

STABILITY AND STABILITY DEGREE OF A CRACKED FLEXIBLE ROTOR  
SUPPORTED ON JOURNAL BEARINGS

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This paper investigates the stability and the stability degree of a flexible cracked rotor supported on different kinds of journal bearings. It is found that no matter what kind of bearings is used, the unstable zones caused by rotor crack locate always within the speed ratio  $\frac{2}{N} \left(1 - \frac{\Delta K_t}{4}\right) < \Omega < \frac{2}{N}$  when gravity parameter  $W_r > 1.0$ ; and locate always within the speed ratio  $\frac{2\Omega_s}{N} \left(1 - \frac{\Delta K_t}{4}\right) < \Omega < \frac{2\Omega_s}{N}$  when  $W_r < 0.1$ , where  $\Delta K_t$  is the crack stiffness ratio,  $N = 1, 2, 3, 4, 5 \dots$  and  $\Omega_s = \left(\frac{1+2\alpha}{2\alpha}\right)^{1/2}$ . When  $0.1 < W_r < 1.0$ , there is a region, where no unstable zones caused by rotor crack exist.

Outside the crack ridge zones, the rotor crack has almost no influence on system's stability and stability degree; while within the crack ridge zones, the stability and stability degree depend both on the crack and system's parameters. In some cases, the system may still be stable even the crack is very large. For small gravity parameter ( $W_r < 0.1$ ), the mass ratio  $\alpha$  has large influence on the position of unstable region, but its influence on the stability degree is small. The influence of fixed Sommerfeld number  $S_0$  on the crack stability degree is small although  $S_0$  has large influence on the stability degree of uncracked rotor.

### NOMENCLATURES

$[A(\Phi)]$  = periodic state matrix

$c_d$  = external damping

$c_p$  = machined clearance of journal bearing

$c_{ij}$ , ( $i, j = x, y$ ) = damping of journal bearing

$C_{ij}$ , ( $i, j = x, y$ ) = dimensionless damping  $\left( = \frac{c_p \omega c_{ij}}{W} \right)$

$D_e$  = external damping ratio  $\left( = \frac{c_d}{2m_d \omega_e} \right)$

$e_u$  = unbalance eccentricity

$f_x, f_y$  = fluid film forces

$F_x, F_y$  = dimensionless fluid film forces  $\left( = \frac{f_{x,y}}{m_d c_p \omega^2} \right)$

$g(t)$  or  $g(\Phi)$  = periodic function,  $0 < g(t) < 1$  and it has the form of Eq.5 for small crack.

$k_c$  = complex stiffness of cracked shaft

$k_s$  = stiffness of the uncracked shaft

$k_\xi, k_\eta$  = stiffness of cracked shaft in  $\xi$  and  $\eta$  directions

$\Delta k_\xi, \Delta k_\eta$  = the largest stiffness change in  $\xi$  and  $\eta$  directions caused by crack

$\Delta K_\xi$  = stiffness change ratio in  $\xi$  (the crack) direction  $\left( = \frac{\Delta k_\xi}{k_s} \right)$

$\Delta K_\eta$  = stiffness change ratio in  $\eta$  (the cross) direction  $\left( = \frac{\Delta k_\eta}{k_s} \right)$

$$\Delta K_1 = \frac{\Delta k_\xi + \Delta k_\eta}{k_s}$$

$$\Delta K_2 = \frac{\Delta k_\xi - \Delta k_\eta}{k_s}$$

$k_{ij}$  ( $i, j = x, y$ ) = stiffness of journal bearing

$K_{ij}$  ( $i, j = x, y$ ) = dimensionless stiffness  $\left( = \frac{c_p k_{ij}}{W} \right)$

$m_b, m_d$  = mass lumped at bearing station and rotor mid-span

$m_p$  = preload factor  $\left( = 1 - \frac{c_b}{c_p} \right)$ , where  $c_b$  is the assembled clearance

$S$  = Sommerfeld number  $\left( = \frac{\mu \omega D L}{2\pi W} \left( \frac{R}{c_p} \right)^2 \right)$ , where  $\mu$  is the viscosity of lubricant

$D$  is the bearing diameter,  $R$  is the bearing radius and  $L$  is the bearing width

$S_0$  = fixed Sommerfeld number  $\left( = S \cdot \left( \frac{\omega_c}{\omega} \right) = \frac{\mu \omega_c D L}{2\pi W} \left( \frac{R}{c_p} \right)^2 \right)$ , it is a design parameter.

$t$  = time

$T$  = period

$U$  = unbalance parameter  $\left( = \frac{e_u}{c_p} \right)$

$W$  = bearing load

$W_g$  = gravity parameter  $\left( = \frac{\delta_g}{c_p} \right)$ , where  $\delta_g = \frac{m_d R}{k_s}$ , it represents the elasticity of the shaft.

$x, y$  = deflection in Cartesian coordinates

$X, Y$  = dimensionless deflection  $\left( = \frac{x}{c_p}, \frac{y}{c_p} \right)$

$z = x + iy$ , complex deflection  $= (\xi + i\eta)e^{i\theta}$ .

$Z = X + iY$

$\bar{Z} = X - iY$

$Z_0$  = steady state response

$\Delta Z$  = perturbation deflection about the steady state response  $Z_0$  ( $= \Delta X + iY$ )

$\alpha$  = mass ratio  $\left( = \frac{m_b}{m_d} \right)$

$\beta$  = crack angle, it is the angle between crack and unbalance

$[\Gamma(T)]$  = transition matrix

$\xi, \eta$  = body fixed rotating coordinates,  $\xi$  is in the direction of crack

$\mu_f$  = Floquet eigenvalue

$|\mu_f|$  = modulus of  $\mu_f$

$\tau$  =  $\omega t$ , dimensionless time

$\varphi$  = initial unbalance angle

$\Phi$  =  $\omega t + \varphi + \beta$

$\omega$  = rotating speed

$\omega_c$  = rigid pin-pin critical speed  $\left( = \left( \frac{k_s}{m_d} \right)^{1/2} \right)$

$\Omega$  = speed ratio  $\left( = \frac{\omega}{\omega_c} \right)$

$(\dot{\phantom{x}})$  =  $\frac{d}{dt}$

$(\prime)$  =  $\frac{d}{d\tau}$  or  $\frac{d}{d\Phi}$

$[ \ ]$  = square matrix

$\{ \}$  = column matrix

## INTRODUCTION

Fatigue cracking of rotor shaft may cause a catastrophic damage to rotating machineries, an example was given by Jack and Patterson (1976). So that a detailed investigation into the behavior of a cracked rotor shaft is very important for diagnosing and preventing rotor cracks.

There are two major research field in cracked rotor analysis, one is the modeling and another is the detecting of the rotor crack. Gasch (1976) considered in body-fixed rotating coordinate the stiffness change due to open and closed cracks and solved the equations of motion of a cracked Jeffcott rotor by an analogue computer. Mays et al (1984) gave an approximate method of estimating the reduction of a section diameter required to model a crack. Nelson and Nataraj (1986) analysed the dynamics of a rotor system with a cracked shaft by the finite element method; Collins et al (1991) considered a rotating Timoshenko shaft with a single transvers crack and analysed the free vibrations and the responses to a single axial impulse and periodic axial impulses. Gasch (1988) et al analysed the dynamic behaviour of a Jeffcott rotor with a cracked hollow shaft and Bernasconi (1986) considered the vibrations of a stepped shaft by using singularity functions. Muszynska (1982) developed a solution for a gaping crack and concluded that the increase in the primary vibration is more than that in the subharmonic vibrations. Papadopoulos and Dimarogonas (1980) pointed out that the informations of subharmonic resonances and the frequency shifting are important for crack identification. Imam et al (1989) presented a very successful on-line crack diagnosis method which can detect cracks of the order of 1 to 2 percent of the shaft diameter deep. Most of the early research results on cracked rotor have been summarized in the book by Dimarogonas and Paipetis (1983) and more literatures on the dynamics of cracked rotors can be found in a reviewing paper by Wauer (1990).

Although more than 100 papers have been published in this field, there are still many problems to be solved. Up till now, only few papers, for example Tamura (1988) and Gasch (1992), analysed the stability of a cracked rotor supported on rigid bearings and it was found that the unstable zones caused by rotor crack are

near the speed ratio  $\frac{\omega}{\omega_c} = \frac{2}{1}, \frac{2}{2}, \frac{2}{3}, \frac{2}{4}, \dots$  and the unstable zone spreading out from  $\frac{\omega}{\omega_c} = 2$  is the broadest, where  $\omega_c$  is the first rigid critical speed. Up till now, no papers about the stability of a cracked flexible rotor supported on fluid film bearings have been published, but the stability and dynamic characteristics of this kind of system are very important as dangerous cracks were mainly reported in rotors running in fluid film bearings.

