horizontal deflection is zero. The explanation for the larger linear deflection is that when the load increases gradually from 0 to 5 lb the bending moment is not increasing at the same rate since the horizontal movement of the end of the cantilever is toward the fixed end. The linear theory, on the other hand, ignores the nonlinear relationship between load and bending moment. This is why the principle of superposition is not valid when applying the exact theory. The exact bending moment is

$$M_{\max} = P(L_1 - \delta_h) = 5(150 - 10.244) = 698.78 \text{ in-lb} \ ,$$

and the maximum bending stress is

$$\sigma_{\max} = \frac{M_{\max} c}{I} = 26{,}833 \text{ lb/in}^2 \ .$$

For comparison purposes, this problem was solved using the finite element computer code SPAR.[5] Input data and results are contained in appendix A. Maximum deflection, moment, and stress results are

$$\delta_v = 50.72 \text{ in} \quad \text{and} \quad \delta_h = 10.72 \text{ in} \ .$$

$$M_{\max} = 696.4 \text{ in-lb and } \sigma_{\max} = 26{,}742 \text{ lb/in}^2 \ .$$

There is less than 5-percent difference in the results of the two methods.

CASE II: Deflection of a Cantilever Beam With a Follower Load P, at the Free End, Whose Line of Action Remains Perpendicular to the Beam Axis[3]

The classical analysis of this case rests with the solution of

$$EI\,\frac{d^2\phi}{ds^2} - P_1\cos\phi - P_2\sin\phi = 0\ ,$$

where $P_1 = P\sin\theta_B$ and $P_2 = P\cos\theta_B$, ϕ is the rotation of any point along the beam and θ_B is the angle from the horizontal axis of the line of action of P. The classical solution to this problem is given below, omitting the derivation.[3] The horizontal deflection at any point along the length of the beam is

$$\delta_x = \frac{1}{k}\,\{\cos\theta_B\,[F(p,m) - F(p,n) + 2E(p,n) - 2E(p,m)] + 2p\sin\theta_B\,(\cos m - \cos n)\}\ ,$$

and the vertical component is

$$\delta_y = \frac{1}{k}\,\{2p\cos\theta_B\,(\cos m - \cos n) - \sin\theta_B\,[F(p,m) - F(p,n) + 2E(p,n) - 2E(p,m)]\}\ .$$

The maximum horizontal deflection at the free end (from the fixed end) is

$$x_c = \frac{1}{k}\,\{\cos\theta_B\,[F(p,m) - K(p) + 2E(p) - 2E(p,m)] + 2p\sin\theta_B\cos m\}\ ,$$

and the maximum vertical deflection is

$$y_c = \delta_v = \frac{1}{k}\,\{2p\cos\theta_B\cos m - \sin\theta_B\,[F(p,m) - K(p) + 2E(p) - 2E(p,m)]\}\ ,$$

where

$$k = \sqrt{\frac{P}{EI}}\ ,$$

as in case I. For this case

$$p = -\sin\frac{1}{2}\,(\theta_B + \gamma)\ ,$$

where

$$\theta_B + \gamma = -90^\circ\ ,$$

and γ is the rotation at the free end relative to the horizontal axis. Also

$$m = \sin^{-1}\left[-\frac{1}{p}\sin\frac{1}{2}\,\theta_B\right]\ ,$$

and

$$n = \sin^{-1}\left[-\frac{1}{p}\sin\frac{1}{2}\,(\theta_B + \phi)\right]\ ,$$

where $\phi = \gamma$ and $\cos n = 0$ at the free end. $E(p) \equiv$ Legendre's complete elliptic integral of the second kind and $E(p,m$ or $n) \equiv$ Legendre's elliptic integral of the second kind

$$E(p,m, \text{ or } n) = \int_0^{m \text{ or } n} (1-p^2 \sin^2 \phi)\, d\phi \ .$$

The solution to this problem can be found by applying the principle of elastic similarity[3] and is presented in the following, omitting some derivation. The length of the beam (deformed or undeformed) is found to be

$$L_1 = \frac{1}{k}\,[K(p)-F(p,m)] \ .$$

The horizontal and vertical components of the deflection of a point $Q(x,y)$ along the beam can be found in terms of the xy-coordinate system. The quantities r and q in the $x'y'$-coordinate system are given as

$$r = \frac{1}{k}\,[F(p,m)-F(p,n)+2E(p,n)-2E(p,m)] \ ,$$

where p, m, and n are defined above. At the free end of the beam, point C,

$$r_C = \frac{1}{k}\,[F(p,m)-K(p)+2E(p)-2E(p,m)] \ ,$$

$$q = h(\cos m - \cos n) = 2p(\cos m - \cos n)/k \ ,$$

and at the free end

$$q_C = \frac{2p}{k}\cos m \ .$$

Hence, at any point Q

$$\delta_x = r\cos\theta_B + q\sin\theta_B \ ,$$

$$\delta_y = q\cos\theta_B - r\sin\theta_B \ .$$

At point C, the free end

$$x_C = r_C\cos\theta_B + q_C\sin\theta_B \ ,$$

and therefore

$$\delta_h = L_1 - x_C \ ,$$

$$y_C = \delta_v = q_C\cos\theta_B - r_C\sin\theta_B \ .$$

The geometrical representation of most of the above variables is shown in figure 2.

An example problem will now be solved to demonstrate the application of these equations. Find the vertical and horizontal deflection at the end of a cantilever beam 20 inches in length with a 0.2 inch square cross section and loaded by a force of 5.347 lb at the end which remains perpendicular to the axis of the beam. The material modulus of elasticity E of 10×10^6 lb/in^2.

$$I = \frac{bt^3}{12} = 1.3333\times10^{-4}\ \text{in}^4 \ ,$$

$\theta_B+\gamma = -90^\circ$, which implies that the load remains perpendicular to the beam axis

