

Analytic Model of Aurorally Coupled Magnetospheric and Ionospheric Electrostatic Potentials

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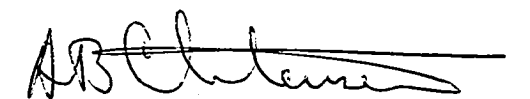
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Analytic Model of Aurorally Coupled Magnetospheric and Ionospheric Electrostatic Potentials

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This paper describes modest but significant improvements on earlier studies of electrostatic potential structure in the auroral region, using the adiabatic auroral arc model. This model has crucial nonlinearities (connected, for example, with aurorally produced ionization) which have hampered analysis; earlier work has either been linear, which I will show is a poor approximation or, if nonlinear, either numerical or too specialized to study parametric dependencies. With certain simplifying assumptions I find new analytic nonlinear solutions fully exhibiting the parametric dependence of potentials on magnetospheric (e.g., cross-tail potential) and ionospheric (e.g., recombination rate) parameters. No purely phenomenological parameters are introduced. The results are in reasonable agreement with observed average auroral potential drops, inverted-V scale sizes, and dissipation rates. The dissipation rate is quite comparable to tail energization and transport rates and should have a major effect on tail and magnetospheric dynamics. This paper gives various relations between the cross-tail potential and auroral parameters (e.g., total parallel currents and potential drops) which can be studied with existing data sets.

1. INTRODUCTION

In this paper I look again at some issues raised over the years concerning the adiabatic auroral arc model [Chiu and Schulz, 1978; Lyons, 1980, 1981; Chiu and Cornwall, 1980; Chiu *et al.*, 1981]. The essence of this model is a linear relation between the auroral parallel current J_{\parallel} and the potential drop $\psi \equiv \psi_i - \psi_e$ between the ionospheric electrostatic potential ψ_i and the equatorial potential ψ_e along an auroral field line (see equation (2) in section 2). In principle, this model can account for all the gross features (e.g., potential drops, auroral size scales, and dissipation rates) of magnetosphere-ionosphere coupling as governed by auroral phenomena, with no adjustable phenomenological parameters or fitting factors, as was pointed out by the author some years ago [Cornwall, 1983]. However, because the auroral model is nonlinear beyond the current-potential relation, it has not yet been possible to give, even in an idealized model, a precise analytic picture of how potential drops and so forth depend on physically determinate parameters such as the polar cap potential drop, ψ_{pc} and the ionospheric recombination rate constant, α . The studies referred to above were either linear or treated nonlinearities numerically for the most part (although Cornwall [1983] did give some nonlinear scaling laws which will be refined here).

In later studies [Cornwall, 1988, 1990], certain exact solutions were found to the nonlinear model equations, but these were too specialized to allow a study of parameter dependence; typically, some combination of independent parameters had to be fixed to allow for a solution. The present investigation is carried out in a similar spirit but with different assumptions and results, which allow most of the interesting physical parameters to be varied freely. In particular, I model the equatorial potential ψ_e as Vasyliunas

[1970] did, which simplifies the nonlinear structure at the price of introducing unphysical discontinuities on the boundary between open and closed field lines. These discontinuities are of little importance to my major results; their primary effect is to make unreliable the detailed shape of $\psi_i - \psi_e$ very near the central auroral field line.

For present purposes one essential nonlinearity of the adiabatic auroral arc model is associated with a density-dependent Pedersen conductivity. (This and other nonlinearities contribute on smaller size scales to auroral structure and instabilities [Cornwall, 1990; Keskinen *et al.*, 1992], which will not concern us here.) It is a strong nonlinearity in the sense that the ionospheric plasma density on the central auroral line can be 10 or more times greater than the unperturbed density. It can, of course, be dealt with numerically, as the early studies referred to previously in this section did, but that is not my purpose here. We will find that it significantly affects the gross size of auroras, increasing their width by a factor of several over those of a linearized analysis [e.g., Chiu *et al.*, 1981; Cornwall, 1983] to a value of several hundred kilometers. A second nonlinearity arises in the relation between the auroral potential drop ψ and the height-integrated ionospheric plasma density, N , which results in nonlinear relations between ψ and ψ_{pc} and between the dissipated auroral power P and ψ_{pc} (the appropriate scaling laws were given earlier by Cornwall [1983]).

