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Renormalization Group Theory of Bolgiano Scaling in Boussinesq Turbulence

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Abstract

Bolgiano scaling in Boussinesq turbulence is analyzed using the Yakhot-Orszag renormalization group. For this purpose, an isotropic model is introduced. Scaling exponents are calculated by forcing the temperature equation so that the temperature variance flux is constant in the inertial range. Universal amplitudes associated with the scaling laws are computed by expanding about a logarithmic theory. Connections between this formalism and the direct interaction approximation are discussed. It is suggested that the Yakhot-Orszag theory yields a lowest order approximate solution of a regularized direct interaction approximation which can be corrected by a simple iterative procedure.

I. Introduction

Application of the Kolmogorov scaling theory of hydrodynamic turbulence to Boussinesq turbulence suggests two regimes. The simpler possibility is that the gravitational coupling can be neglected. Then the temperature is a passive scalar and the velocity and temperature fluctuations have Kolmogorov spectra

$$E \sim \varepsilon^{2/3} k^{-5/3}$$
$$E_T \sim \bar{N} \varepsilon^{-1/3} k^{-5/3}$$

where ε is the kinetic energy dissipation rate, E_T is the spectrum for mean square temperature fluctuations, and $\bar{N}$ is the corresponding dissipation rate. In the passive regime, the correlation $< u_i' T' >$ vanishes by symmetry, and the corresponding spectrum E_H is zero.

Bolgiano[1] identified the second possibility, that velocity and temperature fluctuations are determined by $\bar{N}$ and g. Then dimensional analysis leads to

$$E \sim g^{4/5}\bar{N}^{2/5}k^{-11/5}$$

$$E_H \sim g^{1/5}\bar{N}^{3/5}k^{-9/5}$$

$$E_T \sim g^{-2/5}\bar{N}^{4/5}k^{-7/5}$$

The velocity spectrum falls off more quickly at small scales than a Kolmogorov spectrum; this rapid decay was understood to reflect the conversion of kinetic to potential energy under stable stratification, to which this scaling was assumed to apply exclusively.[2]

It has been suggested[3] that measurements[4] in very high Rayleigh number Rayleigh-Benard convection experiments are consistent with Bolgiano scaling. Although the evidence is not conclusive, this motivates the study of Bolgiano scaling in unstably stratified flow. L'vov and Falkovich have observed[5] that whereas Kolmogorov scaling corresponds to constant energy flux ε, Bolgiano scaling corresponds to constant $\bar{N}$, which can be identified[3] with entropy flux. Their analysis also supports the suggestion[1,2] that Boussinesq turbulence typically exhibits both regimes, with Bolgiano scaling dominant at large scales and Kolmogorov scaling dominant at small scales.

Bolgiano scaling will be considered here from the viewpoint of the Yakhot-Orszag (YO) theory.[6] It will be shown to arise from forcing of the temperature equation alone. A similar calculation for MHD has been made by Fournier et al.[7] However, in this case forcing of the magnetic field equation leads to a Kolmogorov spectrum for the case of constant magnetic energy flux. This apparently occurs because the coupling constant is dimensionless in MHD but dimensional in Boussinesq turbulence.

II. Langevin models for Boussinesq turbulence

The Boussinesq equations in Fourier space are

$$(-i\omega + \nu k^2)u_i = -\frac{1}{2}iP_{imn}(\mathbf{k})\int d\hat{p}\; u_m(\hat{k}-\hat{p})u_n(\hat{p}) + gP_{i3}(\mathbf{k})T(\hat{k})$$

$$(-i\omega + \kappa k^2)T = -\frac{1}{2}ik_p\int d\hat{p}\left[u_p(\hat{k}-\hat{p})T(\hat{p}) + u_p(\hat{p})T(\hat{k}-\hat{p})\right] \tag{1}$$

In Eq. (1), the standard notation,

$$\hat{k} = (\omega, \mathbf{k}) = (\omega, k_1, k_2, k_3)$$
$$P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j / k^2$$
$$P_{imn}(\mathbf{k}) = k_m P_{in}(\mathbf{k}) + k_n P_{im}(\mathbf{k})$$

is used. The buoyancy term coupling u and T is direction dependent. However, if Bolgiano scaling can arise independently of stratification, this dependence should be considered characteristic of the production mechanism only and as absent or irrelevant in the inertial range. Then it is not unreasonable to investigate an isotropic theory by replacing T in Eq. (1) by a convected vector field b_i. Then

$$(-i\omega + \nu k^2) u_i = -\frac{1}{2} i P_{imn}(\mathbf{k}) \int d\hat{p}\; u_m(\hat{k} - \hat{p}) u_n(\hat{p}) + g P_{ip}(\mathbf{k}) b_p$$

$$(-i\omega + \kappa k^2) b_i = -\frac{1}{2} i k_p \int d\hat{p} \big[u_p(\hat{k} - \hat{p}) b_i(\hat{p}) + u_p(\hat{p}) b_i(\hat{k} - \hat{p}) \big] \tag{2}$$