$$p = -\sin\frac{1}{2}(\theta_B + \gamma) = 0.7071 \ ,$$

and

$$k = \sqrt{\frac{P}{EI}} = 6.3327\times10^{-2}\ \text{in}^{-1} \ .$$

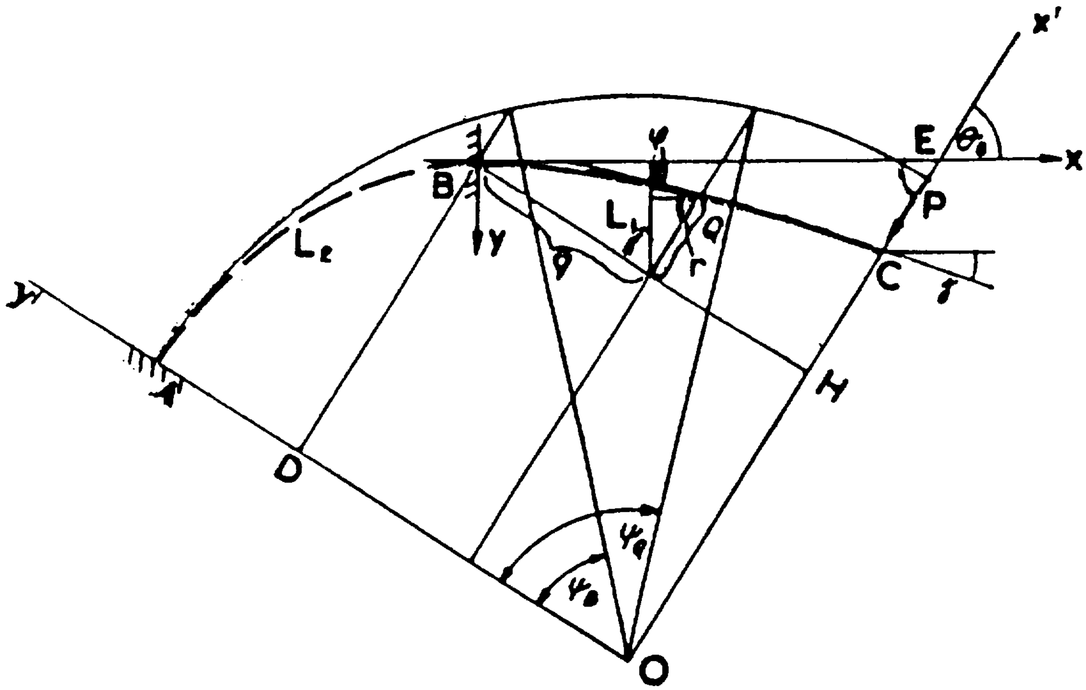

Figure 2. Case II variables and geometry.[2]

Now the equation for length L_1 may be solved for the angle θ_B.

$$L_1 = 15.791\ [K(p)-F(p,m)] = 20 \ .$$

$$K(p) = K(\sin^{-1} p) = K(45^\circ) = 1.8541 \text{ from complete elliptic integral table}$$

$$F(p,m) = 1.8541 - \frac{20}{15.791} = 0.587556 \ .$$

This relationship is satisfied if $m = 32.765^\circ$ from tables of elliptic integrals of the first kind. Therefore, knowing p and m, from the elliptic integral tables

$$E(p,m) = 0.55692 \ ,$$

$$E(p) = 1.3506 \ ,$$

and from the equation for m, $\theta_B = -45^\circ$ which means that the rotation of γ of the free end is equal to -45°. Now all the terms are available to evaluate r_c and q_c

$$r_c = 5.066,$$

$$q_c = 18.77885 \ .$$

Using the absolute value of θ_B, x_c, and y_c are calculated to be

$$x_c \ 5.066(0.7071)+18.77885(0.7071) = 16.861 \text{ inches}$$

and

$$\delta_h = 20.0–16.861 = 3.139 \text{ inches}$$

$$y_c = \delta_v = 18.77885(0.7071)–5.066(0.7071) = 9.696 \text{ inches} \ .$$

The bending moment $M_{\max}$ at the fixed end is equal to Pq_c.

$$M_{\max} = 5.347(18.77885) = 100.41 \text{ in-lb} \ .$$

This problem was solved using the finite element computer code SPAR.[5] A follower force was applied at the free end of the beam. Input data and results are contained in appendix B. Maximum deflection, bending moment, and stress results are

$$\delta_v = 9.697 \text{ inches} \quad \text{and} \quad \delta_h = 3.137 \text{ inches} \ ,$$

$$M_{\max} = 100.42 \text{ in-lb} \quad \text{and} \quad \sigma_{\max} = 75{,}310.0 \text{ lb/in}^2.$$

There is no significant difference between the results of the numerical and exact theoretical methods for this example.

CASE III: Deflection of a Cantilever Beam With Uniformly Distributed Load Whose Direction Remains Vertical[4]

In this case, as in the previous cases, only beams of uniform cross section are considered. The basic assumptions are that the deformations are elastic and that the bending does not alter the length of the beam.

Cases I and II obtain results for a concentrated load. In the case of a uniformly distributed load, the Bernoulli-Euler equation,

$$\frac{1}{\rho} = \frac{d\theta}{ds} = \frac{M}{EI} \ ,$$

gives, with the origin at the free end of the beam,

$$\frac{dM}{ds} = -ws \cos \theta = EI \frac{d^2\theta}{ds^2} \ ,$$

where ρ is the deflected beam curvature, θ is the rotation, s is the distance to any point along the length, M is the bending moment, E is the modulus of elasticity, I is the cross sectional moment of inertia, and w is the load per unit length.

This equation does not lend itself to any simple solution, therefore let

$$\theta = \sum_{n=0}^{\infty} a_n s^n \ ; \ \frac{d\theta}{ds} = \sum_{n=1}^{\infty} n a_n s^{n-1} \ ; \ \frac{d^2\theta}{ds^2} = \sum_{n=2}^{\infty} n(n-1) a_n s^{n-2} \ .$$

The boundary conditions $s = 0$, $\theta = \alpha$, $d\theta/ds = 0$ give $a_0 = \alpha$; $a_1 = 0$. Then

$$\frac{d^2\theta}{ds^2} = -\frac{ws}{EI} (\cos\alpha \cos T - \sin\alpha \sin T) \ , \ T = \sum_{n=2}^{\infty} a_n s^n \ ,$$

where α is the rotation of the beam at the free end.

Expand $\cos T$ and $\sin T$ as

$$\cos T = 1 - \frac{T^2}{2!} + \frac{T^4}{4!} - \frac{T^6}{6!} + \dots \ ,$$

$$\sin T = T - \frac{T^3}{3!} + \frac{T^5}{5!} - \frac{T^7}{7!} + \dots \ ,$$

and substitute into the previous equation. Equating the two expressions for $\frac{d^2\theta}{ds^2}$ and substituting the above expression for T, the coefficients a_n may be determined by equating like coefficients of s. The coefficient a_2 of the constant term is equal to zero. Coefficients of s through s^7 are:

$$a_3 = -\frac{w \cos\alpha}{6EI} \ , \ \text{(coefficient of } s)$$

$$a_4 = a_5 = 0 \ , \ \text{(coefficients of } s^2 \text{ and } s^3)$$

$$a_6 = \frac{a_3 w \sin\alpha}{30EI} = -\frac{w^2 \sin\alpha \cos\alpha}{180E^2I^2} \ , \ \text{(coefficient of } s^4)$$

$$a_7 = a_8 = 0 \ , \ \text{(coefficients of } s^5 \text{ and } s^6)$$

$$a_9 = \frac{a_6 w \sin\alpha}{72EI} \ , \ \text{(coefficient of } s^7)$$

and

$$a_{3n+4} = a_{3n+5} = 0 \quad \text{for} \quad n = 0, 1, 2, 3, \dots \ . \ \text{(ref. 4)}$$

Thus,

$$\theta = \alpha + \sum_{k=1}^{\infty} a_{3k} s^{3k} \ .$$

Since $dy/ds = \sin\theta$,

$$y = \int_0^s \sin\theta \, ds = T_1 \sin\alpha + T_1 \cos\alpha \ ,$$

$$T_1 = s - \frac{a_3^2 s^7}{14} - \frac{a_3 a_6 s^{10}}{10} - \dots \ ,$$

$$T_2 = \frac{a_3 s^4}{4} + \frac{a_6 s^7}{7} + \frac{(a_9 - a_3^3/6) s^{10}}{10} + \dots \ .$$