Our major results on the relations between the auroral size, ψ , P , and ψ_{pc} are given in equations (43)–(49); with no adjustable parameters, they give values within a factor of 2 or better of observed values for typical values of ψ_{pc} .

These results have interest not only in themselves but also for the future studies they suggest. Section 1 concludes with some remarks in this direction, setting the present work in a larger context involving tail and magnetospheric convection dynamics.

The standard picture of magnetospheric convection is largely based on the work of Vasyliunas [1970] (see also Fejer [1964]; Schield *et al.* [1969] for important precursors). In this picture, magnetic field lines are equipotentials, and the two-dimensional electrostatic potential, mapped to the

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ionosphere, is governed by ionospheric current conservation. The effective height-integrated conductivities receive contributions both from the ionosphere and from the ring current, and there are parallel currents driven by the convection discontinuity at the boundary between open and closed field lines. This picture has been modeled with considerable success in computer simulations [e.g., Wolf, 1970; Wolf *et al.*, 1991].

Of course, field lines in the auroral zone are not equipotentials, as many studies have shown [Evans, 1974; Mizera and Fennell, 1977; Reiff *et al.*, 1988; Lindqvist and Marklund, 1990]. The convection modelers have not yet added the physics discussed in this paper to their codes, in part because it adds substantial complexity and, perhaps, in part because it seems at first glance to have impact localized to the auroral zone.

I suggest that the analytic treatment of this paper could be used, at least at first, as a substitute for adding complex codes to the convection models. My results could be used as phenomenological input to existing model codes. I further suggest that some treatment of auroral dissipation will be essential, if the modelers are ever to hope to capture the main features of tail and magnetospheric dynamics, even in regions remote from the auroral zone. The reason is that auroral-ionospheric Joule dissipation is [Cornwall, 1983] comparable to energization and transport rates in the tail and ring current and so plays an important role in global energy balance. (A similar point was made by W. Liu, as cited by Huang *et al.* [1989].)

Consider the following simple argument. The energy per unit ionospheric area stored in a dipolar flux tube is $\sim L^4 R_E P$, where R_E is the radius of the Earth and $P \sim 4 \times 10^{-9}$ ergs cm^{-3} is a typical plasma sheet pressure, while the Joule dissipation rate $\Sigma_p E_\perp^2$ is of the order of 10 ergs cm^{-2} s or more during an aurora. So the time scale τ on which an aurora could drain this flux tube of energy is

$$\tau \approx \frac{L^4 R_E P}{\Sigma_p E_\perp^2} \sim 10^2 - 10^3 \text{ s} \quad (1)$$

not large compared to tail transport and energization times. (Note that the tail energy is stored mostly in ions, while the aurora transfers energy from the magnetosphere to the ionosphere mostly via energetic electrons.) It has been long known [Erickson and Wolf, 1980; Schindler and Birn, 1982] that there are serious difficulties in creating a steady state loss-free model of tail convection. This suggested the possibility of nonsteady convection as discussed by the above authors or lossy convection (as indicated, for example, by an effective adiabatic index of $\gamma < 1$ [Spence *et al.*, 1987]). An important contributor to loss is cross-tail drift [Kivelson and Spence, 1988; Spence and Kivelson, 1990], and equation (1) suggests that ionospheric Joule dissipation may be just as important.

I am currently in the process of constructing semianalytic models of the effect of auroral dissipation on tail dynamics, which may be of some use. Ultimately, I hope that the computer modelers will take over and vastly improve my simple efforts.