L'vov[3] suggests that the *locality* of inertial range interactions, expressed analytically as the finiteness of the DIA integral expression for $\bar{N}$ developed later in Sect. III, supports the possibility of Bolgiano scaling in unstably stratified flow: locality implies that inertial range scales interact among themselves and not with production range scales. This property of locality might also be invoked to justify Eq. (2) as a model of Boussinesq turbulence. Nevertheless, some caution in replacing the Boussinesq equations Eq. (1) by the isotropic model Eq. (2) is required, because vanishing of the correlation $< u_i' T' >$ in isotropic turbulence is a kinematic requirement for Eq. (1), whereas the correlation $< u_i' b_i' >$ in Eq. (2) need not vanish in this case. Accordingly, we leave more complete consideration of the relation between true Boussinesq turbulence and the isotropic model Eq. (2) for later investigations.

The direct interaction approximation (DIA) for Boussinesq turbulence has been formulated by Kraichnan[8]. DIA for Eq. (2) can be written as a system of generalized Langevin equations[9]

$$\dot{u}_i(\mathbf{k}, t) = \int_0^t ds\; \eta(\mathbf{k}, t, s) u_i(\mathbf{k}, s) + f_i^u(\mathbf{k}, t) + g b_i(\mathbf{k}, t)$$

$$\dot{b}_i(\mathbf{k}, t) = \int_0^t ds\; \eta^b(\mathbf{k}, t, s) b_i(\mathbf{k}, s) + \int_0^t ds\; \eta^{bu}(\mathbf{k}, t, s) u_i(\mathbf{k}, s) + f_i^b(\mathbf{k}, t) \tag{3}$$

where the damping functions are defined by

$$\begin{aligned}
\eta(\mathbf{k},t,s) &= \frac{1}{4}\int_{\mathbf{k}=\mathbf{p}+\mathbf{q}} d\mathbf{p}\ d\mathbf{q}\ P_{rmn}(\mathbf{k})P_{mrs}(\mathbf{p})P_{ns}(\mathbf{q})G(p,t,s)Q^{u}(q,t,s) \\
\eta^{b}(\mathbf{k},t,s) &= \frac{1}{4}\int_{\mathbf{k}=\mathbf{p}+\mathbf{q}} d\mathbf{p}\ d\mathbf{q}\ k_p q_l P_{lp}(\mathbf{p})G^{b}(p,t,s)Q^{u}(q,t,s) \\
\eta^{bu}(\mathbf{k},t,s) &= \frac{1}{4}\int_{\mathbf{k}=\mathbf{p}+\mathbf{q}} d\mathbf{p}\ d\mathbf{q}\ \{k_p P_{prs}(\mathbf{p})G(p,t,s)Q^{h}_{sr}(q,t,s) \\
&\quad + k_p q_r G^{b}(q,t,s)Q^{h}_{pr}(p,t,s)\}
\end{aligned} \tag{4}$$

and the force correlations are

$$\begin{aligned}
< f_i^u f_j^u > &= \frac{1}{4}\int_{\mathbf{k}=\mathbf{p}+\mathbf{q}} d\mathbf{p}\ d\mathbf{q}\ P_{imn}(\mathbf{k})P_{jrs}(\mathbf{k})P_{mr}(\mathbf{p})P_{ns}(\mathbf{q})Q^{u}(p,t,s)Q^{u}(q,t,s) \\
< f_i^u f_j^b > &= \frac{1}{4}\int_{\mathbf{k}=\mathbf{p}+\mathbf{q}} d\mathbf{p}\ d\mathbf{q}\ P_{imn}(\mathbf{k})k_p\ \{P_{mp}(\mathbf{p})Q^{u}(p,t,s)Q^{h}_{nj}(q,t,s) \\
&\quad + P_{np}(\mathbf{q})Q^{h}_{mj}(p,t,s)Q^{u}(q,t,s)\} \\
< f_i^b f_j^b > &= \frac{1}{4}\int_{\mathbf{k}=\mathbf{p}+\mathbf{q}} d\mathbf{p}\ d\mathbf{q}\ \{k_p k_l P_{lp}(\mathbf{p})Q^{u}(p,t,s)Q^{b}(q,t,s) \\
&\quad + k_p k_l Q^{h}_{lr}(p,t,s)Q^{h}_{pr}(q,t,s)\}
\end{aligned} \tag{5}$$

The scalar Green's functions G and G^b in Eq. (4) are defined as is usual in DIA by

$$\begin{aligned}
G &= \frac{1}{d-1}\frac{\delta u_i}{\delta f_j^u}P_{ij} \\
G^b &= \frac{1}{d}\frac{\delta b_i}{\delta f_j^b}\delta_{ij}
\end{aligned}$$