Similarly,

$$x = T_1 \cos\alpha - T_2 \sin\alpha \ .$$

Finally, solving for M in the Bernoulli-Euler equation and differentiating the expression for θ with respect to s gives

$$M = EI \sum_{k=1}^{\infty} 3ka_{3k}s^{3k-1} \ ,$$

$$\alpha = -\sum_{k=1}^{\infty} a_{3k}L^{3k} \ ,$$

where, for this case, the imaginary beam length is zero and L is the actual beam length. With $s = L$, $\alpha = wL^3/6EI$, the first approximation to the maximum deflection is

$$y \doteq s\sin\alpha + \frac{1}{4}a_3 s^4 \cos\alpha \doteq \frac{wL^4}{8EI} \ , \ (\sin\alpha \doteq \alpha \text{ and } \cos\alpha \doteq 1)$$

and the moment is

$$M_{\max} \doteq EI(3a_3 s^2) \doteq \frac{wL^2}{2} \ ,$$

which are in agreement with small deflection theory.

The following example illustrates these equations. Given a thin cantilever beam 20 inches long and constant cross section of 0.2 inches square, find the uniformly distributed load which causes a 1.0 radian rotation at the free end and find the maximum deflection and stress for that load. The modulus of elasticity is 10×10^6 lb/in^2.

$$I = \frac{bt^3}{12} = 1.333\times10^{-4}\,\text{in}^4 \ ,$$

is the area moment of inertia and

$$\alpha = -\sum_{k=1}^{\infty} a_{3k}L^{3k} = -a_3L^3 - a_6L^6 - a_9L^9 \ldots$$

$$= \frac{wL^3\cos\alpha}{6EI} + \frac{w^2L^6\cos\alpha\sin\alpha}{180E^2I^2} + \frac{w^3L^9\cos\alpha\sin^2\alpha}{12{,}960E^3I^3} \ldots \ .$$

Let $\zeta = \dfrac{wL^3}{EI}$ and substitute in the above equation. After moving α to the right hand side and dividing by $\dfrac{\cos\alpha\sin^2\alpha}{12{,}960}$ the expression is

$$\zeta^3 + \frac{72}{\sin\alpha}\zeta^2 + \frac{2{,}160}{\sin^2\alpha}\zeta - \frac{12{,}960\alpha}{\sin^2\alpha\cos\alpha} = 0 \ .$$

Solving this equation for ζ with $\alpha = 1.0$ radian gives $\zeta = 8.7422$. Therefore, the uniformly distributed load w is equal to 1.45667 lb/in. Continuing with the solution

$$a_3 = -9.84\times10^{-5} \ ,$$

$$a_6 = -3.01623\times10^{-9} \ ,$$

$$a_9 = -3.852139\times10^{-14} \ ,$$

and

$$T_1 = 20.0-0.88526-0.30392 = 18.81082$$

$$T_2 = -3.936-0.55154+0.12316 = -4.36438 \ .$$

Hence, the maximum vertical deflection is

$$y_L = \delta_v = 18.81082 \sin(1.0) + (-4.36438) \cos(1.0) = 13.47 \text{ in.}$$

The horizontal distance of the free end from the fixed end is

$$x_L = 18.81082 \cos(1.0) - (-4.36438) \sin(1.0) = 13.836 \text{ in.}$$

Therefore, the maximum horizontal deflection is

$$\delta_h = 20.0-13.836 = 6.164 \text{ in.}$$

The bending moment at the fixed end is

$$M_{\max} = EI(3a_3L^2+6a_6L^5+9a_9L^8+...)$$

$$= 1.333\times10^3 (-0.11808-0.05791-0.00888)$$

$$= -246.43 \text{ in-lb} \ .$$

Thus, the maximum bending stress is

$$\sigma_{\max} = \frac{M_{\max}c}{I} = 184{,}822.5 \text{ lb/in}^2 \ .$$

The SPAR[5] finite element results are:

$$\delta_v = 13.314 \text{ in} \ ,$$

$$\delta_h = 6.0485 \text{ in} \ ,$$

and

$$M_{\max} = -227.27 \text{ in-lb} \ .$$

The difference in sign is due to the different coordinate system origins. Note that the finite element results are all less than the closed form solution with the largest difference being in $M_{\max}$ (± 8 percent). This difference may be reduced if the total load (wL = 29.13 lb) is concentrated at midlength and multiplied by the deformed distance to the fixed end (8.225 in) to give $M_{max} = -239.62$ in-lb. Complete results are contained in appendix C.

These may be compared with linear beam theory results:

$$\delta_v = 21.8555 \text{ in} ,$$

$$\delta_h = 0.0 \text{ in} ,$$

and

$$M_{max} = 291.33 \text{ in-lb}.$$

SUMMARY

Solutions obtained by the methods presented in this paper are accurate, but lengthy and time consuming. Therefore, the following nomograms, for cases I, II, and III, are presented in figures 3, 4, and 5, respectively, for the analyst who requires a faster but less accurate result. They are derived from the closed-form solution equations. Abscissa (x-axis) and ordinate (y-axis) values are nondimensional. Abscissa parameters are noted in the legend to the right of each nomogram. Ordinate values are $\frac{PL^2}{EI}$ for figures 3 and 4 and $\frac{wL^3}{EI}$ for figure 5. The parameter L for both axes is equal to the actual beam length L_1. See the list of symbols for parameter definitions.

Finite element method results are nearly identical to the closed-form solutions as shown in all three cases. This instills confidence in each method.

Other load cases for which closed-form solutions were found are thermal,[6] two concentrated vertical loads,[3] and end moment plus vertical force.[1]