2. FINDING THE ELECTROSTATIC POTENTIAL

It is well known [Vasyliunas, 1970] that the usual sort of two-dimensional electrostatic potential in and near the polar

cap calls for nonvanishing J_\parallel , driven by a nonvanishing divergence of horizontal ionospheric currents. Later, Cornwall [1983] identified these parallel currents with auroral currents according to the standard prescription of the adiabatic auroral model:

$$J_\parallel = -Q(\psi_i - \psi_e) \equiv -Q\psi \quad (2)$$

where $\psi_{i,e}$ is the ionospheric or equatorial potential along a given field line and Q is a parameter of the order of $ne^2(m_e \bar{V}_e)^{-1}$ with n as the plasma sheet electron density and \bar{V}_e as a characteristic plasma sheet electron velocity. This led to the prediction [Cornwall, 1983] that

$$\psi \sim (4 - 6) \frac{\Lambda \psi_{pc}}{R_E} \quad (3)$$

where

$$\Lambda = (\Sigma_p/Q)^{1/2} \sim 100 \text{ km} \quad (4)$$

is the characteristic length scale of the adiabatic auroral arc model and ψ_{pc} is the polar cap potential drop. (Actually, (2) holds only when ψ_{pc} is large enough to give $\psi \geq 1$ kV; for smaller ψ the necessary J_\parallel can be furnished by other means).

The earlier work was largely linear, and even the linear analysis was not carried to completion. In this section I will give the full linear analysis and also an exact nonlinear solution to the model equations. In some respects the linear analysis is not a good approximation to the exact solution (it has the wrong spatial scale), but it can be plausibly fudged to give global results (e.g., the central potential drop) quite similar to the exact analysis. Moreover, it illustrates how I turn the equatorial potential ψ_e , which appears as a source term in the nonlinear equations, into a boundary condition.

The two fundamental equations of the adiabatic auroral arc model are

$$\nabla_\perp \cdot (\Sigma \cdot \nabla_\perp \psi_i) = -J_\parallel = Q\psi \quad (5)$$

$$\frac{\partial N}{\partial t} + V_E \cdot \nabla_\perp N = \frac{\Gamma}{e} Q\psi - \alpha(N^2 - N_o^2) \quad (6)$$

where N is the height-integrated ionospheric plasma density, Σ is the height-integrated ionospheric conductivity tensor, Γ is the number of electron-ion pairs produced per incident auroral electron, α is a height-integrated dissociative recombination coefficient, αN_o^2 summarizes nonauroral sources of ionospheric ionization, and V_E is the electric drift velocity such that

$$V_E = \frac{-c}{B^2} (\nabla \psi \times B). \quad (7)$$

The return current region lying just outside the auroral region is governed by different physics, which I will not attempt to model in detail here (for an earlier attempt, see Chiu *et al.* [1981]). Presumably, $\psi - \psi_e$ goes through zero at the edges of the aurora and then becomes slightly negative in the return current region, but with $|\psi - \psi_e|$ much smaller than in the auroral region. In the simplified analysis I will give, $\psi - \psi_e$ approaches zero only at asymptotically large distances from the zero, and so the return current is also far away. I emphasize that this is a mere artifact, which unless it were mistakenly taken literally, has no effect on our conclusions. In reality, the return current region lies just adjacent to the auroral region, only a few auroral scale lengths, Λ , away.

Equations (5–7) contain a vast number of effects which are readily studied with a computer but whose analytic treatment is either difficult or impossible. These include various instabilities, as well as nonlinearities, the primary effect of which is to increase the density N and conductivity Σ in the auroral region. This conductivity enhancement has a number of important and well-known consequences for convection, which will be studied elsewhere in connection with auroral-magnetospheric coupling. For now, I want to find an analytically treatable treatment of this conductivity nonlinearity in the spirit of earlier [Cornwall, 1988, 1990] analytic solutions to similar auroral problems. These earlier solutions are not useful in the present context, because they do not allow for a priori specification of the input driving potential ψ_e . Fortunately, ψ_e is not completely arbitrary, and progress can be made by modeling ψ_e as Vasyliunas [1970] and Cornwall [1983] did; ψ_e consists of two separate solutions to Laplace's equation (reflecting charge neutrality), joined on a closed contour representing the auroral zone and continuous across this contour. However, $\nabla\psi_e$ is discontinuous across the contour. This discontinuity leads to J_{\parallel} proportional to a Dirac delta function. The input potential ψ_e then only appears as a boundary condition on a homogeneous nonlinear equation, some special solutions to which can be found analytically.