In these equations, the correlation functions are defined by

$$\begin{aligned}
< u_i(t)u_j(s) > &= \int d\mathbf{p}\ Q^{u}_{ij}(\mathbf{p},t,s) \\
< u_i(t)b_j(s) > &= \int d\mathbf{p}\ Q^{h}_{ij}(\mathbf{p},t,s) \\
< b_i(t)b_j(s) > &= \int d\mathbf{p}\ Q^{b}_{ij}(\mathbf{p},t,s)
\end{aligned}$$

Isotropy implies that

$$\begin{aligned}
Q^{u}_{ij}(\mathbf{p},t,s) &= Q^{u}(p,t,s)P_{ij}(\mathbf{p}) \\
Q^{b}_{ij}(\mathbf{p},t,s) &= Q^{b}(p,t,s)\delta_{ij}
\end{aligned}$$

The YO theory can be understood as a solution of these equations in a special limiting case. First, the geometric coefficients are all evaluated in the *distant interaction limit* k/p and $k/q \to 0$. Second, the triangle integrals are all *regularized* by restricting the integration to the region in which either $p \geq k$ or $q \geq k$. Finally, the time dependence is also evaluated in the distant interaction limit. Namely, assuming arbitrary scaling in the inertial range $G(p,t,s) = G(ap^r(t-s))$ for some positive exponent r and suitable dimensional parameter a, the similarity form for G satisfies

$$G(0) = 1$$

$$G(\tau) < 1 \text{ for } \tau > 0$$

$$\int_0^\infty G(\tau)d\tau < \infty$$

therefore

$$p^r G(p^r(t-s)) \sim \delta(t-s) \text{ for } p \to \infty$$

by standard properties[10] of delta functions. In Eqs. (4) and (5), this same argument is applied to products of Green's functions and time correlation functions in the distant interaction limit. Therefore, in this limit, the damping is Markovian and the forcing is white noise in time. Fluctuation dissipation relations relating the correlation and response functions are valid in this limit.

In Bolgiano scaling, Eq. (5) implies that the forcings satisfy the scaling relations

$$\begin{aligned} &< f^u f^u > \ \sim \ < u_i b_i > \ \sim \ k^{-19/5} \\ &< f^b f^b > \ \sim k^{-3} \end{aligned} \tag{6}$$

The velocity force spectrum is steeper than the k^{-3} force corresponding to a Kolmogorov spectrum;[5] accordingly, Kolmogorov scaling will apply at sufficiently small scales. The *Bolgiano scale* k_B can then be understood as the scale beyond which Kolmogorov forcing dominates the velocity equation. Thus,

$$\epsilon k_B^{-3} \sim g^{6/5} \bar{N}^{3/5} k_B^{-19/5}$$

Since in view of Eq. (6) the DIA velocity force scales like the cross correlation in Bolgiano scaling, it is reasonable to assume that the forcing of the velocity equation is due entirely to the coupling to the b field. With these assumptions, DIA reduces to the "correspondence principle" of the Yakhot-Orszag theory[6] for Bolgiano scaling in Boussinesq turbulence, namely that the Bolgiano scaling regime is described by the noninear Langevin model

$$(-i\omega + \nu k^2)u_i = -\frac{1}{2} i P_{imn}(\mathbf{k}) \int d\hat{p}\; u_m(\hat{k}-\hat{p})u_n(\hat{p}) + gP_{ip}(\mathbf{k})b_p$$

$$(-i\omega + \kappa k^2)b_i = -\frac{1}{2} i k_p \int d\hat{p}\big[u_p(\hat{k}-\hat{p})b_i(\hat{p}) + u_p(\hat{p})b_i(\hat{k}-\hat{p})\big] - \gamma k^2 u_i + f_i \qquad (7)$$

in which the force correlation is

$$< f_i(\hat{k})f_j(\hat{k}') > = D\delta(\hat{k}+\hat{k}')\delta_{ij}k^{-y} \qquad (8)$$

and an arbitrary scaling exponent replaces the value $y = d$ appropriate for Bolgiano scaling.[6]

The additional transport coefficient γ corresponds to the damping function η^{bu} in the DIA Langevin model. This added term does not affect the RG calculations of ν and κ. The physical case, defined so that the stirring coefficient D has the same units as $\bar{N}$, corresponds to $y = d$ and $\epsilon = 8$. Although the damping terms have been replaced by the original nonlinear terms of the equations of motion, their effect will be evaluated perturbatively in the same "one loop" approximation used in the DIA Eq. (4).