## EQUATIONS OF MOTION AND PERIODIC STATE MATRIX FOR STABILITY ANALYSIS

Considering a simple flexible Jeffcott rotor with a cracked shaft supported on journal bearings (see Fig.1), the equations of motion of the system can be written as:

$$\begin{cases} m_d \ddot{z}_d + c_d \dot{z}_d + k_c [(z_d - z_b), t] \cdot (z_d - z_b) = m_d g + m_d e_u \cdot \omega^2 e^{i(\omega t + \phi)} \\ m_b \ddot{z}_b + \frac{1}{2} k_c [(z_d - z_b), t] \cdot (z_b - z_d) = m_b g - (f_x + i f_y) \end{cases} \quad (1)$$

where  $m_b$  and  $m_d$  are the mass lumped at bearing station and rotor mid-span,  $z_d$  and  $z_b$  are the complex deflections of disk and bearing centers;  $\omega$  is the rotating speed;  $k_c [(z_d - z_b), t]$  is the stiffness coefficient which is a function of  $z_d - z_b$  and changes with time  $t$  due to the opening and closing of the rotor crack and,  $f_x$  and  $f_y$  are fluid film forces of journal bearing. There are

$$\begin{cases} z_d = x_d + iy_d \\ z_b = x_b + iy_b \end{cases}$$

### (1) Stiffness Forces

Supposing that the vibration deflection is small enough and the influence of deflection difference  $z_d - z_b$  on shaft stiffness  $k_c$  can be omitted, then the shaft stiffness  $k_c$  changes only with time  $t$  and

$$k_c [(z_d - z_b), t] \approx k_c(t) \quad (2)$$

Supposing further that the weight deflection is dominant, then  $k_c(t)$  changes periodically with time and can be written in  $\xi$  and  $\eta$  directions as:

$$\begin{cases} k_\xi = k_s - g(t) \Delta k_\xi \\ k_\eta = k_s - g(t) \Delta k_\eta \end{cases} \quad (3)$$

where  $g(t)$  is a periodic function of time and  $0 < g(t) < 1$ .  $\Delta k_\xi$  and  $\Delta k_\eta$  are constant and represent the stiffness change caused by rotor crack. There is

$$z = x + iy = (\xi + i\eta) e^{i\Phi}$$

where  $\Phi = \omega t + \phi + \beta$ ,  $z$  is  $z_d$  or  $z_b$ . The stiffness force  $k_c(t) \cdot z$  can be written as:

$$\begin{aligned} k_c(t) \cdot z &= \frac{1}{2} [(k_\xi + k_\eta) \cdot z + (k_\xi - k_\eta) \cdot \bar{z} e^{2i\Phi}] \\ &= [k_s - \frac{1}{2} g(t) (\Delta k_\xi + \Delta k_\eta)] \cdot z - \frac{1}{2} g(t) (\Delta k_\xi - \Delta k_\eta) \cdot \bar{z} \cdot e^{2i\Phi} \end{aligned} \quad (4)$$

For small crack, the opening and closing of the crack can be represented by a rectangular function  $g(t)$ , and  $g(t)$  can be written in a Fourier series of  $\Phi$  as:

$$g(t) = g(\Phi) = \left(\frac{2}{\pi}\right) \left[ \frac{\pi}{4} + \cos\Phi - \frac{1}{3} \cos 3\Phi + \frac{1}{5} \cos 5\Phi - \dots \right] \quad (5)$$

where  $\Phi = \omega t + \phi + \beta$ .

## (2) Linearised Fluid Film Forces and Coefficients

The vibration response of the system can be split to

$$\begin{cases} z_d = z_{do} + \Delta z_d \\ z_b = z_{bo} + \Delta z_b \end{cases}$$

where  $z_{do}$  and  $z_{bo}$  are the steady state response of the system,  $\Delta z_d$  and  $\Delta z_b$  are the perturbation deflections.

For small dynamic perturbation, the nonlinear fluid film forces  $f_x$  and  $f_y$  can be linearised about the steady state response ( $x_{bo}, y_{bo}$ ) as:

$$\begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} f_{xo} \\ f_{yo} \end{bmatrix} + \begin{bmatrix} k_{xx} & k_{xy} \\ k_{yx} & k_{yy} \end{bmatrix} \begin{bmatrix} \Delta x_b \\ \Delta y_b \end{bmatrix} + \begin{bmatrix} c_{xx} & c_{xy} \\ c_{yx} & c_{yy} \end{bmatrix} \begin{bmatrix} \Delta \dot{x}_b \\ \Delta \dot{y}_b \end{bmatrix} \quad (6)$$

The nondimensional forms of  $f_x$  and  $f_y$  can be written as

$$\begin{bmatrix} F_x \\ F_y \end{bmatrix} = \frac{1}{m_d c_p \omega^2} \begin{bmatrix} f_x \\ f_y \end{bmatrix} = \frac{W}{m_d c_p \omega^2} \left\{ \frac{1}{W} \begin{bmatrix} f_{xo} \\ f_{yo} \end{bmatrix} + \begin{bmatrix} K_{xx} & K_{xy} \\ K_{yx} & K_{yy} \end{bmatrix} \begin{bmatrix} \Delta X_b \\ \Delta Y_b \end{bmatrix} + \begin{bmatrix} C_{xx} & C_{xy} \\ C_{yx} & C_{yy} \end{bmatrix} \begin{bmatrix} \Delta X'_b \\ \Delta Y'_b \end{bmatrix} \right\} \quad (7)$$

The nondimensional stiffness and damping coefficients  $K_{ij}$  and  $C_{ij}$  ( $i, j = x, y$ ) are

$$K_{ij} = \frac{c_p}{W} k_{ij}, \quad C_{ij} = \frac{c_p}{W} c_{ij}$$

and they can be obtained from the book by Someya (1989) and the book by Vance (1988). These coefficients are only functions of Sommerfeld number  $S$  or eccentricity ratio  $\epsilon_e = \frac{e_x}{c_p}$  for a given set of bearing parameters, where the Sommerfeld number is

$$S = \frac{\mu \omega D L}{2\pi W} \left( \frac{R}{c_p} \right)^2 \quad (8)$$

As the rotating speed  $\omega$  and the bearing parameters are mixed up in the Sommerfeld number  $S$ , it is difficult to separate in the stability analysis the influences of rotating speed and bearing parameters on the nondimensional stiffness and damping coefficients. In order to solve this problem, the fixed Sommerfeld number  $S_o$  is introduced and it is defined as

$$S_o = S \cdot \left( \frac{\omega_c}{\omega} \right) = \frac{\mu \omega_c D L}{2\pi W} \left( \frac{R}{c_p} \right)^2 \quad (9)$$

That is, the rotating speed  $\omega$  in Sommerfeld number  $S$  (see Eq.8) is replaced with the rigid pin-pin critical speed  $\omega_c$   $\left( = \left( \frac{k_t}{m_d} \right)^{1/2} \right)$ .  $S_o$  is thus the Sommerfeld number in the rigid pin-pin critical speed and does not change with the rotating speed. Now the stiffness and damping coefficients are functions of the fixed Sommerfeld number  $S_o$  and the rotating speed ratio  $\Omega \left( = \frac{\omega}{\omega_c} \right)$ .

In this paper, three types of journal bearings are used for stability analysis, the 2 Axial Grooved Cylindrical Bearing, the 4 Lobe Bearing and the 5 Pad Tilting Pad Bearing. Fig.2 shows the diagrams of these journal bearings and the stiffness and damping coefficients of these three bearings are taken from Someya's book (1989).

For the simple Jeffcott rotor considered here, there is

$$W = \frac{m_d R}{2} + m_b R = m_d R \cdot \left( \frac{1}{2} + \alpha \right)$$

So that

$$\frac{W}{m_d c_p \omega^2} = \frac{W_f}{\Omega^2} \left( \frac{1}{2} + \alpha \right) \quad (10)$$

where  $W_f = \frac{g}{c_p \omega^2} = \frac{\delta_f}{c_p}$  is the gravity parameter.