Now for the details. I begin by simplifying the ion conductivity equation (6), dropping the terms on the left-hand side. These terms are primarily important [Cornwall, 1990; Keskinen *et al.*, 1992] for auroral instabilities on time scales of ~ 100 s, but on longer time scales it is a reasonable approximation to assume rough equilibrium between recombination and auroral precipitation. Then (6) yields

$$\psi \equiv \psi_i - \psi_e = \frac{\alpha e}{\Gamma Q} (N^2 - N_o^2) \quad (8)$$

and (5) is rewritten in terms of ψ :

$$\nabla_{\perp} \cdot (\Sigma \cdot \nabla_{\perp} \psi) - Q\psi = -\nabla_{\perp} \cdot (\Sigma \cdot \nabla_{\perp} \psi_e). \quad (9)$$

Under the assumption that the various parameters (α , Γ , N_o) which appear in (8) are constants, the Hall term drops out of the left-hand side of (9), since then both Σ and ψ are functions of the single function N . Thus (9) becomes

$$\nabla_{\perp} \cdot (\Sigma_p \nabla_{\perp} \psi) - Q\psi = -\nabla_{\perp} \cdot (\Sigma \cdot \nabla_{\perp} \psi_e). \quad (10)$$

Now we need to specify the source ψ_e and the right-hand side of (10). In the equipotential case ($\psi = 0$) the right-hand side vanishes almost everywhere except on the convection boundary between open and closed lines, where it is a delta function. I give a simple example of such a source term, based on constant Σ and ψ_e as two different solutions of Laplace's equation, joined at the convection boundary. Use spherical coordinates with the magnetic dipole axis as the z axis and the azimuthal angle ϕ defined to be $\pi/2$ at dawn and $-\pi/2$ at dusk; θ is the magnetic colatitude. Let

$$\psi_e = \frac{1}{2} \psi_{pc} \frac{(\tan \frac{1}{2} \theta) \sin \phi}{\tan \frac{1}{2} \theta_{pc}} \quad \theta < \theta_{pc} \quad (11)$$

$$\psi_e = \frac{1}{2} \psi_{pc} \frac{\sin \psi_{pc} \sin \phi}{\sin \theta} \quad \pi - \theta_{pc} > \theta > \theta_{pc}. \quad (12)$$

(For the southern polar cap, replace θ with $\pi - \theta$ and $\tan \frac{1}{2} \theta_{pc}$ with $\cot \frac{1}{2} \theta_{pc}$ in (11). Here ψ_{pc} is the dawn-to-dusk potential across the polar cap, and the convection boundary is at $\theta = \theta_{pc}$ (or $\theta = \pi - \theta_{pc}$ in the south). Both (11) and (12) satisfy Laplace's equation in the angular variables. The form (11) represents an essentially constant dawn-dusk electric field over the polar cap, as one sees from the stereographic projection of the unit sphere from the south pole to the $x - y$ plane tangent to the north pole, with

$$\begin{aligned} \tan \frac{1}{2} \theta &\rightarrow \rho; \quad x = \rho \cos \phi \\ \phi &\rightarrow \phi; \quad y = \rho \sin \phi. \end{aligned} \quad (13)$$

Then (11), in the projected variables, is $\psi \sim y$, yielding a constant field.