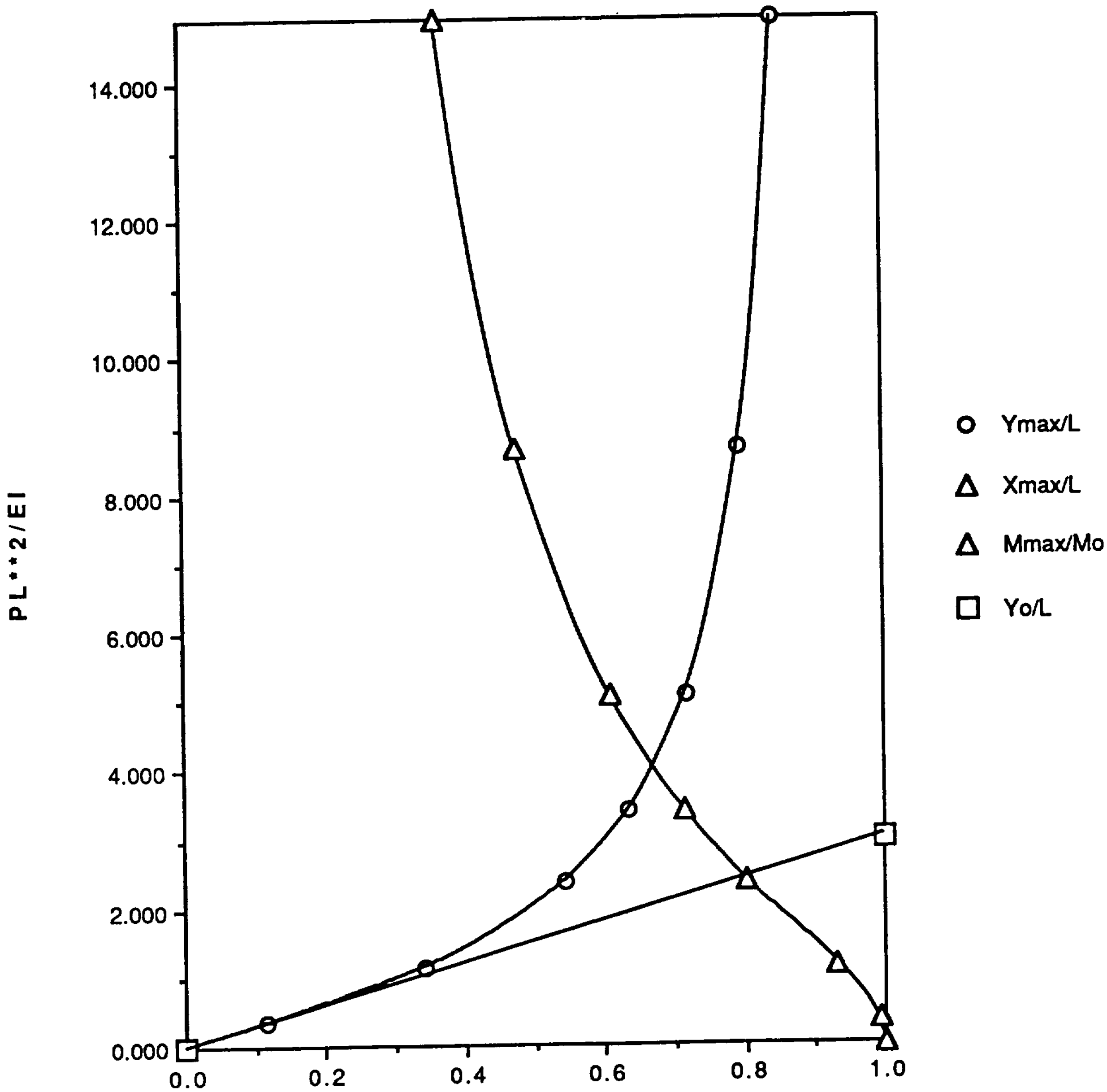

Figure 3. Cantilever beam with vertical end force.

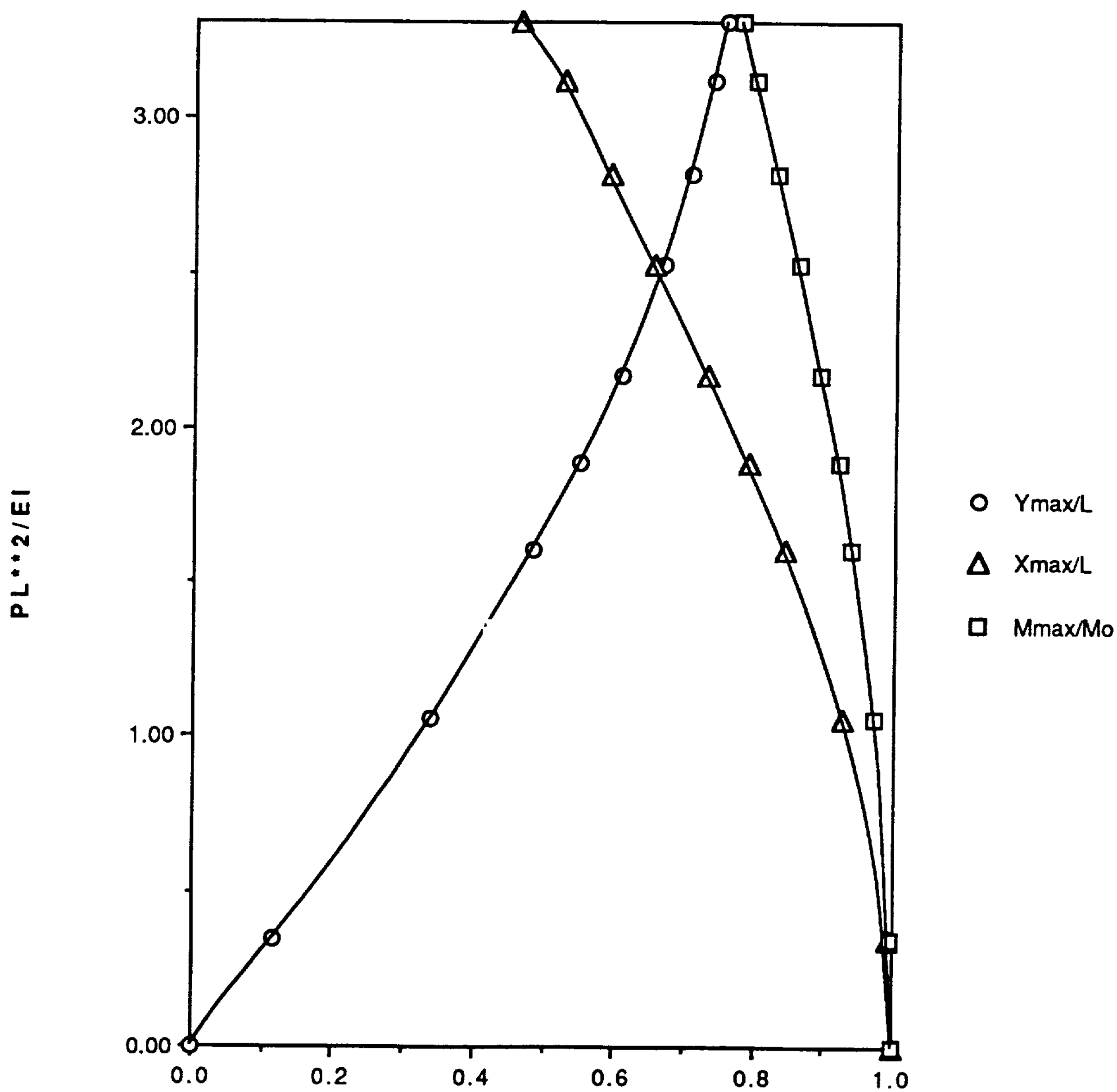

Figure 4. Cantilever beam with follower end force.