### (3) Nondimensional Equations of Motion of the System

Substituting Eq.4 and Eq.6 into Eq.1, dividing both sides by  $m_d c_p \omega^2$  and then substituting into Eq.7 and Eq.10, we can get at last the nondimensional forms of Eq.1 as

$$\begin{cases} Z''_d + \frac{2D_f}{\Omega} Z'_d + \frac{1}{\Omega^2} \left\{ \left[ 1 - \frac{1}{2} g(t) \Delta K_1 \right] (Z_d - Z_b) - \right. \\ \left. - \frac{1}{2} g(t) \Delta K_2 \cdot (\bar{Z}_d - \bar{Z}_b) \cdot e^{2i\theta} \right\} = \frac{W_f}{\Omega^2} + U e^{K_1 + i\theta} \\ Z''_b + \frac{1}{2\alpha\Omega^2} \left\{ \left[ 1 - \frac{1}{2} g(t) \Delta K_1 \right] (Z_b - Z_d) - \frac{1}{2} g(t) \Delta K_2 \cdot (\bar{Z}_b - \bar{Z}_d) \cdot e^{2i\theta} \right\} + \\ + \frac{W_f (1 + 2\alpha)}{2\alpha\Omega^2} \left[ \left( \frac{f_{x0}}{W} + K_{xx} \Delta X_b + K_{xy} \Delta Y_b + C_{xx} \Delta X'_b + C_{xy} \Delta Y'_b \right) + \right. \\ \left. i \left( \frac{f_{y0}}{W} + K_{yx} \Delta X_b + K_{yy} \Delta Y_b + C_{yx} \Delta X'_b + C_{yy} \Delta Y'_b \right) \right] = \frac{W_f}{\Omega^2} \end{cases} \quad (11)$$

where  $(\cdot)' = \frac{d}{d\tau}$  and  $\Delta K_1 = \frac{\Delta K_x + \Delta K_y}{k_x}$ ,  $\Delta K_2 = \frac{\Delta k_x - \Delta k_y}{k_x}$ .

In the steady state case, there are

$$Z_d = Z_{d0}, Z_b = Z_{b0}, Z''_d = Z'_d = 0, Z''_b = Z'_b = 0, f_x = f_{x0} \text{ and } f_y = f_{y0}$$

As  $Z_{d0}$  and  $Z_{b0}$  are also solutions of Eq.(11) and

$$Z_d = Z_{d0} + \Delta Z_d, Z_b = Z_{b0} + \Delta Z_b$$

the perturbation equations of motion of the system can be got from Eq.(11) as:

$$\begin{cases} \Delta Z''_d + \frac{2D_f}{\Omega} \Delta Z'_d + \frac{1}{\Omega^2} \left\{ \left[ 1 - \frac{1}{2} \Delta K_1 g(t) \right] (\Delta Z_d - \Delta Z_b) - \right. \\ \left. - \frac{1}{2} \Delta K_2 g(t) (\Delta \bar{Z}_d - \Delta \bar{Z}_b) \cdot e^{2i\theta} \right\} = 0 \\ \Delta Z''_b + \frac{1}{2\alpha\Omega^2} \left\{ \left[ 1 - \frac{1}{2} \Delta K_1 g(t) \right] (\Delta Z_b - \Delta Z_d) - \frac{1}{2} \Delta K_2 g(t) (\Delta \bar{Z}_b - \Delta \bar{Z}_d) e^{2i\theta} \right\} + \\ + \frac{W_f (1 + 2\alpha)}{2\alpha\Omega^2} \left[ (K_{xx} \Delta X_b + K_{yy} \Delta Y_b + C_{xx} \Delta X'_b + C_{xy} \Delta Y'_b) + \right. \\ \left. i (K_{yx} \Delta X_b + K_{yy} \Delta Y_b + C_{yx} \Delta X'_b + C_{yy} \Delta Y'_b) \right] = 0 \end{cases}$$

The above equations can be written in Cartesian coordinates as ( $g(t)$  is written in the form of  $g(\Phi)$ ):  
rotor:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta X''_d \\ \Delta Y''_d \end{bmatrix} + \begin{bmatrix} \frac{2D_f}{\Omega} & 0 \\ 0 & \frac{2D_f}{\Omega} \end{bmatrix} \begin{bmatrix} \Delta X'_d \\ \Delta Y'_d \end{bmatrix} + \frac{1}{\Omega^2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} -$$

$$\frac{1}{2}g(\Phi) \cdot \begin{Bmatrix} \Delta K_1 + \Delta K_2 \cos 2\Phi & \Delta K_2 \sin 2\Phi \\ \Delta K_2 \sin 2\Phi & \Delta K_1 - \Delta K_2 \cos 2\Phi \end{Bmatrix} \begin{bmatrix} \Delta X_d - \Delta X_b \\ \Delta Y_d - \Delta Y_b \end{bmatrix} = 0$$

bearing:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta X''_b \\ \Delta Y''_b \end{bmatrix} + \frac{1}{2\alpha\Omega^2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{2}g(\Phi) \cdot \begin{Bmatrix} \Delta K_1 + \Delta K_2 \cos 2\Phi & \Delta K_2 \sin 2\Phi \\ \Delta K_2 \sin 2\Phi & \Delta K_1 - \Delta K_2 \cos 2\Phi \end{Bmatrix} \begin{bmatrix} \Delta X_b - \Delta X_d \\ \Delta Y_b - \Delta Y_d \end{bmatrix} + \frac{W_r(1+2\alpha)}{2\alpha\Omega^2} \left\{ \begin{bmatrix} K_{xx} & K_{xy} \\ K_{yx} & K_{yy} \end{bmatrix} \begin{bmatrix} \Delta X_b \\ \Delta Y_b \end{bmatrix} + \begin{bmatrix} C_{xx} & C_{xy} \\ C_{yx} & C_{yy} \end{bmatrix} \begin{bmatrix} \Delta X'_b \\ \Delta Y'_b \end{bmatrix} \right\} = 0 \quad (12)$$

#### (4) Periodic State Matrix of the System

Taking the notes of

$$\begin{aligned} X_1 &= \Delta X_d, \quad X_2 = \Delta Y_d, \quad X_3 = \Delta X_b, \quad X_4 = \Delta Y_b, \\ X_5 &= \Delta X'_d, \quad X_6 = \Delta Y'_d, \quad X_7 = \Delta X'_b, \quad X_8 = \Delta Y'_b, \end{aligned}$$

supposing that  $\varphi$  and  $\beta$  do not change with time, and denoting  $(\cdot) = d/d\Phi$ , the first order state equations of motion of the system can be obtained from Eq.12 and then can be written in matrix form as

$$\{X\}' = [A(\Phi)]\{X\} \quad (14)$$

where the vector  $\{X\}$  is defined as  $\{X\} = (X_1, X_2, \dots, X_8)^T$

and  $[A(\Phi)]$  is the periodic state matrix and has a period of  $2\pi$ .

$$[A(\Phi)] = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{P_1(\Phi)}{\Omega^2} & \frac{Q_1(\Phi)}{\Omega^2} & \frac{P_1(\Phi)}{\Omega^2} & \frac{Q_1(\Phi)}{\Omega^2} & \frac{2\mu}{\Omega} & 0 & 0 & 0 \\ \frac{Q_1(\Phi)}{\Omega^2} & \frac{P_2(\Phi)}{\Omega^2} & \frac{Q_1(\Phi)}{\Omega^2} & \frac{P_2(\Phi)}{\Omega^2} & 0 & \frac{2\mu}{\Omega} & 0 & 0 \\ \frac{P_1(\Phi)}{2\alpha\Omega^2} & \frac{Q_1(\Phi)}{2\alpha\Omega^2} & \frac{K_{xx} + P_1(\Phi)}{2\alpha\Omega^2} & \frac{K_{xy} - Q_1(\Phi)}{2\alpha\Omega^2} & 0 & 0 & \frac{C_{xx}}{2\alpha\Omega^2} & \frac{C_{xy}}{2\alpha\Omega^2} \\ \frac{Q_1(\Phi)}{2\alpha\Omega^2} & \frac{P_2(\Phi)}{2\alpha\Omega^2} & \frac{K_{yx} - Q_1(\Phi)}{2\alpha\Omega^2} & \frac{K_{yy} + P_2(\Phi)}{2\alpha\Omega^2} & 0 & 0 & \frac{C_{yx}}{2\alpha\Omega^2} & \frac{C_{yy}}{2\alpha\Omega^2} \end{bmatrix} \quad (15)$$

where

$$\begin{cases} P_1(\Phi) = 1 - \frac{1}{2}\Delta K_1 g(\Phi) - \frac{1}{2}\Delta K_2 g(\Phi) \cos 2\Phi \\ P_2(\Phi) = 1 - \frac{1}{2}\Delta K_1 g(\Phi) + \frac{1}{2}\Delta K_2 g(\Phi) \cos 2\Phi \\ Q_1(\Phi) = \frac{1}{2}\Delta K_2 g(\Phi) \cdot \sin 2\Phi \end{cases} \quad (16)$$

The stability of the periodically time-variant system of Eq.14 can be analysed by the well known

Floquet's method.