I will now make the idealization in (10) of saving only the most singular term, which is a delta function. It is this which allows further progress to be made, since the nonlinear terms on the left-hand side of (10) depend on ψ_e only through boundary conditions at $\theta = \theta_{pc}$, $\pi - \theta_{pc}$. The result is that (10) becomes, in the northern hemisphere,

$$\begin{aligned} \nabla_{\perp} \cdot (\Sigma_p \nabla_{\perp} \psi) - Q\psi &= \frac{\psi_{pc} \Sigma_p \sin \phi}{2R_E^2 \sin \theta} \\ &\cdot (1 + \cos \theta) \delta(\theta - \theta_{pc}), \end{aligned} \quad (14)$$

where, on the right-hand side, everything depending on θ (including Σ_p) is to be evaluated at $\theta = \theta_{pc}$. I will not explicitly write out the contributions from the southern hemisphere, which are easily supplied by symmetry.

Note that in the idealization of saving only the most singular source term, the specific forms (11) and (12) for ψ_e and the assumption of constant Σ when $\theta \neq \theta_{pc}$ are irrelevant; all that matters is the coefficient of the delta function in (14).

To find a solution to (14) two solutions to the homogeneous version of this equation must be found, matched in value at $\theta = \theta_{pc}$, and the discontinuity in θ derivatives adjusted to match the delta function. The two solutions are chosen so that each decays exponentially with angular distance from θ_{pc} . Far from $\theta = \theta_{pc}$ the conductivity Σ_p approaches, it is assumed, a constant value, at least in the sense that Σ_p varies more slowly (owing, for example, to day-night effects) than it does due to precipitation in the auroral zone. In that case, (14) is a linear equation, straightforwardly solved. Later in this section the linear version of (14) is solved, including the delta function source; this both yields the necessary linear solutions far from the auroral zone and illustrates the matching procedure used to accommodate the delta function.

2.1. Linear Case

The equation to be solved for constant Σ_p is

$$\begin{aligned} \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2} - \lambda^2 \psi \\ = K \frac{\sin \phi}{\sin \theta} \delta(\theta - \theta_{pc}), \end{aligned} \quad (15)$$

where

$$K = \frac{1}{2} \psi_{pc} (1 + \cos \theta_{pc}) \quad (16)$$

$$\lambda = R_E \Lambda = R_E (Q/\Sigma_p)^{1/2}. \quad (17)$$

Since Λ , the width of the (idealized) auroral zone, is about 100 km, $\lambda \gg 1$. We seek a solution of the form

$$\psi = K \sin \phi G(\theta, \theta_{pc}), \quad (18)$$

where $G(\theta, \theta')$ is a Green's function for Legendre's equation with $m = 1$, $l(l+1) = -\lambda^2$. That is,

$$l = -\frac{1}{2} \pm \frac{1}{2} (1 - 4\lambda^2)^{1/2} = -\frac{1}{2} \pm i\lambda. \quad (19)$$

In the limit of large λ the exact solution for G can be formally expressed as a linear combination of the conical functions $P_{-1/2+i\lambda}^1(\cos \theta)$, $Q_{-1/2+i\lambda}^1(\cos \theta)$ [e.g., Abramowitz and Stegun, 1964]. However, it is difficult to find useful forms for these rather obscure special functions, and we choose instead to give an approximate solution in terms of the well-known functions K_1 , I_1 (Hankel and Bessel functions of imaginary argument). For small θ the homogeneous solutions to (15) are $K_1(\lambda\theta)$ and $I_1(\lambda\theta)$, valid when $\lambda\theta^3 \ll 1$. Since $\lambda\theta^3 \gg 1$, these do not cover a useful range in θ . A better approach is to seek homogeneous solutions of the form, for example, $F(\theta)K_1[\lambda R(\theta)]$, determining F and R so that dangerous terms in the differential equation proportional to λ or λ^2 are exactly cancelled. A straightforward computation shows that this requires

$$R = \theta, F = \left(\frac{\theta}{\sin \theta} \right)^{1/2} \quad (20)$$

With this choice the fractional derivation of $FK_1(\lambda R)$ from solving (15) is $\theta^2/36$ for small θ and $O(\lambda^{-2})$ for large θ .