Suppose that this procedure were applied, not to the model equation (2) for the vector field b, but to the original equation (1) for T. Then it is easy to check that the corrections to viscosity and diffusivity would be tensors. However, the consistency of this calculation is uncertain. For as inspection of the DIA Langevin equations shows, anisotropic damping and correlations must correspond to an anisotropic force correlation with nonzero velocity and cross correlations. Only in the case of an isotropic theory do symmetry conditions dictate the form of the force correlation. But it is also easily verified that the isotropic part of the viscosity and diffusivity corrections computed in this manner would coincide with the results of the present calculation.

The RG calculations follow as in Ref. 6. They lead to recurrences

$$\frac{d\nu}{dk} = \frac{1}{4} g^2 \mathbf{D} k^{-\epsilon-1}\nu^{-4}\frac{2}{d(d+2)}\left[A(a)(d^2-d+6-\epsilon) + B(a) - C(a)\right] \qquad (9)$$

$$\frac{d\kappa}{dk} = \frac{1}{2}(1-\frac{1}{d})g^2\mathbf{D}k^{-\epsilon-1}\nu^{-4}D(a) \qquad (10)$$

in which

$$\epsilon = 8 + y - d$$

$$a = \kappa/\nu$$

$$\mathbf{D} = 2\pi D S_d$$

As in the YO analysis of the passive scalar, the coefficients $A, \cdots, D$ are rational functions of a:

$$\begin{aligned}
A(a) &= \frac{a+2}{4a(a+1)^2} \\
B(a) &= \frac{a^2+3a+4}{4a(a+1)^3} \\
C(a) &= \frac{7a^2+21a+12}{4a(a+1)^3} \\
D(a) &= \frac{2a+1}{4a^2(a+1)^2}
\end{aligned} \tag{11}$$

Solving the recurrence relation for ν when $d = 3$,

$$\frac{1}{5}(\nu^5 - \nu_B^5) = \frac{1}{4\epsilon}\frac{2}{15}g^2\mathbf{D}[A(a)(d^2 - d + 6 - \epsilon) + B(a) - C(a)](k^{-\epsilon} - k_B^{-\epsilon})$$

The initial conditions are $\nu = \nu_B$ when $k = k_B$ correspond to the highest wavenumber governed by Bolgiano scaling. The effective viscosity ν_B will be given by Kolmogorov scaling and not in general by the molecular viscosity.

In the YO theory, an expansion is made about the logarithmic theory $\epsilon = 0$ recognizing that evaluation of integrals beyond the one loop level will introduce further corrections of order ϵ^n with $n \geq 1$ to the constants. At lowest order, consistency demands that all such corrections be dropped, including those already computed in the present one loop approximation. Dropping these corrections and setting $\epsilon = 8$,

$$\nu = \frac{1}{2}\left[\frac{2}{3}g^2\mathbf{A}\mathbf{D}\right]^{1/5}k^{-8/5}$$

where

$$\mathbf{A} = A(a)(d^2 - d + 6) + B(a) - C(a)$$

The quantity a, an inverse Prandtl number, is evaluated by assuming it is scale independent and can be evaluated by setting the viscosity ν proportional to the conductivity κ in the inertial range. This leads to a quartic equation for a

$$3a^4 + 9a^3 - 2a^2 - 15a - 5 = 0 \tag{12}$$

which has only one positive root

$$a = 1.291 \tag{13}$$

Then from Eq.,

$$\mathbf{A} = .7929$$

It must be stressed that in this procedure, ϵ *is never set to any value other than* 8. These results could also be derived directly from the simplified DIA Langevin model without introducing recurrence relations following Woodruff.[11] By regularizing the triangle integrals, crossover to a dominant sweeping interaction[11] when $\epsilon = 7$ is suppressed.

The inverse Prandtl number of Eq. (13) is very close to the value 1.3929 obtained for forced convection by Yakhot and Orszag[6] (compare also Ref. 7). The close proximity of these numbers is consistent with a common assumption[12] that the turbulent Prandtl number is independent of Richardson number in the atmospheric surface layer. Although experimental evidence does not support this assumption unambiguously,[12] the theoretical value of Eq. (13) is within the range $1.00 < a < 1.43$ observed in some recent measurements of buoyant plumes.[13]