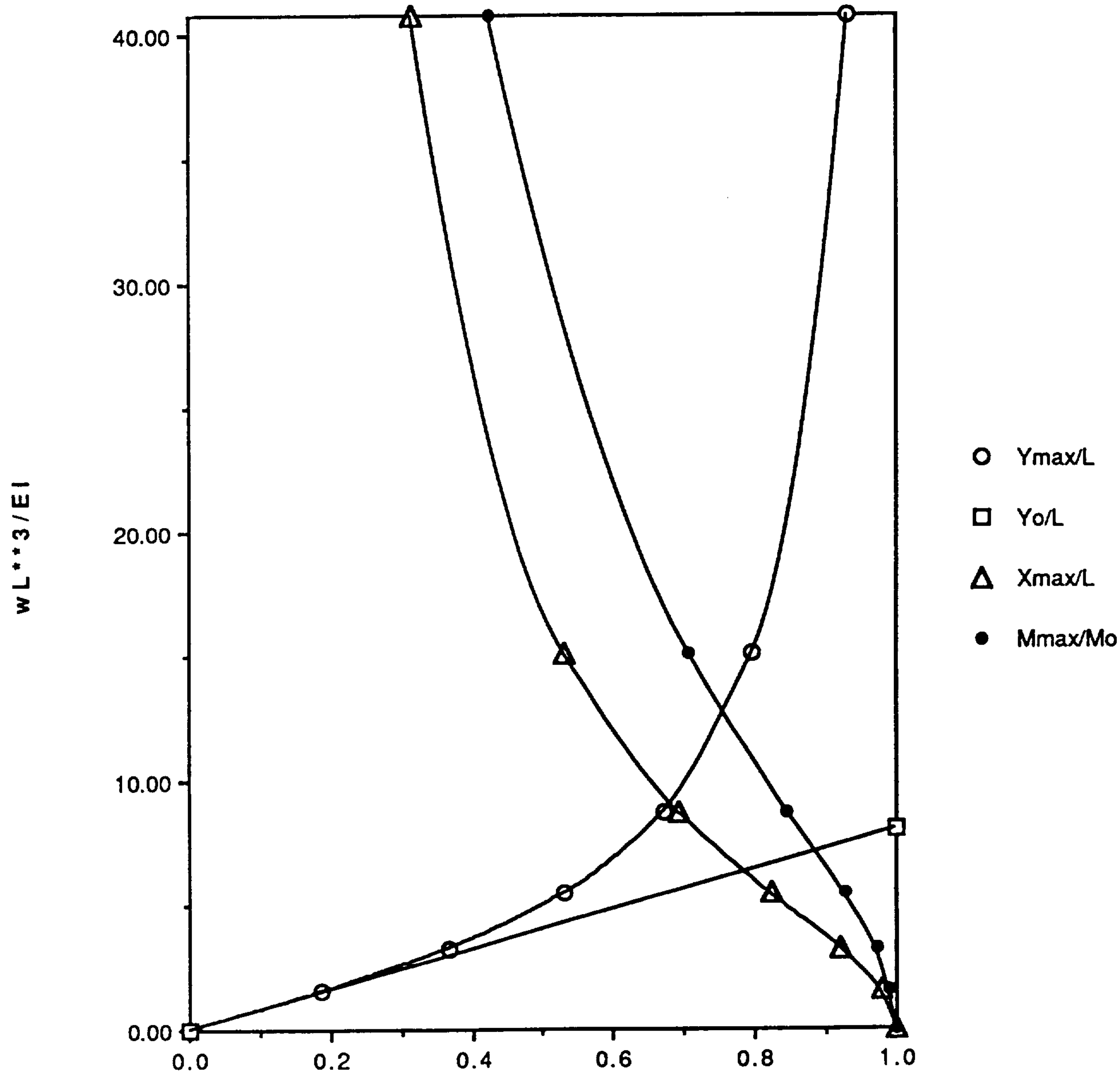

Figure 5. Cantilever beam with uniformly distributed load.

REFERENCES

1. Frisch-Fay, R.: "On Large Deflections." Austral. J. Appl. Sci., vol. 10, 1959, p. 418.

2. Frisch-Fay, R.: "Particular Cases of Large Deflections." Austral. J. Appl. Sci., vol. 11, 1960, p. 443.

3. Frisch-Fay, R.: "A New Approach to the Analysis of the Deflection of Thin Cantilevers." Journal of Applied Mechanics, vol. 28, 1961, p. 87.

4. Rohde, F.V.: "Large Deflections of a Cantilever Beam With Uniformly Distributed Load." Quart. Appl. Math., vol. 11, 1953, p. 337.

5. Engineering Analysis Language (EAL), Engineering Information Systems, Inc., 14395 Saratoga Ave., Saratoga, CA 95070.

6. Wilson, P.E.: "Large Thermal Deflection of a Cantilever Beam." AIAA Journal, vol. 1, 1963, p. 1451.

APPENDIX A

Case I Finite Element Model Input and Results

```
*XQT TAB
START 51 2 4 6$
TITLE'LARGE DEFLECTION OF CANTILEVER BEAM WITH VERTICAL END FORCE
JLOC$
1 0. 0. 0. 0. 0. 150. 51 1$
MATC$
1 30.+6 .3 .298 1.0-5$
MREF$
1 1 1 1 1.$
BA$
DSY 1 3.2552-1 0. 3.2552-3 0. .625 1.302-2$.25X2.5 RECT
1. 1. 1. .125 1.25 .125 -1.25 -.125 1.25 -.125 -1.25$
CONS=1$
ZERO 1 3 5:1$
*XQT ELD
E21$
1 2 1 50$
*XQT E
*XQT EKS
*XQT U1
*(GNA OPTIONS)
!SS=.25$INITIAL STEP SIZE
!SSMIN=.1$MINUMUM STEP SIZE
*XQT AUS
ALPHA:1 CASE TITL 3$
1'VERTICLE FORCE REMAINS VERTICLE AT JOINT 51,WL**2/EI=1.152
$1'FORCE FOLLOWS ROTATION OF JOINT 51,(WL**2/EI)=1.604
SYSVEC:2 APPL FORC:I=1:J=51:-5.000$(WL**2/EI)=1.152
$SYSVEC:2 FOLL FORC:I=1:J=51:-5.3470$(WL**2/EI)=1.604
$1'5LB FORCE APPLIED AT 45DEG TO BEAM
$SYSVEC:2 APPL FORC:I=1:J=51:-3.5355:I=3:J=51:-3.5355$
*CALL(18 NL JCL)
*CALL(GNA MAIN)$PERFORM NONLINEAR ANALYSIS
*CALL(GNA MOTI)$COMPUTE NONLINEAR DISPLACEMENTS
*CALL(GNA REAC)$COMPUTE NONLINEAR REACTIONS
*CALL(GNA PCH)$PRINT CONVERGENCE HISTORY
*XQT VPRT
FORMAT=4$
TPRINT NONL MOTI 3 1$
TPRINT NONL REAC 3 1$
*XQT DCU
TOC 1$
TOC 2$
*XQT  EXIT
```