It is easy to find the appropriate linear combinations to match the delta function in (15), with the final result that the solution to the linear problem of (15) is

$$\psi = -\frac{K \sin \phi}{2} \left(\frac{\theta \theta_{pc}}{\sin \theta \sin \theta_{pc}} \right)^{1/2} [K_1(\lambda\theta)I_1(\lambda\theta_{pc})H(\theta - \theta_{pc}) + I_1(\lambda\theta)K_1(\lambda\theta_{pc})H(\theta_{pc} - \theta)], \quad (21)$$

where $H(x)$ is the unit step function ($H(x) = 1, x > 0; = 0, x < 0$). This potential falls off exponentially (roughly as $\exp[-\lambda|\theta - \theta_{pc}|]$) on both sides of θ_{pc} . Note that $\lambda\theta_{pc} \gg 1$, so that in the vicinity of the auroral zone one should use large-argument asymptotic expansions to evaluate the Bessel functions. When this is done, one finds for the potential drop at $\theta = \theta_{pc}$, $\sin \phi = -1$:

$$\psi_c \equiv \psi(\theta = \theta_{pc}) = \frac{\Lambda \psi_{pc}}{4R_E \sin \theta_{pc}} = \Lambda E_{\perp}. \quad (22)$$

In extending this linear solution to a nonlinear solution the point is that Λ depends on Σ_p and thus on the height-integrated plasma density N , which by (8) depends on ψ . A simple-minded approach to this nonlinearity is to use (8) to express N (and Σ_p) in terms of ψ , and then (22) becomes an algebraic equation for ψ_c :

$$\psi_c = \frac{\sin \phi \psi_{pc}}{4R_E \sin \theta_{pc}} \left(\frac{1}{Q} \frac{\partial \Sigma_p}{\partial N} \right)^{1/2} \left(N_o^2 + \frac{\Gamma Q \psi_c}{\alpha e} \right)^{1/4}. \quad (23)$$

(In the limit $N_o \rightarrow 0$ the scaling law $\psi_c \sim \psi_{pc}^{4/3}$ was already given by Cornwall [1983].) We will compare this equation later to the nonlinear solution developed next.

2.2. Nonlinear Case

The nonlinearities are important only in the immediate vicinity of the auroral zone, which has a small width compared to R_E . Therefore I will use the flat earth approximation in the equations, with the x coordinate perpendicular to the auroral zone and y along it. The inhomogeneous delta function such as in (14) need not be written explicitly, since its only role is to furnish a boundary condition at the auroral zone. Equation (8) is used to eliminate ψ in favor of N , assuming that N_o and $\alpha/\Gamma Q$ in this equation are constants. With the observation that Σ_p is linear in N , so that $\partial \Sigma_p / \partial N$ is also constant, the homogeneous version of (10) is

$$\nabla_{\perp} \cdot (N \nabla_{\perp} N^2) - Q(\partial \Sigma_p / \partial N)^{-1} (N^2 - N_o^2) = 0 \quad (24)$$

or

$$\nabla_{\perp}^2 n^3 - \frac{3}{2\Lambda_o^2} (n^2 - 1) = 0 \quad (25)$$

$$\Lambda_o^2 = \frac{\Sigma_{po}}{Q}, \Sigma_{po} = \Sigma_p(N = N_o), n = \frac{N}{N_o}. \quad (26)$$

Nonlinear equations similar to (24) have been studied before analytically [Cornwall, 1988, 1990] and numerically [Keskinen et al., 1992] with full two-dimensional dependence. However, the known two-dimensional analytic solutions are too restricted for the present purpose, and my present strategy is to save in (24) and (25) only gradients in the x direction (across the aurora, roughly north-south). Thus I am not modeling auroral blobs and transient (i.e., unstable) local structures. In any event, in the absence of structural magnetospheric forcing the gradient structure which is more or less persistent is the x direction gradient which is saved. There are also examples [Cornwall, 1988] of Kelvin-Helmholtz stable fully two-dimensional structures in which the x and y direction length scales are essentially the same and roughly equal to the scale length I will find.