Spectra are defined as usual by

$$E_u(p) = \frac{1}{2}\oint d\mathbf{p} \int_{-\infty}^{\infty} d\omega \; Q^u_{ii}(\hat{p})$$

$$E_h(p) = \frac{1}{2}\oint d\mathbf{p} \int_{-\infty}^{\infty} d\omega \; Q^h_{ii}(\hat{p})$$

$$E_b(p) = \frac{1}{2}\oint d\mathbf{p} \int_{-\infty}^{\infty} d\omega \; Q^b_{ii}(\hat{p})$$

and evaluated at lowest order in perturbation theory as

$$\begin{aligned}
E_u &= \frac{d-1}{2}\oint d\mathbf{p}\int_{-\infty}^{\infty} d\omega \;\; | G(\hat{p}) |^2 | G_b(\hat{p}) |^2 \, g^2 D p^{-y} \\
&= \frac{8}{2a(a+1)}(\frac{2}{3}\mathbf{A})^{-3/5} g^{4/5}\mathbf{D}^{2/5} p^{-11/5} \\
E_h &= \frac{d-1}{2}\oint d\mathbf{p}\int_{-\infty}^{\infty} d\omega \; G(\hat{p}) \, | G_b(\hat{p}) |^2 \, g D p^{-y} \\
&= \frac{4}{2a(a+1)}(\frac{2}{3}\mathbf{A})^{-2/5} g^{1/5}\mathbf{D}^{3/5} p^{-9/5} \\
E_b &= \frac{d}{2}\oint d\mathbf{p}\int_{-\infty}^{\infty} d\omega \;\; | G_b(\hat{p}) |^2 \, D p^{-y} \\
&= \frac{1}{a}(\frac{2}{3}\mathbf{A})^{-1/5} g^{-2/5}\mathbf{D}^{4/5} p^{-7/5}
\end{aligned}$$

Then

$$
\begin{aligned}
\nu(k) &= a_\nu g^{2/5}\mathbf{D}^{1/5}k^{-8/5} \\
E_u(k) &= a_u g^{4/5}\mathbf{D}^{2/5}k^{-11/5} \\
E_h(k) &= a_h g^{1/5}\mathbf{D}^{3/5}k^{-9/5} \\
E_b(k) &= a_b g^{-2/5}\mathbf{D}^{4/5}k^{-7/5}
\end{aligned}
\tag{14}
$$

where

$$
\begin{aligned}
a_\nu &= 0.440 \\
a_u &= 1.310 \\
a_h &= 0.576 \\
a_b &= 0.679
\end{aligned}
$$

Introducing an inverse integral scale k_f so that the energy is contained entirely in the inertial range $k_f \leq k < \infty$, the single point turbulence quantities are

$$
\begin{aligned}
K &= \frac{1}{2} < u_p u_p > = \frac{5}{6} a_u g^{4/5}\mathbf{D}^{2/5}k_f^{-6/5} \\
H &= \frac{1}{2} < u_p b_p > = \frac{5}{4} a_h g^{1/5}\mathbf{D}^{3/5}k_f^{-4/5} \\
C &= \frac{1}{2} < b_p b_p > = \frac{5}{2} a_b g^{-2/5}\mathbf{D}^{4/5}k_f^{-2/5} \\
\nu_T &= \nu(k_f) = a_\nu g^{1/5}\mathbf{D}^{2/5}k_f^{-8/5}
\end{aligned}
\tag{15}
$$

In these equations, corrections due to a Kolmogorov regime at small scales have been ignored.

III. Elimination of the force amplitude

The expressions for spectra and single point turbulence quantities developed above all contain the quantity $\mathbf{D}$. In the YO theory, this quantity is expressed in terms of the dissipation rate $\bar{N}$ through the DIA inertial range energy balance,[14]

$$
0 = ik_p \int_{\mathbf{k}=\mathbf{p}+\mathbf{q}} d\mathbf{p}d\mathbf{q} < \left[u_p(\mathbf{p},t)b_i(\mathbf{q},t) + u_p(\mathbf{q},t)b_i(\mathbf{p},t)\right] b_i(-\mathbf{k},t) >
$$

Assuming a steady state, let

$$S(\mathbf{k},\mathbf{p},\mathbf{q}) = ik_p < \left[u_p(\mathbf{p},t)b_i(\mathbf{q},t) + u_p(\mathbf{q},t)b_i(\mathbf{p},t)\right] b_i(-\mathbf{k},t) >$$

Integrating the DIA energy balance over all time separations,

$$\begin{aligned} S(\mathbf{k},\mathbf{p},\mathbf{q}) = \frac{1}{4}\{ & \; 2\Theta(k,p,q)k_p P_{pmn}(\mathbf{p})P_{mi}(\mathbf{k})P_{ni}(\mathbf{q})Q^h(k)Q^h(q) \\ & + \Theta(k,p,q)\left[k_p q_l P_{li}(\mathbf{k})P_{pi}(\mathbf{p})Q^h(k)Q^h(p) - k_p k_l P_{pi}(\mathbf{p})P_{li}(\mathbf{q})Q^h(p)Q^h(q)\right] \\ & + \Theta(k,p,q)\left[k_p q_l P_{lp}(\mathbf{p})\delta_{ii}(\mathbf{k})Q^u(p)Q^b(k) - k_p k_l P_{pl}(\mathbf{p})\delta_{ii}(\mathbf{q})Q^u(p)Q^b(q)\right] \; \} \\ & + (p,q) \end{aligned} \tag{16}$$

where (p,q) denotes interchange of wavenumbers p and q and the time scale Θ is evaluated using fluctuation dissipation relations, valid in the present case of Markovian damping and white noise forcing,

$$\begin{aligned} < u_i(t)b_j(s) > &= \int dsG(k,t,s) < u_i(s)b_j(s) > \\ < u_i(s)b_j(t) > &= \int dsG^b(k,t,s) < u_i(s)b_j(s) > \end{aligned}$$

for $t > s$. The result is

$$\Theta(k,p,q) = \int_0^\infty d\sigma \; G^u(p,\sigma)G^b(k,\sigma)G^b(q,\sigma)$$