## FLOQUET'S THEORETICAL AND NUMERICAL METHODS USED FOR STABILITY ANALYSIS

The most straightforward method for dealing with the stability of a periodic system is perhaps the Floquet's method (see eg., Caesari, 1970), which states that the knowledge of the state transition matrix over one period is sufficient for determining the stability of the equilibrium position  $\{X\} = 0$  of a periodic system

$$\{X\}' = [A(\Phi)]\{X\} \quad (17)$$

where  $[A(\Phi)] = [A(\Phi + T)]$  and  $T$  is the period. The transition matrix  $[\Gamma(T)]$  is defined by

$$\{X(T)\} = [\Gamma(T)]\{X(0)\}$$

When the transition matrix  $[\Gamma(T)]$  has been obtained, the Floquet's eigenvalues of the system can be calculated from the following eigen-function

$$|[\Gamma(T)] - \mu_f [I]| = 0 \quad (18)$$

where  $\mu_f$  is the Floquet eigenvalue and  $[I]$  is the unit matrix.

The stability of the equilibrium position  $\{X\} = 0$  of the periodic system (17) can be determined by the value of the Floquet eigenvalue  $\mu_f$  that is, if all  $|\mu_f| < 1$ , the  $\{X\} = 0$  position is stable; if one or more  $|\mu_f| > 1$ , the  $\{X\} = 0$  position is unstable and the  $\{X\} = 0$  position is the critical case if one or more  $|\mu_f| = 1$ . The physic meaning of the Floquet eigenvalue  $\mu_f$  is clear, it shows the factor by which a given initial amplitude increases or decreases after one period of motion.

Although the stability of the equilibrium position  $\{X\} = 0$  of system (17) can be determined from the transition matrix  $[\Gamma(T)]$  of the system, unfortunately, there is no general analytical method for calculating  $[\Gamma(T)]$ . But there are several efficient numerical methods although they take too much computer time (Fridemann, 1977). One of them is the so called Hsu's method and the essential aspects of this method are given in Fridemann's paper (1977), the following is the result of this method.

The period  $T$  is divided into  $K$  intervals denoted by  $\Phi_k$  ( $k = 1, 2, \dots, K$ ) with  $0 = \Phi_0 < \Phi_1 < \dots < \Phi_K = T$ . The  $k$ th interval  $(\Phi_{k-1}, \Phi_k)$  is denoted by  $\tau_k$ . In the  $k$ th interval, a constant matrix  $[B_k]$  is defined by

$$[B_k] = \int_{\Phi_{k-1}}^{\Phi_k} [A(\Phi)] d\Phi \quad \Phi \in \tau_k \quad (19)$$

and the final approximation for the transition matrix  $[\Gamma(T)]$  at the end of one period can be calculated by

$$[\Gamma(T)] = \prod_{i=1}^K \exp[B_i] \approx \prod_{i=1}^K \left\{ [I] + \sum_{j=1}^j \frac{[B_i]^j}{j!} \right\} \quad (20)$$

## STABILITY AND STABILITY DEGREE OF A CRACKED FLEXIBLE ROTOR SUPPORTED ON JOURNAL BEARINGS

Based on the Floquet's theorem and the Hsu's numerical method, the Floquet eigenvalues of Eq.15 are calculated numerically for speed ratio  $0.2 < \Omega < 5.0$ , crack stiffness ratio  $0.0 < \Delta K_c < 0.8$ , gravity parameter  $0.001 < W_r < 10.0$ , fixed Sommerfeld number  $0.1 < S_o < 2.3$  and mass ratio  $0.05 < \alpha < 0.45$ . The stability and the stability degree of a cracked flexible rotor supported on journal bearings are analysed. At first, the following phraseologies are defined with Fig.5 as an example.

**crack ridge:** a protruding ridge of Floquet eigenvalue caused by rotor crack.

**crack ridge zone:** the zone where the crack ridge locates, for example from A to B in Fig.5. The position



and extent of the crack ridge zone is given by Eq.(21) and Eq.(22).

**crack unstable zone:** unstable zone caused by the crack, it is the zone where  $|\mu_f| > 1.0$ . It locates within the crack ridge zone.

**stability degree:** the degree of stability which is determined by the largest Floquet eigenvalue  $|\mu_f|_m$ , the larger the  $|\mu_f|_m$  is, the smaller the stability degree. When  $|\mu_f|_m < 1$ , the equilibrium position of the system is stable, when  $|\mu_f|_m > 1$ , the equilibrium position of the system is unstable and the equilibrium position of the system is the critical case when  $|\mu_f|_m = 1$ .

**diagram of stability degree:** the 3-D diagram of Floquet eigenvalue. Its Z-axis is the largest Floquet eigenvalue  $|\mu_f|_m$ , Y-axis is the speed ratio  $\Omega$  and X-axis is the system parameter (the fixed Sommerfeld number  $S_o$ , the gravity parameter  $W_g$ , the mass ratio  $\alpha$  or the stiffness change ratio  $\Delta K_f$ ).

**1.0-Level:** The level of the Floquet eigenvalue  $|\mu_f|_m = 1$ , it is the limit of stability. Above this level, the system is unstable.

### (1) The Stability and Stability Degree of Uncracked Rotor Shaft

It should be noticed that now the periodic system (17) is simplified to a time-invariant one and the Floquet's method is also simplified to the normal eigenvalue problem.

Journal bearings are superior to rolling element bearings in high-speed operation. They have high damping ability, high load-carrying capacity and low friction. In addition, silent operation and long life can be attained. Therefore, journal bearings of various types are widely used in high-speed rotating machineries. But an inappropriately designed journal bearing may not only increase the vibration amplitude, but bring also the oil whirl or oil whip, that is, a self-excited lateral vibration of rotating shaft caused by oil film in journal bearings. Generally, multilobe bearings are more stable than circular bearings. A tilting-pad bearing is always stable, because it has no cross-coupling terms of oil film coefficients.

The gravity parameter  $W_g$  is a very heavy parameter in the stability analysis. Fig.3 shows the influences of gravity parameter on the stability degree. It is found that there is a valley of Floquet eigenvalue for gravity parameter of mid-value ( $0.05 < W_g < 1.0$ ) and small speed ratio  $\Omega$ . As  $\Omega$  increases, this valley becomes more narrow and more shallow. The tilting pad bearing has the longest valley and the cylindrical bearing has the shortest one. The system will be the most stable if the parameters are selected in the bottom of the valley. In the left side (the small gravity parameter side) of the valley, there is a hill of Floquet eigenvalue for cylindrical and multilobe bearings. This hill is very important for system's stability as it determines the stability limit. Besides, the hill of the cylindrical bearing is much higher and wider than the multilobe bearing, that is why the cylindrical bearing is more unstable than the multilobe bearing. The tilting pad bearing is always stable as it has no such a hill.

Although the multilobe bearing is generally more stable than the cylindrical bearing, it may be more unstable than the cylindrical bearing in the case of small fixed Sommerfeld number  $S_o$  and small gravity parameter  $W_g$ .

Another heavy parameter in the stability analysis is the fixed Sommerfeld number  $S_o$ . Fig.4 shows the influences of  $S_o$  on the stability degree in different gravity parameter  $W_g$ . For multilobe bearing and for cylindrical bearing, when  $S_o > 0.4$  or  $W_g > 0.1$ , it is found that the threshold speed ratio of stability increases as  $W_g$  increases.

The variation of stability degree with fixed Sommerfeld number  $S_o$  depends largely on the value of gravity parameter  $W_g$ . For small  $W_g$ , the stability degree decreases as  $S_o$  increases; while for large  $W_g$ , the variation of stability degree with  $S_o$  is very small.