So I will ignore the y variation and replace (25) with

$$F'' - \frac{3}{2} \Lambda_o^{-2} (F^{2/3} - 1) = 0, \quad (27)$$

where $F = n^3$ and prime indicate $\partial/\partial x$. This equation can be reduced to quadratures; I quote the solution in terms of the original variable n :

$$\int_n^{n_c} \frac{dn n^2}{(n-1)} \left[\left(n + \frac{2}{3} \right)^3 + \frac{10}{27} \right]^{-1/2} = \pm \frac{x}{5^{1/2} \Lambda_o}. \quad (28)$$

This integral supplies the boundary conditions that at $x = 0$, $n = n_c \equiv N_c/N_o$, where N_c is the central ionospheric density in the auroral zone; typically, $n_c \gg 1$. Another constant of integration has been supplied so that as $|x| \rightarrow \infty$, $n \rightarrow 1$ at an exponential rate:

$$n \rightarrow 1 + C_1 \exp(-x/\Lambda_o) + O[\exp(-2x/\Lambda_o)]. \quad (29)$$

We will soon see that the constant C_1 in (29) is determined by n_c , which is in turn determined by matching a discontinuity in $n'(x=0)$ to $J_{||}$, as in the linear case.

The integral in (28) can be reduced to a combination of integrals of rational functions plus an elliptic integral of the third kind [e.g., *Whittaker and Watson*, 1952] and so can be said to yield an exact solution in terms of known functions, at least in principle. But this form is of little use, because it gives x in terms of n while we want n in terms of x . Nevertheless, for the dedicated reader who wishes to pursue it, I quote the change of variables which yields a standard elliptic integral form:

$$u = \frac{n + \frac{2}{3} + \frac{1}{3} 10^{1/3}(1 - 3^{1/2})}{n + \frac{2}{3} + \frac{1}{3} 10^{1/3}(1 + 3^{1/2})}. \quad (30)$$

To determine the constant C_1 , one may use (29) to replace x/Λ_o in (28) to find

$$\ln C_1 = \lim_{n \rightarrow 1} 5^{1/2} \int_n^{n_c} \frac{dn n^2}{(n-1)} \left[\left(n + \frac{2}{3} \right)^3 + \frac{10}{27} \right]^{-1/2} + \ln(n-1). \quad (31)$$

By adding and subtracting an integral, which can be explicitly done and which removes the singularities in the integrand of (31) at both endpoints, one can find

$$C_1 = \frac{4}{5} \left(\frac{\beta-1}{\beta+1} \right) \exp [2(\beta-1) + \gamma] \quad (32)$$

where

$$\beta = (5n_c - 4)^{1/2} \quad (33)$$

$$\begin{aligned} \gamma(n_c) = 5^{1/2} \int_1^{n_c} \frac{dn}{n-1} \left\{ n^2 \left[\left(n + \frac{2}{3} \right)^3 + \frac{10}{27} \right]^{-1/2} - \left(n - \frac{4}{5} \right)^{1/2} \right\} \\ = \gamma(n_c = \infty) + \frac{6}{5} (5^{1/2}) n_c^{-1/2} + O(n_c^{-1}) \end{aligned} \quad (34)$$

$$\gamma(n_c = \infty) = -2.50$$

where the last form of (34) is useful for large n_c .

Rather than continue with analyzing the integral (28), one finds a more useful form of the solution for large and small $|x|$ by direct solution of the differential equation. For small $|x|$ this amounts to an expansion in powers of $n_c^{-2} \ll 1$ and leads to

$$\begin{aligned} n = n_c(1 \pm ax)^2 \left[1 + \frac{5}{9} n_c^{-2}(1 \pm ax)^{-4} \right. \\ \left. - \frac{50}{567} n_c^{-4}(1 \pm ax)^{-8} + \dots \right] \end{aligned} \quad (35)$$

where

$$a = [2\Lambda_o(5n_c)^{1/2}]^{-1} \quad (36)$$

and the upper sign in (35) is for $x < 0$ and the lower sign for $x > 0$. This form of the solution is useful out to $a|x| \approx 1 - n_c^{-1/2}$, where $n = O(1)$.