Following Kraichnan,[14]

$$\bar{N} = \int_{k\le k_0} d\mathbf{k} \int_{p,q\ge k_0} d\mathbf{p} \; d\mathbf{q} \; S(\mathbf{k},\mathbf{p},\mathbf{q}) - \int_{2k_0\ge k\ge k_0} d\mathbf{k} \int_{p,q\le k_0} d\mathbf{p} \; d\mathbf{q} \; S(\mathbf{k},\mathbf{p},\mathbf{q}) \tag{17}$$

must be finite and independent of k_0 when inertial range scalings are substituted in S. The triangle condition $\mathbf{k} = \mathbf{p} + \mathbf{q}$ is understood in the integrals. Convergence must be checked in the cases $p \to 0$ and $q \to 0$. We find

$$\lim_{q\to 0} \oint d\mathbf{q} \; S(\mathbf{k},\mathbf{p},\mathbf{q}) = O(q^4)Q(q)$$

therefore

$$\int_{q\sim 0} S(\mathbf{k},\mathbf{p},\mathbf{q}) \sim \int dq \; q^{2-\alpha} \; < \infty$$

because in all integrals, $\alpha < 3$.

To evaluate the flux integral, define the quantity $T(k,p,q)$ by

$$\int dp\; dq\; T(k,p,q) = \int_{\mathbf{k}=\mathbf{p}+\mathbf{q}} d\mathbf{p}\; d\mathbf{q}\; S(\mathbf{k},\mathbf{p},\mathbf{q})$$

The scalings are such that

$$T(\lambda k, \lambda p, \lambda q) = \lambda^{-3} T(k,p,q)$$

a result equivalent to the choice of a k^{-3} force. This implies[15] that the inertial range transfer does not depend on the wavenumber k_0 and permits the transfer integral to be reduced to a double integral over a finite region following Kraichnan.[14] Numerical integration of Eqs. (16)-(17) leads to

$$\bar{N} = 4.01\mathbf{D}$$

This result can be used following YO to eliminate the force amplitude $\mathbf{D}$ from Eq. (14) for the spectra and Eq. (15) for single point quantities. The results for the scale dependent viscosity and spectra are

$$\begin{aligned} \nu(k) &= 0.333 g^{2/5} \bar{N}^{1/5} k^{-8/5} \\ E_u(k) &= 0.752 g^{4/5} \bar{N}^{2/5} k^{-11/5} \\ E_h(k) &= 0.250 g^{1/5} \bar{N}^{3/5} k^{-9/5} \\ E_b(k) &= 0.336 g^{-2/5} \bar{N}^{4/5} k^{-7/5} \end{aligned} \tag{18}$$

For the single point turbulence quantities:

$$\begin{aligned} K &= 0.627 g^{4/5} \bar{N}^{2/5} k_f^{-6/5} \\ H &= 0.313 g^{1/5} \bar{N}^{3/5} k_f^{-4/5} \\ C &= 0.840 g^{-2/5} \bar{N}^{4/5} k_f^{-2/5} \end{aligned} \tag{19}$$

These results lead to a formula for the turbulent viscosity in terms of single point quantities,

$$\nu_T = 0.633 \frac{KC}{\bar{N}} \tag{20}$$

which can be applied in transport models. In problems in which the heat transfer H is constant, the relations

$$H^2 = 0.186KC$$
$$\bar{N} = 12.8gK^{-2}H^3 = 1.03gC^{3/2}K^{-1/2}$$

may also be useful.