## (2) The Stability and Stability Degree of Cracked Rotor Shaft

It is well known that the stability and stability degree of an uncracked flexible rotor supported on journal bearing change greatly for different types of journal bearings. What will happen if the rotor has a small crack? Of course, the crack will have influence on the stability of the uncracked system and will cause new unstable regions, but how large is the influence and where are these new unstable regions? These questions are important for detecting the crack and preventing the system from an accident. In order to answer these questions, the Floquet eigenvalues of Eq.15 are calculated. For the reason of simple, the cross stiffness change caused by crack is omitted in the following analysis, that is  $\Delta K_{xy} = 0$  is assumed, because  $\Delta K_{xy}$  is very small compared with  $\Delta K_t$ .

Based on the large quantities of numerical results, it is found that the position of crack ridge zones and then the position of crack unstable zones have no relation with the type of journal bearings, that is, they do not depend on the fixed Sommerfeld number  $S_j$ . The position of crack ridge zones depend only on the speed ratio and the stiffness change ratio for large gravity parameter ( $W_g > 1.0$ ), and depend also on the mass ratio for small gravity parameter ( $W_g < 0.05$ ).

Fig.5 to Fig.7 show the positions and the extents of the crack ridge zones. When  $W_g > 1.0$ , the mass ratio has no influence and the crack ridge zones locate always within the two lines of  $\Omega = \frac{2}{N}$  and  $\Omega = \frac{2}{N} \left( 1 - \frac{\Delta K_t}{4} \right)$ , where  $N = 1, 2, 3, 4, 5, \dots$ . The widths of the crack ridge zones are separately equal to or less than  $\frac{\Delta K_t}{2N}$  (see Fig.5). So that the crack unstable zones locate within

$$\frac{2}{N} \left( 1 - \frac{1}{4} \Delta K_t \right) < \Omega < \frac{2}{N} \quad (N = 1, 2, 3, 4, 5, \dots) \quad (21)$$

When  $W_g < 0.05$ , the crack ridge zones depend largely on mass ratio and they locate always within the two lines of  $\Omega = \frac{2\Omega_*}{N}$  and  $\Omega = \frac{2\Omega_*}{N} \left( 1 - \frac{1}{4} \Delta K_t \right)$ , where  $\Omega_* = \left( \frac{1 + 2\alpha}{2\alpha} \right)^{1/2}$  and  $N = 1, 2, 3, 4, 5, \dots$ . The widths of the crack ridge zones are separately equal to or less than  $\frac{\Delta K_t}{2N} \Omega_*$  (see Fig.6). So that the crack unstable zones locate within

$$\frac{2\Omega_*}{N} \left( 1 - \frac{1}{4} \Delta K_t \right) < \Omega < \frac{2\Omega_*}{N} \quad (N = 1, 2, 3, 4, 5, \dots) \quad (22)$$

In this paper, the shaft stiffness is supposed to change linearly with crack depth, so that the width of the crack ridge zone increases also linearly with crack stiffness change ratio due to the reduction of shaft stiffness. Besides, the larger the speed ratio and the stiffness change ratio are, the wider the crack ridge zone is (see Fig.7). Therefore, the instability caused by crack in larger speed ratio is more dangerous and should be paid more attention to.

It is known that the gravity parameter  $W_g$  is a very heavy parameter in the stability analysis of uncracked rotor shaft and it is found in the above discussions that when  $W_g < 0.05$ , the mass ratio  $\alpha$  has large influence on the crack ridge zone, while when  $W_g > 1.0$ ,  $\alpha$  has almost no influence on the crack ridge zone. What will happen if  $0.05 < W_g < 1.0$ ? Fig.8 shows the influence of gravity parameter  $W_g$  on the stability and stability degree (please compare Fig.8 with Fig.3). It is found that in a region of  $W_g$  within  $0.05 < W_g < 1.0$ , the crack has almost no influence on the stability degree no matter how large the crack is. This is because that in this region, the diagram of stability degree has either a valley or a hill. The cylindrical bearing and multilobe bearing have generally a valley at small speed ratio and a hill at large speed ratio, while the tilting pad bearing has always a

valley although this valley becomes smaller as the speed ratio increases.

It is found that the position of crack ridge zones almost do not change with the fixed Sommerfeld number  $S_o$ , but the extent of crack ridge zones decrease with the increasing of  $S_o$  when the gravity parameter is small ( $W_r < 0.05$ ) and has almost no change or increase a little with the increasing of  $S_o$  when the gravity parameter is large ( $W_r > 1.0$ ) (see Fig.9). When comparing with the uncracked rotor case, it is found that the change of crack ridge zone with  $S_o$  is not due to the crack itself, but due mainly to the influence of  $S_o$  on the stability degree of uncracked system.

Although the crack ridge zones almost do not change with the fixed Sommerfeld number  $S_o$ , the crack unstable zones become a little bit larger as  $S_o$  increases. This is also due to the change of the stability degree of uncracked system with  $S_o$ .

Fig.10 shows the influences of crack stiffness change ratio on the stability and stability degree. It is found again that the position of the crack ridge zones depend not on the types of journal bearings and, the crack unstable zones locate always within the crack ridge zones, but the stability outside the crack ridge zones depends mainly on the type of bearings used. Although the positions of crack ridge zones do not change with gravity parameter  $W_r$  and mass ratio  $\alpha$ , the crack unstable zones depend both on the stiffness change ratio and the system parameters. When  $W_r$  is small, there is an extra crack unstable zones in high speed ratio due to the influence of mass ratio. When  $W_r = 0.1$ , almost no crack ridge zones exist, because the system is now in the valley of the Floquet eigenvalue. So that, as it is mentioned before, the crack has almost no influence on system's stability in some combinations of system parameters. As  $W_r$  increases, the crack ridge zones appear again, but the crack ridge zone in high speed ratio is disappeared because now the mass ratio has no more influence on the crack ridge zones.

It is found that for tilting pad bearing, the crack ridge in small gravity parameter case is much higher and therefore is more unstable than in large gravity parameter case.

## CONCLUSIONS

This paper investigates the stability and the stability degree of a flexible cracked rotor supported on different kinds of journal bearings. The influences of the crack stiffness ratio  $\Delta K_c$ , the fixed Sommerfeld number  $S_o$ , the gravity parameter  $W_r$  and the mass ratio  $\alpha$  are analysed. It is found that no matter what kind of journal bearings is used, the position of crack ridge zones and then the position of crack unstable zones depend only on the speed ratio and the stiffness change ratio for large gravity parameter ( $W_r > 1.0$ ), and depend also on the mass ratio for small gravity parameter ( $W_r < 0.05$ ). The position and extent of the crack ridge zones are given by Eq.21 for  $W_r > 1.0$  and by Eq.22 for  $W_r < 0.05$ . When  $0.1 < W_r < 1.0$ , there is a region, where no unstable zones caused by rotor crack exist.

The larger the speed ratio and the stiffness change ratio are, the wider the crack ridge zone is. Therefore, the instability caused by crack in larger speed ratio is more dangerous.

Outside the crack ridge zones, the stability and stability degree of the cracked system are almost the same with those of the uncracked system, that is, they have almost no relation with the crack and depend mainly on system's parameters; while within the crack ridge zones, they depend mainly on crack stiffness change ratio, but also on system's parameters. In some cases, the system may still be stable even the crack is very large.

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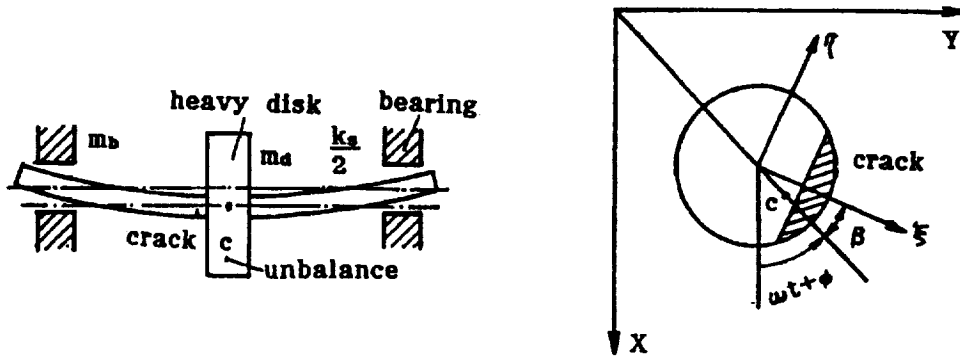
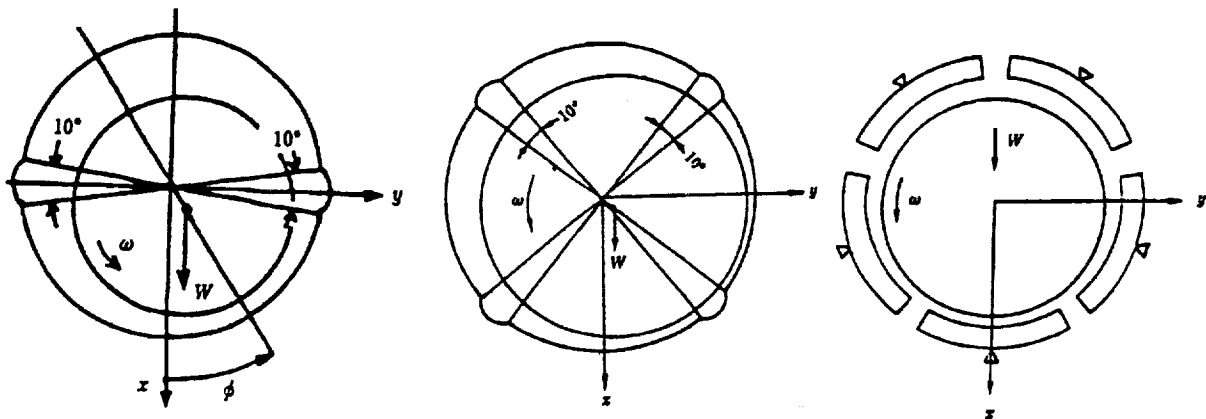