1NONL MOTI ID= 3/ 1/ 1

VERTICAL FORCE REMAINS VERTICAL AT JOINT 51,WL**2/EI=1.152

0JOINT,	1	3	5
1	0.00000000000E+00*	0.00000000000E+00*	0.00000000000E+00*
2	-0.31857754683E-01	-0.16914902874E-03	-0.21162496967E-01
3	-0.12649687230E+00	-0.16622557833E-02	-0.41864320417E-01
4	-0.28249615236E+00	-0.57209222089E-02	-0.62105769066E-01
5	-0.49841337723E+00	-0.13500994217E-01	-0.81887325754E-01
6	-0.77278915413E+00	-0.26074280129E-01	-0.10120964441E+00
7	-0.11041505218E+01	-0.44430409009E-01	-0.12007353726E+00
8	-0.14910143236E+01	-0.69478809769E-01	-0.13847996242E+00
9	-0.19318903496E+01	-0.10205079293E+00	-0.15643001165E+00
10	-0.24252842514E+01	-0.14290171809E+00	-0.17392489860E+00
11	-0.29697002346E+01	-0.19271323119E+00	-0.19096594727E+00
12	-0.35636435348E+01	-0.25209555687E+00	-0.20755458086E+00
13	-0.42056226814E+01	-0.32158983217E+00	-0.22369231102E+00
14	-0.48941515589E+01	-0.40167046915E+00	-0.23938072739E+00
15	-0.56277512696E+01	-0.49274753477E+00	-0.25462148752E+00
16	-0.64049518078E+01	-0.59516913773E+00	-0.26941630721E+00
17	-0.72242935511E+01	-0.70922381277E+00	-0.28376695115E+00
18	-0.80843285782E+01	-0.83514289392E+00	-0.29767522393E+00
19	-0.89836218206E+01	-0.97310286921E+00	-0.31114296150E+00
20	-0.99207520554E+01	-0.11232277101E+01	-0.32417202286E+00
21	-0.10894312749E+02	-0.12855911700E+01	-0.33676428220E+00
22	-0.11902912758E+02	-0.14602190465E+01	-0.34892162137E+00
23	-0.12945176894E+02	-0.16470914032E+01	-0.36064592265E+00
24	-0.14019746368E+02	-0.18461447480E+01	-0.37193906191E+00
25	-0.15125279105E+02	-0.20572741636E+01	-0.38280290212E+00
26	-0.16260449956E+02	-0.22803353896E+01	-0.39323928707E+00
27	-0.17423950802E+02	-0.25151468530E+01	-0.40325003555E+00
28	-0.18614490557E+02	-0.27614916464E+01	-0.41283693576E+00
29	-0.19830795083E+02	-0.30191194532E+01	-0.42200173997E+00
30	-0.21071607019E+02	-0.32877484193E+01	-0.43074615963E+00
31	-0.22335685529E+02	-0.35670669711E+01	-0.43907186059E+00
32	-0.23621805979E+02	-0.38567355808E+01	-0.44698045870E+00
33	-0.24928759544E+02	-0.41563884789E+01	-0.45447351563E+00
34	-0.26255352752E+02	-0.44656353159E+01	-0.46155253500E+00
35	-0.27600406971E+02	-0.47840627736E+01	-0.46821895870E+00
36	-0.28962757845E+02	-0.51112361282E+01	-0.47447416350E+00
37	-0.30341254675E+02	-0.54467007664E+01	-0.48031945784E+00
38	-0.31734759763E+02	-0.57899836569E+01	-0.48575607894E+00
39	-0.33142147708E+02	-0.61405947796E+01	-0.49078519006E+00
40	-0.34562304672E+02	-0.64980285136E+01	-0.49540787799E+00
41	-0.35994127607E+02	-0.68617649883E+01	-0.49962515075E+00
42	-0.37436523454E+02	-0.72312713982E+01	-0.50343793553E+00
43	-0.38888408318E+02	-0.76060032856E+01	-0.50684707678E+00
44	-0.40348706615E+02	-0.79854057924E+01	-0.50985333452E+00
45	-0.41816350198E+02	-0.83689148861E+01	-0.51245738283E+00
46	-0.43290277467E+02	-0.87559585596E+01	-0.51465980858E+00
47	-0.44769432463E+02	-0.91459580111E+01	-0.51646111020E+00
48	-0.46252763949E+02	-0.95383288050E+01	-0.51786169684E+00
49	-0.47739224475E+02	-0.99324820168E+01	-0.51886188750E+00
50	-0.49227769441E+02	-0.10327825367E+02	-0.51946191046E+00
51	-0.50717356151E+02	-0.10723764343E+02	-0.51966190289E+00

1NONL REAC ID= 3/ 1/ 1

VERTICLE FORCE REMAINS VERTICLE AT JOINT 51,WL**2/EI=1.152

0JOINT,	1	3	5
1	0.50000000003E+01*	-0.25268581005E-07*	0.69638117838E+03*
2	0.36690892841E-08	-0.74119820420E-07	0.14224497136E-08
3	0.64273665857E-08	-0.59487510396E-07	0.45147316996E-08
4	-0.83442991160E-08	0.18679165254E-06	0.93059497885E-08
5	-0.11067002853E-07	0.14089676045E-06	0.15839759726E-07
6	0.43960983023E-08	-0.47157846861E-07	0.22981112124E-07
7	0.46079767976E-07	-0.34994293571E-06	0.30788214644E-07
8	-0.46776936667E-07	0.37285039037E-06	0.41447492549E-07
9	0.40927636487E-08	-0.19209443135E-07	0.52939867601E-07
10	0.61998234659E-07	-0.33212069083E-06	0.62398612499E-07
11	0.28959021320E-06	-0.13972768631E-05	0.71911927080E-07
12	-0.45219339283E-06	0.22184180677E-05	0.86478394223E-07
13	0.22711591727E-06	-0.99310839715E-06	0.10020448826E-06
14	0.97048395601E-07	-0.34056570586E-06	0.11163683666E-06
15	-0.74678942571E-06	0.28669526852E-05	0.13378121366E-06
16	0.56335676347E-06	-0.20977331591E-05	0.15072146198E-06
17	-0.40877176819E-06	0.13891253109E-05	0.16328704078E-06
18	0.66424686186E-06	-0.21892571347E-05	0.17038837541E-06
19	-0.23180173345E-06	0.76486211739E-06	0.17915408534E-06
20	-0.15186500804E-06	0.45445375184E-06	0.19668186724E-06
21	0.27901017496E-06	-0.78814326746E-06	0.20764309738E-06
22	0.76635097407E-06	-0.20648057751E-05	0.20850166038E-06
23	-0.17863103046E-05	0.47841309528E-05	0.22809763323E-06
24	0.31924620270E-06	-0.87805593201E-06	0.25432018447E-06
25	0.16538629210E-05	-0.41145539368E-05	0.24293876777E-06
26	-0.12766291317E-05	0.31440978928E-05	0.24980181479E-06
27	-0.35793355006E-06	0.81169097785E-06	0.26901398087E-06
28	0.14615012557E-05	-0.33550627957E-05	0.25135341275E-06
29	-0.18538326464E-05	0.41735536966E-05	0.26310954127E-06
30	0.23572582791E-05	-0.51586649736E-05	0.26704765332E-06
31	-0.28504115499E-05	0.61073636789E-05	0.26730049285E-06
32	0.21206145893E-05	-0.44715303057E-05	0.27229089028E-06
33	-0.64698108018E-06	0.13434514541E-05	0.25650751923E-06
34	0.55747928731E-06	-0.11334025390E-05	0.24536075216E-06
35	-0.10969274017E-05	0.21946723913E-05	0.24617384042E-06
36	-0.32130926996E-06	0.61182657252E-06	0.26677662390E-06
37	0.68545543663E-06	-0.13588690658E-05	0.24983910407E-06
38	0.15038994885E-05	-0.28481178322E-05	0.21968662622E-06
39	-0.15744060593E-05	0.29410096701E-05	0.19704930310E-06
40	0.60735658795E-06	-0.10983124328E-05	0.20132392820E-06
41	-0.11038678933E-05	0.19922122131E-05	0.19832623366E-06
42	0.17359014726E-05	-0.31660113165E-05	0.16631474864E-06
43	-0.19883071451E-05	0.35696666179E-05	0.14507349988E-06
44	0.20173751072E-05	-0.36301386149E-05	0.11937481759E-06
45	-0.20609390727E-05	0.36645720160E-05	0.10267331163E-06
46	0.22568194052E-05	-0.40014328361E-05	0.94547885965E-07
47	-0.16601169731E-05	0.29174758918E-05	0.79199708125E-07
48	-0.15050845840E-05	0.26605719123E-05	0.90053390522E-07
49	0.43321427263E-05	-0.77020095101E-05	0.21684513740E-07
50	-0.32892274111E-05	0.58249666746E-05	-0.11591055227E-07
51	-0.49999991927E+01	-0.14412546392E-05	0.18818013814E-07