IV. The transport coefficient γ

The transport coefficient γ can be computed after ν and κ because it does not enter the recurrence relations for these quantities. We find

$$\begin{aligned}\frac{d\gamma}{dp} = \frac{1}{4}\frac{g\mathbf{D}p^{d-1}}{\nu^3p^{6+y}}\frac{1}{d(d+2)}\Big\{&\frac{7a^2+4a+1}{4a^2(a+1)^3} \\ &+ \frac{1}{4a^2(a+1)^3}[7a^2+4a+1-(3a+1)(a+1)(6+y)] \\ &+ d\frac{3a+1}{4a^2(a+1)} + \frac{1}{2a(a+1)^2} + \frac{2}{a(a+1)^3} \\ &- \frac{1}{2a(a+1)^3}[(a+1)(y+6)-2] + (d+2)\frac{1}{2a(a+1)^2}\Big\}\end{aligned}$$

Note that γ does not scale like ν and κ; instead, using $\nu \sim p^{-\epsilon/5}$, we find

$$\gamma \sim p^{3\epsilon/5-(\epsilon-2)}$$

indicating that γ is irrelevant for $\epsilon < 5$. Accordingly, to evaluate the amplitude for γ, it is appropriate to consider ν and a as known, but to expand otherwise about $\epsilon = 5$, or $y = 0$. This procedure leads to

$$\gamma = 0.00263g^{-1/5}\bar{N}^{2/5}k^{-6/5}$$

It appears that this quantity can be neglected for practical purposes.

V. Iterative solution of the DIA Langevin equations

Let us compare the present calculation using the YO theory with the original DIA Langevin equation. The YO "correspondence principle" can readily be understood in terms of this Langevin model. The necessity of a stirring force is immediate from DIA, and the

k^{-3} scaling is equivalent to Orszag's transparency condition[15] on the inertial range. The YO theory begins by generalizing this model: the DIA response equation is satisfied by a family of scaling solutions depending on the force scaling exponent y. Then the theory computes perturbatively about the logarithmic (Gaussian, asymptotically free) theory in which $\epsilon = 8 + y - d = 0$. DIA does not suggest this procedure since only $\epsilon = 8$ is consistent with the DIA response equation. However, if it is agreed that this procedure is appropriate, the calculation of amplitudes in the physical case is consistent: in this calculation, ϵ is never set to any value other than 8. The numerical accuracy of this procedure remains uncertain.

The stirring force amplitude is eliminated by appeal to the full DIA response equation which is neither regularized nor evaluated in the distant interaction limit. However, even this calculation is incomplete because the time dependence has been calculated assuming Markovian damping acting against a white noise in time forcing. Finally, the viewpoint of DIA, the correspondence principle for Boussinesq turbulence was incomplete because it assumed that f^u could be neglected in comparison to $< ub >$. The consistency of this assumption has not been demonstrated.

All of these considerations suggest that we have only computed a first approximation to DIA. In abbreviated notation, this solution is

$$
\begin{aligned}
G_1(k,\sigma) &\sim \exp(-\sigma k^{2/5}) \\
G_1^b(k,\sigma) &\sim \exp(-\sigma k^{2/5}) \\
Q_1^u(k,\sigma) &\sim k^{-21/5} G_1(k,\sigma) \\
Q_1^h(k,\sigma) &\sim k^{-19/5} G_1(k,\sigma) \text{ for } \sigma > 0 \\
Q_1^h(k,\sigma) &\sim k^{-19/5} G_1^b(k,\sigma) \text{ for } \sigma < 0 \\
Q_1^b(k,\sigma) &\sim k^{-17/5} G_1^b(k,\sigma)
\end{aligned}
$$

where time stationarity has been assumed and $\sigma = t - s$. The DIA Langevin model can be used to generate corrected damping and forcing:

$$\eta_1(k,\sigma) \sim \int G_1 Q_1^u$$

$$\eta_1^b(k,\sigma) \sim \int G_1^b Q_1^u$$

$$\eta_1^{bu}(k,\sigma) \sim \int G_1 Q_1^h$$

$$F_1^{uu} =< f_1^u f_1^u > \sim Q_1^u Q_1^u$$

$$F_1^{ub} =< f_1^u f_1^b > \sim Q_1^u Q_1^h$$

$$F_1^{bb} =< f_1^b f_1^b > \sim Q_1^h Q_1^h$$

With these corrected dampings and forcings, new response functions and correlation functions $G_2, G_2^b, Q_2^u, Q_2^h, Q_2^b$ can be calculated assuming that the damping and forcing are known:

$$\dot{G}_2 + \eta_1 * G_2 = 0$$

$$\dot{G}_2^b + \eta_1^b * G_2^b + \eta_1^{bu} * G_2^b = 0$$

$$\dot{Q}_2^u + \eta_1 * Q_2^u = G_1 * F_1^{uu}$$

$$\dot{Q}_2^b + \eta_1^b * Q_2^b + \eta_1^{bu} * Q_2^u = G_1^b * F_1^{bb}$$

This iterative procedure answers the objections raised above. At the second iteration, the distant interaction limit will be corrected,the Markovian time dependence will be modified (hence the fluctuation dissipation relation will not be satisfied) and the complete DIA forcing will be computed. Regularization of the response equation cannot be avoided in this kind of theory. This procedure introduces an arbitrary parameter into the theory since we can also regularize by requiring $p > ak$ for any positive constant a.[16] We have followed YO by setting $a = 1$.