Fig.1 Schematic diagram of a cracked flexible rotor supported on journal bearings



Cylindrical, with  
two axial grooves  
( $L/D=0.5$ )

4 lobes  
( $L/D=0.5$ ,  $m_p=1/2$ )

Tilting pad, 5 pads  
(LOP,  $L/D=0.5$ ,  $m_p=0$ )

Fig.2 Bearing geometry

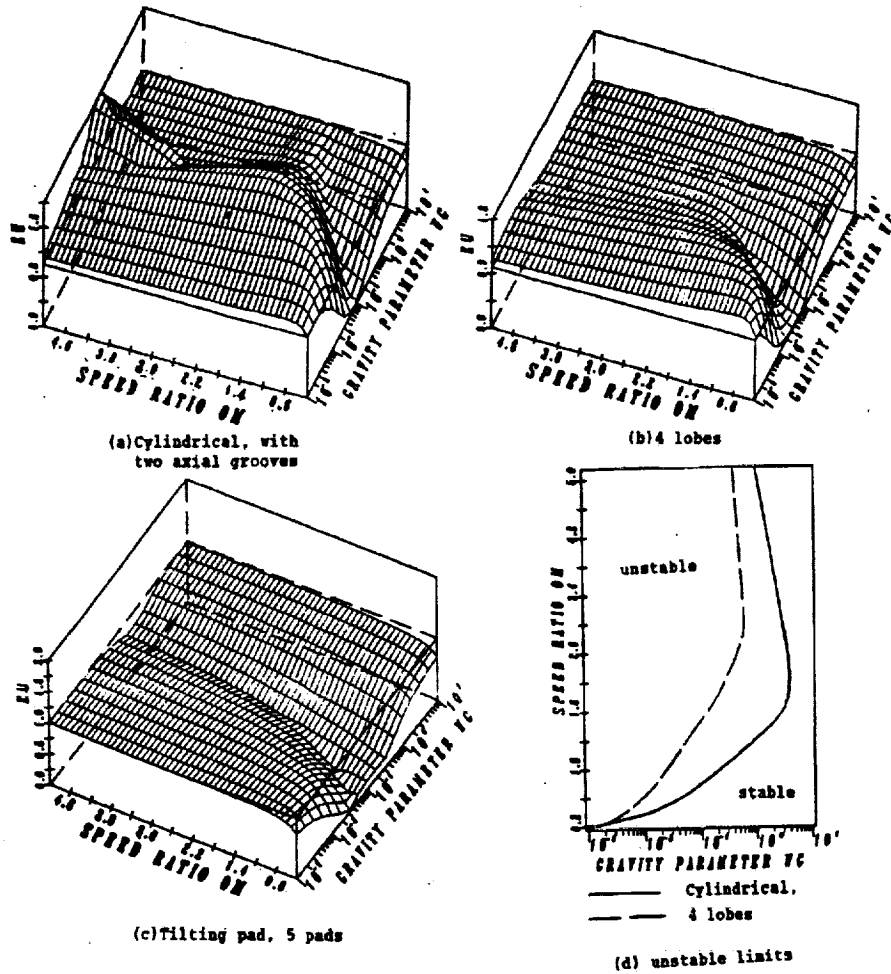
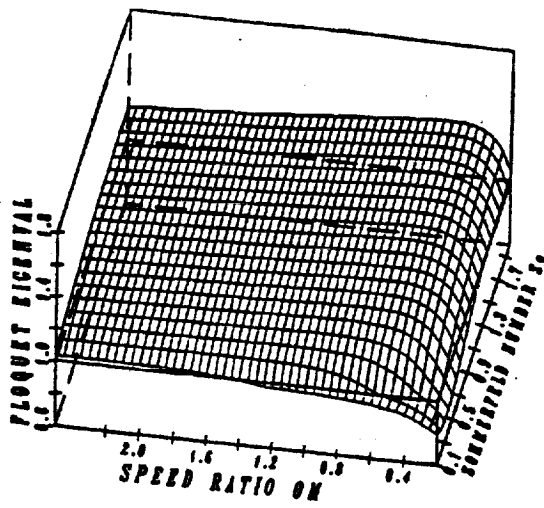
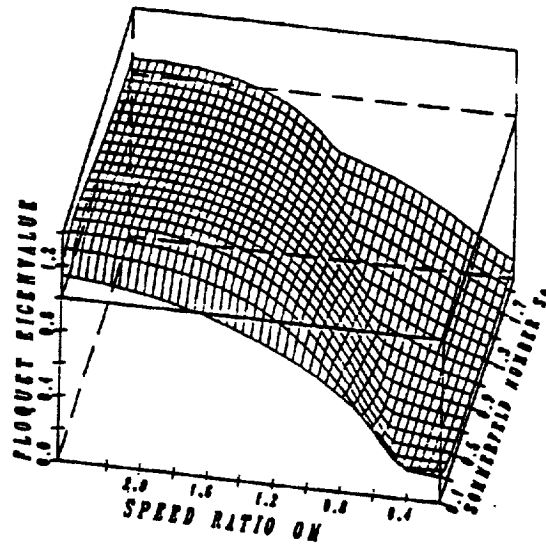


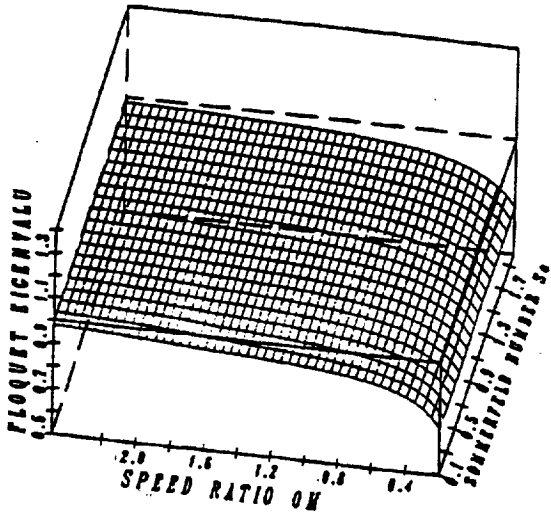
Fig.3 Influence of gravity parameter on the diagram of stability degree, without crack case ( $\Delta K_z = 0.0, \Delta K_y = 0.0, D_s = 0.01, S_s = 2.0, \alpha = 0.2$ )



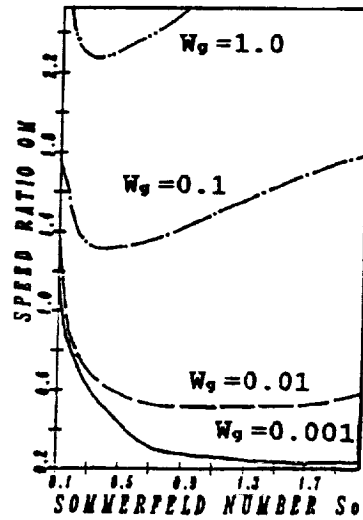
(a)  $W_g = 0.001$



(b)  $W_g = 0.1$



(c)  $W_g = 5.0$



(d) unstable limits

Fig.4 4—Lobes bearing, influence of gravity parameter and fixed Sommerfeld number on the diagram of stability degree, without crack case ( $\Delta K_c = 0.0$ ,  $\Delta K_p = 0.0$ ,  $D_s = 0.01$ ,  $\alpha = 0.2$ )