APPENDIX B

Case II Finite Element Model Input and Results

```
*XQT TAB
START 21 2 4 6$
TITLE'LARGE DEFLECTION OF CANTILEVER BEAM WITH FOLLOWER END FORCE
JLOC$
1 0. 0. 0. 0. 0. 20. 21 1$
MATC$
1 10.+6 .3 .101 1.0-5$
MREF$
1 1 1 1 1.$
BA$
GIVN 1 1.333-4 0. 1.333-4 0. .04 2.256-4$.2X.2 RECT
CONS=1$
ZERO 1 3 5:1$
*XQT ELD
E21$
1 2 1 20$
*XQT E
*XQT EKS
*XQT AUS
ALPHA:1 CASE TITL 4$
1'FORCE FOLLOWS ROTATION OF JOINT 21,(WL**2/EI)=1.604
SYSVEC:2 FOLL FORC:I=1:J=21:-5.3470$(WL**2/EI)=1.604
*XQT VPRT
FORMAT=4$
TPRINT 2 FOLL FORC 1$
*XQT U1
*(GNA OPTIONS)$
!SS=.25$INITIAL STEP SIZE
!SSMIN=.1$MINUMUM STEP SIZE
*CALL(18 NL JCL)$
*CALL(GNA MAIN)$PERFORM NONLINEAR ANALYSIS
*CALL(GNA MOTI)$COMPUTE NONLINEAR DISPLACEMENTS
*CALL(GNA REAC)$COMPUTE NONLINEAR REACTIONS
*CALL(GNA PCH)$PRINT CONVERGENCE HISTORY
*XQT VPRT
FORMAT=4$
TPRINT NONL MOTI 4 1$
TPRINT NONL REAC 4 1$
*XQT DCU
TOC 1$
TOC 2$
*XQT  EXIT
```

```
1NONL    MOTI                                                    ID=    4/ 1/   1
 FORCE FOLLOWS ROTATION OF JOINT 21,(WL**2/EI)=1.604
0JOINT,          1                         3                         5
      1  0.00000000000E+00*  0.00000000000E+00*  0.00000000000E+00*
      2 -0.37157973498E-01  -0.69969773255E-03  -0.73860944922E-01
      3 -0.14646568109E+00  -0.67001513649E-02  -0.14468755475E+00
      4 -0.32427657011E+00  -0.22643220231E-01  -0.21230195271E+00
      5 -0.56654657690E+00  -0.52441447316E-01  -0.27654930043E+00
      6 -0.86894995208E+00  -0.99267856809E-01  -0.33729667836E+00
      7 -0.12269858297E+01  -0.16556585669E+00  -0.39443165037E+00
      8 -0.16360727156E+01  -0.25307563728E+00  -0.44786059467E+00
      9 -0.20916291373E+01  -0.36287334164E+00  -0.49750687929E+00
     10 -0.25891396451E+01  -0.49541942456E+00  -0.54330895366E+00
     11 -0.31242061404E+01  -0.65061294532E+00  -0.58521841939E+00
     12 -0.36925851152E+01  -0.82784898817E+00  -0.62319813381E+00
     13 -0.42902118230E+01  -0.10260769158E+01  -0.65722039046E+00
     14 -0.49132126763E+01  -0.12438576857E+01  -0.68726521161E+00
     15 -0.55579073104E+01  -0.14794189620E+01  -0.71331877963E+00
     16 -0.62208017952E+01  -0.17307072172E+01  -0.73537202717E+00
     17 -0.68985744385E+01  -0.19954364139E+01  -0.75341940002E+00
     18 -0.75880555437E+01  -0.22711332009E+01  -0.76745780185E+00
     19 -0.82862023713E+01  -0.25551788255E+01  -0.77748572680E+00
     20 -0.89900704394E+01  -0.28448481858E+01  -0.78350258303E+00
     21 -0.96967821874E+01  -0.31373466046E+01  -0.78550820899E+00
1NONL    REAC                                                    ID=    4/ 1/   1
 FORCE FOLLOWS ROTATION OF JOINT 21,(WL**2/EI)=1.604
0JOINT,          1                         3                         5
      1  0.37805317898E+01*  0.37812848421E+01*  0.10041585312E+03*
      2 -0.36051339030E-05   0.80082172810E-05   0.10538043625E-04
      3 -0.36758077268E-05   0.57586682639E-05   0.84672487901E-05
      4 -0.35562574175E-05   0.40763069665E-05   0.69430980147E-05
      5 -0.33287210750E-05   0.27880235418E-05   0.58234495555E-05
      6 -0.30727364155E-05   0.18756233026E-05   0.49970376494E-05
      7 -0.27791258141E-05   0.11335455356E-05   0.43781419663E-05
      8 -0.25004520902E-05   0.60761937755E-06   0.39015853872E-05
      9 -0.22134820304E-05   0.18376311476E-06   0.35194223074E-05
     10 -0.19275187515E-05  -0.15286477211E-06   0.31957883948E-05
     11 -0.16780235241E-05  -0.36599938256E-06   0.29054508559E-05
     12 -0.14201456468E-05  -0.55291329558E-06   0.26305763186E-05
     13 -0.11698341180E-05  -0.69208105134E-06   0.23588468139E-05
     14 -0.99904874661E-06  -0.70954232548E-06   0.20832990231E-05
     15 -0.77717252442E-06  -0.78031921873E-06   0.17983695670E-05
     16 -0.64516829895E-06  -0.73406904074E-06   0.15023351807E-05
     17 -0.50128378716E-06  -0.70083995827E-06   0.11957694142E-05
     18 -0.44193721944E-06  -0.57964617921E-06   0.87838986929E-06
     19 -0.31290403859E-06  -0.52649294086E-06   0.55287887335E-06
     20 -0.31154568093E-06  -0.34264307622E-06   0.22326662474E-06
     21 -0.37804968735E+01  -0.37813031365E+01   0.11187353266E-08
```