APPENDIX: Integrals in calculation of renormalized transport coefficients

The following notation is used:

$$G(\hat{k}) = (-i\omega + \nu k^2)^{-1}$$

$$G_b(\hat{k}) = (-i\omega + \kappa k^2)^{-1}$$

Viscosity corrections are calculated from

$$k^2\frac{d\nu}{dp} = -\frac{1}{2}iP_{imn}(\mathbf{k})\int_{-\infty}^{\infty} d\omega \oint d\mathbf{p}$$

$$(-\frac{1}{2}iG(\hat{k}-\hat{p})P_{mrs}(\mathbf{k}-\mathbf{p})g^2P_{sn}(\mathbf{p})G(\hat{p})G(-\hat{p})G_b(\hat{p})G_b(-\hat{p})p^{-y}$$

$$(-\frac{1}{2}iG(\hat{p})P_{nrs}(\mathbf{p})g^2P_{ms}(\mathbf{k}-\mathbf{p})G(\hat{p}-\hat{k})G(\hat{k}-\hat{p})G_b(\hat{p}-\hat{k})G_b(\hat{k}-\hat{p}) \mid \mathbf{k}-\mathbf{p}\mid^{-y}$$

where $\oint d\mathbf{p}$ denotes integration over a $d-1$ dimensional sphere of radius p. The integrals are evaluated in the limit $k/p \to 0$. The frequency integrations in this limit are

$$\int_{-\infty}^{\infty} d\omega\ G(\hat{k}-\hat{p})G(\hat{p})G(-\hat{p})G_b(\hat{p})G_b(-\hat{p}) = \frac{2\pi}{\nu^4p^8}\frac{1}{4a(a+1)^3}\times$$
$$\{(a+2)(a+1)+k_\nu p_\nu p^{-2}(a^2+3a+2)\}$$

and the second frequency integral can be found by replacing $\mathbf{p}$ by $\mathbf{k}-\mathbf{p}$ in this result.

Similarly,

$$k^2\frac{d\kappa}{dp} = -\frac{1}{4}k_p\int_{-\infty}^{\infty} d\omega \oint d\mathbf{p}$$

$$\mid G(\hat{p}-\hat{k})\mid^2 \mid G_b(\hat{p}-\hat{k})\mid^2 G_b(\hat{p})p_mD\delta_{mp} \mid \mathbf{k}-\mathbf{p}\mid^{-y}$$

$$\mid G(\hat{p})\mid^2\mid G_b(\hat{p})\mid^2 G_b(\hat{k}-\hat{p})Dp^{-y}\delta_{pm}$$

The frequency integrals are evaluated by interchanging ν and κ, or equivalently by replacing a by $1/a$. Also,

$$k^2\frac{d\gamma}{dp} = -\frac{1}{4}k_p\int_{-\infty}^{\infty} d\omega \oint d\mathbf{p}$$

$$G(\hat{p}-\hat{k})\mid G_b(\hat{p}-\hat{k})\mid^2 G_b(\hat{p})gDp_mP_{pi}(\mathbf{k}-\mathbf{p})\mid \mathbf{k}-\mathbf{p}\mid^{-y}$$

$$+G(\hat{p})\mid G_b(\hat{p})\mid^2 G_b(\hat{k}-\hat{p})gD(k_m-p_m)P_{pi}(\mathbf{p})p^{-y}$$

$$+G(\hat{k}-\hat{p})G(-\hat{p})\mid G_b(\hat{p})\mid^2 gDP_{pmn}(\mathbf{k}-\mathbf{p})P_{ni}(\mathbf{p})p^{-y}$$

$$+G(\hat{p})G(\hat{p}-\hat{k})\mid G(\hat{p}-\hat{k})\mid^2 gDP_{pmn}(\mathbf{p})P_{ni}(\mathbf{p}-\mathbf{k})\mid \mathbf{p}-\mathbf{k}\mid^{-y}$$

In evaluating products of projection operators in the distant interaction limit, the first order Taylor series

$$P_{ij}(\mathbf{p}-\mathbf{k}) = P_{ij}(\mathbf{p}) + p^{-2}k_\nu P_{\nu ij}(\mathbf{p}) + \cdots$$

is useful.

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Bolgiano scaling in Boussinesq turbulence is analyzed using the Yakhot-Orszag renormalization group. For this purpose, an isotropic model is introduced. Scaling exponents are calculated by forcing the temperature equation so that the temperature variance flux is constant in the inertial range. Universal amplitudes associated with the scaling laws are computed by expanding about a logarithmic theory. Connections between this formalism and the direct interaction approximation are discussed. It is suggested that the Yakhot-Orszag theory yields a lowest order approximate solution of a regularized direct interaction approximation which can be corrected by a simple iterative procedure.

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