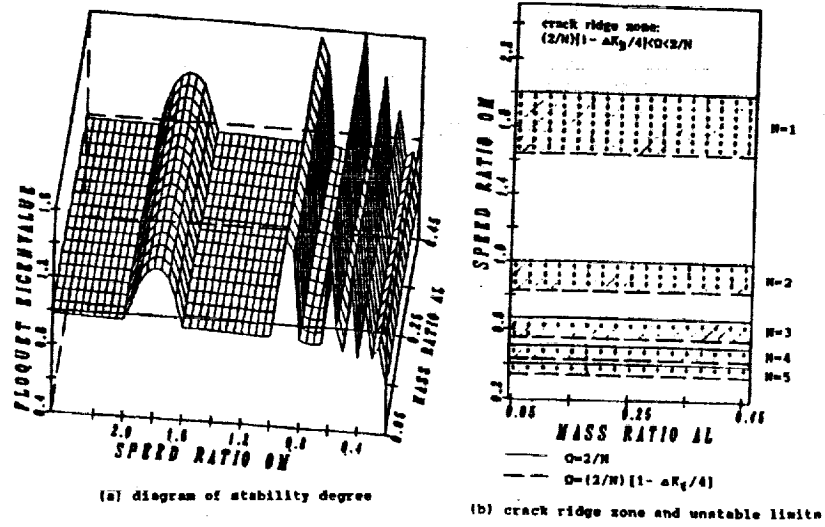


Fig.5 Position of crack ridge zones, influence of mass ratio for large gravity parameter (4-lobes bearing,  $\Delta K_c=0.7$ ,  $\Delta K_g=0.0$ ,  $D_c=0.01$ ,  $S_o=0.1$ ,  $W_g=5.0$ )

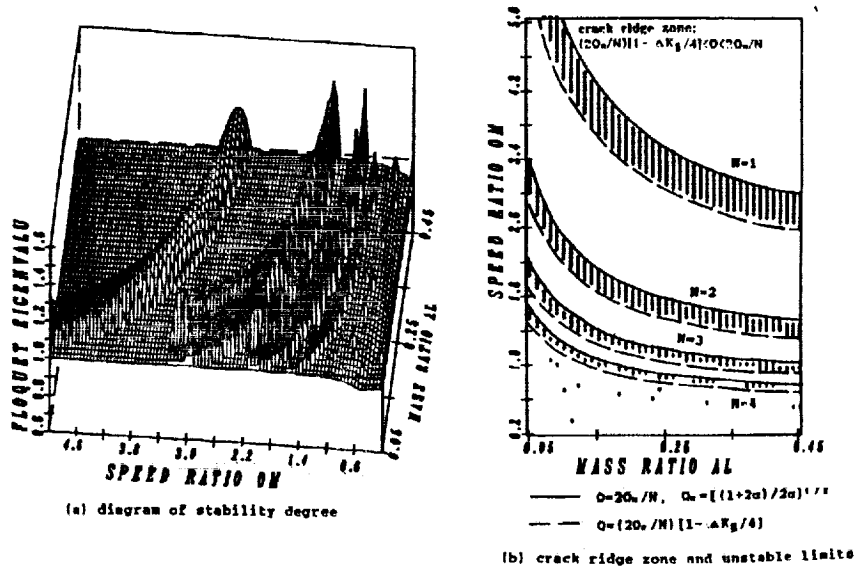


Fig.6 Position of crack ridge zones, influence of mass ratio for small gravity parameter (Tilting pad bearing,  $\Delta K_c=0.5$ ,  $\Delta K_g=0.0$ ,  $D_c=0.01$ ,  $S_o=0.1$ ,  $W_g=0.001$ )



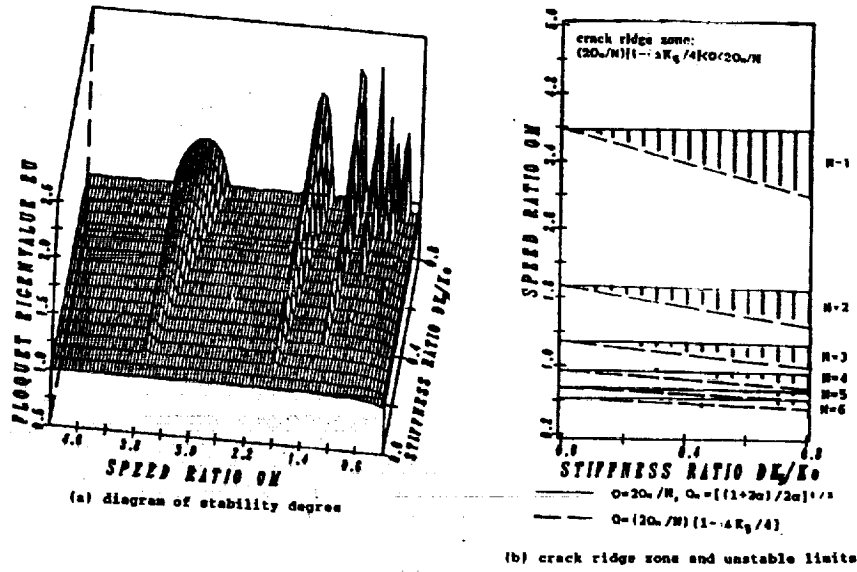


Fig.7 Position of crack ridge zones, influence of stiffness ratio (Tilting pad bearing,  $\Delta K_y = 0.0, D_e = 0.01, S_0 = 0.1, \alpha = 0.2, W_g = 0.001$ )

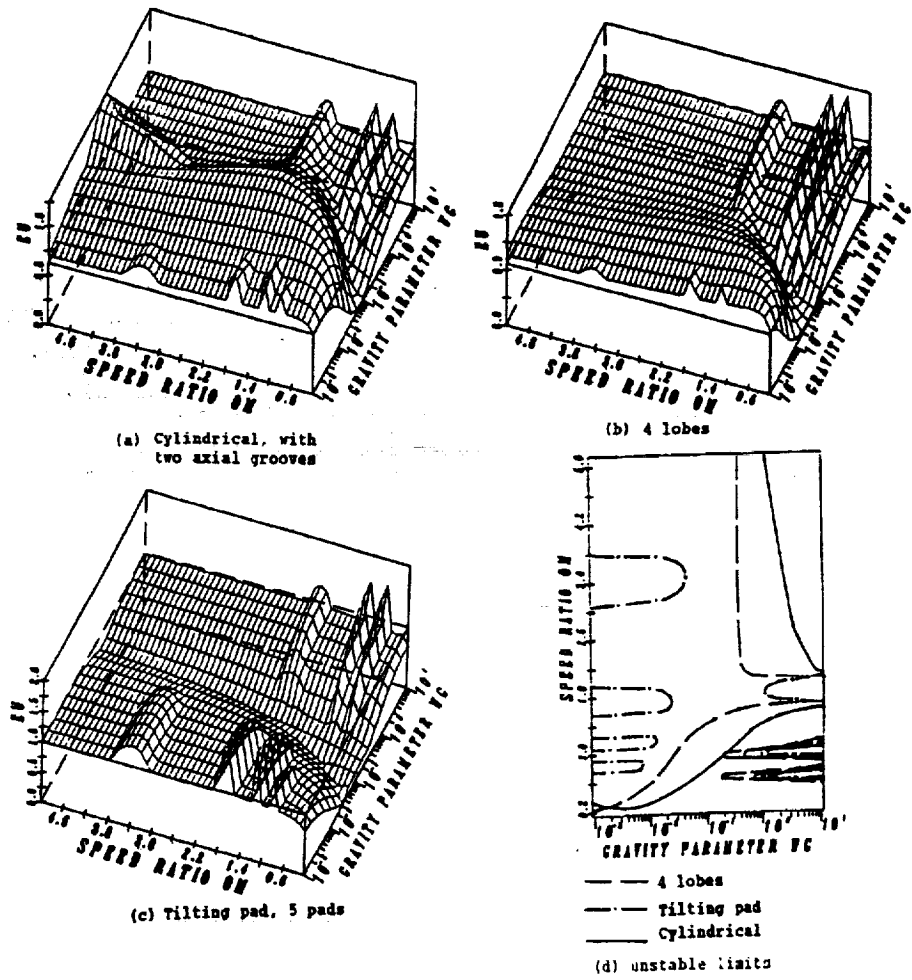


Fig.8 Influence of gravity parameter on the diagram of stability degree ( $\Delta K_z = 0.7, \Delta K_y = 0.0, D_e = 0.01, S_0 = 2.0, \alpha = 0.2$ )