APPENDIX C

Case III Finite Element Model Input and Results

PRECEDING PAGE BLANK NOT FILMED

PAGE 24 INTENTIONALLY BLANK

```
*XQT TAB
START 21 2 4 6$
TITLE'VERIFY CANT. BEAM WITH UNIFORM DIST. LOAD LARGE DISP.
JLOC$
1 0. 0. 0. 0. 0. 20. 21 1$
MATC$
1 10.+6 .3 .101 1.0-5$
MREF$
1 1 1 1 1.$
BA$
GIVN 1 1.333-4 0. 1.333-4 0. .04 2.256-4$.2X.2 RECT
CONS=1$
ZERO 1 3 5:1$
*XQT ELD
E21$
1 2 1 20$
*XQT E
*XQT EKS
*XQT AUS
ALPHA:1 CASE TITL 4$
1'UNIFORMLY DISTRIBUTED LOAD,(WL**3/EI)=8.7422$
SYSVEC:2 APPL FORC:I=1:J=1:21:-.728335:J=2,20:-1.45667$(WL**3/EI)=8.7422
$SYSVEC:2 APPL FORC:I=1:J=11:-29.1334$(WL**3/EI)=8.7422
*XQT VPRT
FORMAT=4$
TPRINT 2 APPL FORC 1$
*XQT U1
*(GNA OPTIONS)$
!SS=.25$INITIAL STEP SIZE
!SSMIN=.1$MINUMUM STEP SIZE
*CALL(18 NL JCL)$
*CALL(GNA MAIN)$PERFORM NONLINEAR ANALYSIS
*CALL(GNA MOTI)$COMPUTE NONLINEAR DISPLACEMENTS
*CALL(GNA REAC)$COMPUTE NONLINEAR REACTIONS
*CALL(GNA PCH)$PRINT CONVERGENCE HISTORY
*XQT VPRT
FORMAT=4$
TPRINT NONL MOTI 4 1$
TPRINT NONL REAC 4 1$
*XQT DCU
TOC 1$
TOC 2$
*XQT  EXIT
```

1NONL MOTI ID= 4/ 1/ 1

UNIFORMLY DISTRIBUTED LOAD,(WL**3/EI)=8.7422$

0JOINT,	1	3	5
1	0.00000000000E+00*	0.00000000000E+00*	0.00000000000E+00*
2	-0.81538520445E-01	-0.33239993945E-02	-0.15987638662E+00
3	-0.31066211238E+00	-0.29910802471E-01	-0.29929428526E+00
4	-0.66394461155E+00	-0.94370047518E-01	-0.41992736056E+00
5	-0.11195835966E+01	-0.20417463579E+00	-0.52358954518E+00
6	-0.16584046886E+01	-0.36171831143E+00	-0.61209148738E+00
7	-0.22641411169E+01	-0.56601274444E+00	-0.68715456592E+00
8	-0.29233073868E+01	-0.81397242615E+00	-0.75036631988E+00
9	-0.36248825814E+01	-0.11013321818E+01	-0.80316359804E+00
10	-0.43599328755E+01	-0.14232741968E+01	-0.84683307687E+00
11	-0.51212429944E+01	-0.17748413757E+01	-0.88252187638E+00
12	-0.59029891089E+01	-0.21512011733E+01	-0.91125342951E+00
13	-0.67004644591E+01	-0.25478087340E+01	-0.93394551116E+00
14	-0.75098579908E+01	-0.29605044488E+01	-0.95142852387E+00
15	-0.83280812514E+01	-0.33855701642E+01	-0.96446291227E+00
16	-0.91526369711E+01	-0.38197602179E+01	-0.97375506377E+00
17	-0.99815225441E+01	-0.42603177514E+01	-0.97997134508E+00
18	-0.10813162069E+02	-0.47049828067E+01	-0.98375009348E+00
19	-0.11646361236E+02	-0.51519960813E+01	-0.98571147435E+00
20	-0.12480279932E+02	-0.56001005123E+01	-0.98646516492E+00
21	-0.13314417842E+02	-0.60485418247E+01	-0.98661584764E+00

1NONL REAC ID= 4/ 1/ 1

UNIFORMLY DISTRIBUTED LOAD,(WL**3/EI)=8.7422$

0JOINT,	1	3	5
1	0.28405064739E+02*	0.37493751393E-06*	0.22727046458E+03*
2	-0.14566690868E+01	0.40573603144E-07	0.24162636691E-06
3	-0.14566693891E+01	0.17144086151E-06	0.10343565009E-05
4	-0.14566696583E+01	0.18945757674E-06	).15467057892E-05
5	-0.14566698756E+01	0.17425380250E-06	0.18266136976E-05
6	-0.14566700055E+01	0.92202856905E-07	0.19306394279E-05
7	-0.14566701042E+01	0.40793789680E-07	0.19090766727E-05
8	-0.14566701539E+01	-0.23910054381E-07	0.18009495761E-05
9	-0.14566701822E+01	-0.70063922664E-07	0.16351889371E-05
10	-0.14566701849E+01	-0.10873932155E-06	0.14357237887E-05
11	-0.14566701863E+01	-0.12625688228E-06	0.12203508959E-05
12	-0.14566701684E+01	-0.14065609122E-06	0.10027918051E-05
13	-0.14566701445E+01	-0.14568882634E-06	0.79395965713E-06
14	-0.14566701346E+01	-0.12832246199E-06	0.60264358126E-06
15	-0.14566701210E+01	-0.11230199704E-06	0.43350303258E-06
16	-0.14566701227E+01	-0.75494916985E-07	0.29091620490E-06
17	-0.14566700643E+01	-0.82225238174E-07	0.17748456571E-06
18	-0.14566700444E+01	-0.60133000801E-07	0.92594305556E-07
19	-0.14566700701E+01	-0.81951867514E-08	0.36932846115E-07
20	-0.14566700385E+01	0.66707173119E-09	0.81222477633E-08
21	-0.72833500346E+00	-0.23405597460E-08	-0.33626434970E-10

APPROVAL

A CASE STUDY OF ANALYSIS METHODS FOR LARGE DEFLECTIONS OF A CANTILEVER BEAM

By L.D. Craig

The information in this report has been reviewed for technical content. Review of any information concerning Department of Defense or nuclear energy activities or programs has been made by the MSFC Security Classification Officer. This report, in its entirety, has been determined to be unclassified.

J.C. Blair

J.C. BLAIR
Director, Structures and Dynamics Laboratory

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