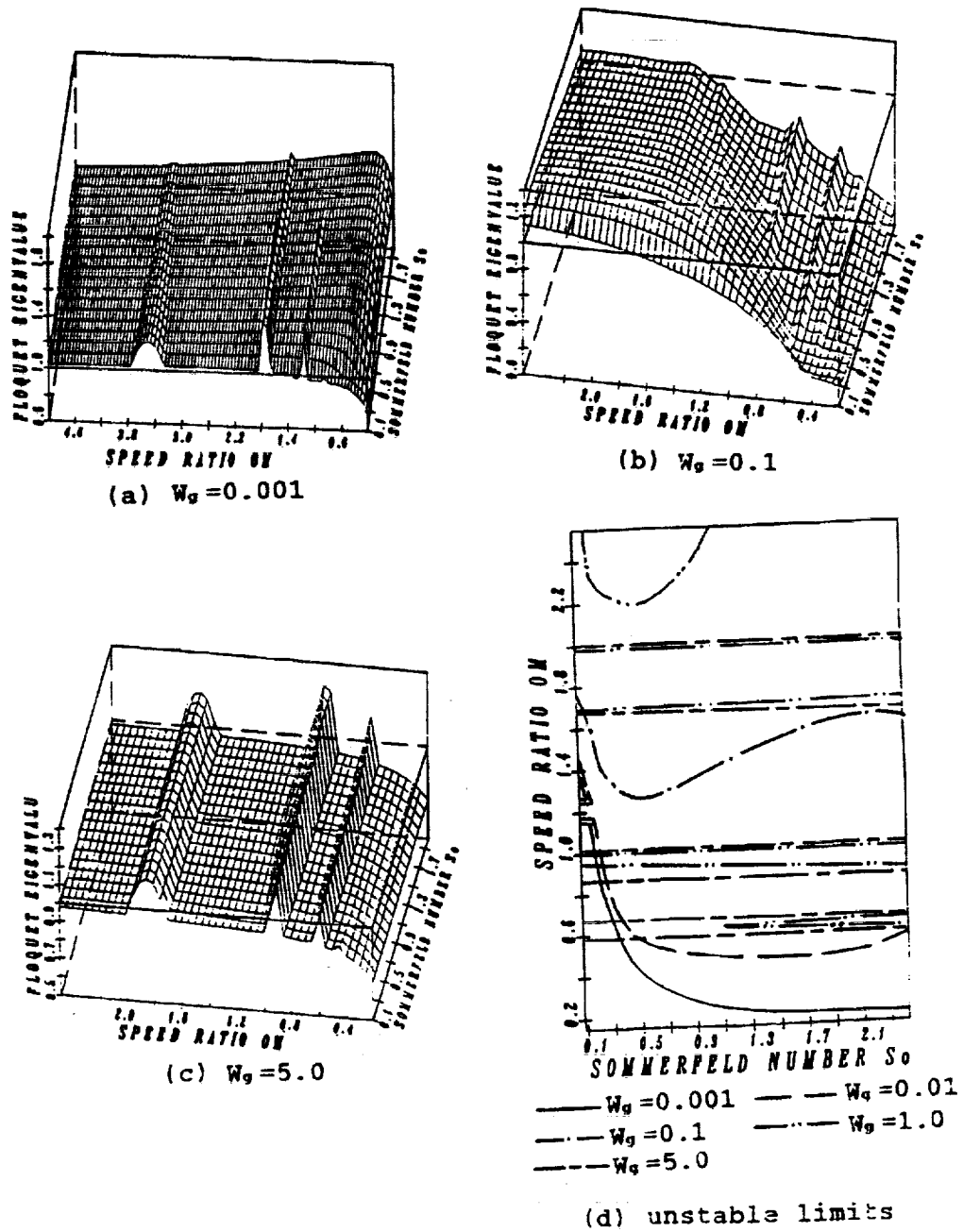


Fig.9 4-Lobes bearing, influence of fixed Sommerfeld number and gravity parameter on the diagram of stability degree ( $\Delta K_t = 0.5$ ,  $\Delta K_y = 0.0$ ,  $D_e = 0.01, \alpha = 0.2$ )

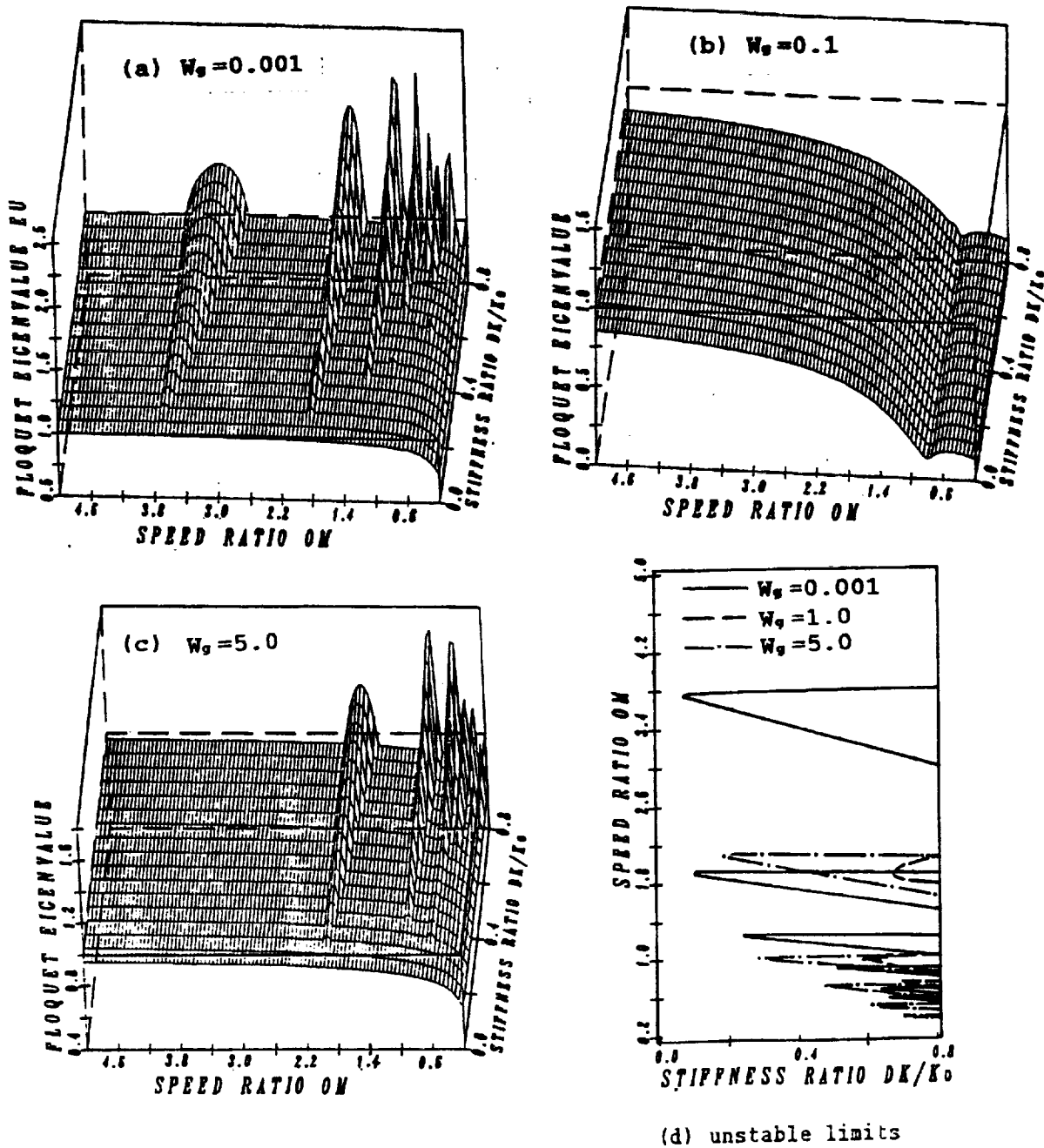


Fig.10 Tilting pad bearing, influence of stiffness change ratio and gravity parameter on the diagram of stability degree ( $\Delta K_q = 0.0$ ,  $D_e = 0.01$ ,  $S_o = 2.0$ ,  $\alpha = 0.2$ )