For large $|x|$ a similar analysis leads to

$$\begin{aligned} n = 1 + C_1 e^{\pm x/\Lambda_o} - (7/6)[C_1 \exp(\pm x/\Lambda_o)]^2 + (101/48) \\ \cdot (C_1 e^{\pm x/\Lambda_o})^3 + \dots \end{aligned} \quad (37)$$

with the same sign convention as in (36). Here C_1 is the same constant as in (32). Note that it is qualitatively (but not necessarily quantitatively) accurate to estimate C_1 by finding a point x_1 where the slopes and values of the large $|x|$ and small $|x|$ forms of n in (35)–(37) match. Just saving the first nontrivial terms yields

$$x_1 = (20)^{1/2} \Lambda_o (n_c^{1/2} - C_2^{1/2}) \equiv (20)^{1/2} (\Lambda_c - C_2^{1/2} \Lambda_o) \quad (38)$$

$$C_1 = (C_2 - 1) \exp(x_1/\Lambda_o) \quad (39)$$

$$C_2 = \frac{1}{10} (11 + 21^{1/2}) = 1.56. \quad (40)$$

It follows that for large n_c (either from the above equations or from the more accurate (32)), the transition from the nonlinear regime (40) to the linear regime (saving only the first term of (37)) sets in at a value of x of the order of $5\Lambda_c$, where Λ_c is the auroral scaling length $(\Sigma_p/Q)^{1/2}$ evaluated at the center of the auroral zone. In effect, x_1 is the distance from the center of the aurora to the unperturbed ionosphere, and is several times larger than one would have supposed from a linear analysis.

The scaling $x_1 \sim (20)^{1/2} \Lambda_c$ is one of several key results following from the nonlinear analysis. The second result is a formula for the central density N_c or, equivalently, for $\psi_c \equiv \psi(x=0)$ using (8). It is derived as follows: First, the inhomogeneous term for the nonlinear equation (27) is supplied from the fundamental equation (14), using $x = R_E(\theta_{pc} - \theta)$ and the relation between ψ and N in (8). The result for the duskside aurora where $\sin \phi = -1$ is

$$\begin{aligned} (n^3)^n - \frac{3}{2\Lambda_o^2} (n^2 - 1) = - \left(\frac{3\Gamma Q}{2\alpha e N_o^2 \Sigma_{po}} \right) \\ \cdot \left(\frac{\psi_{pc} \Sigma_{pc} (1 + \cos \theta_{pc})}{2R_E \sin \theta_{pc}} \right) \delta(x) \equiv -\Delta \delta(x). \end{aligned} \quad (41)$$

One inserts the solution (35) to find

$$12an_c^3 \left(1 + \frac{5}{9} n_c^{-2} - \frac{125}{567} n_c^{-4} + \dots \right) = \Delta. \quad (42)$$

It turns out that $n_c \gg 1$, so only the first term of this expansion is saved. Then, using the definition of a in (36) and rearranging, one finds

$$N_c^{3/2} = \frac{5^{1/2} \Gamma}{8\alpha e} \left(Q \frac{\partial \Sigma_p}{\partial N} \right)^{1/2} \frac{\psi_{pc} (1 + \cos \theta_{pc})}{R_E \sin \theta_{pc}}. \quad (43)$$

Putting in nominal ionospheric numbers ($\Gamma = 100$, $\alpha = 10^{-14} \text{ cm}^2 \text{ sec}^{-1}$, $Q = 0.1 \text{ cm}^{-1} \text{ sec}^{-1}$, and $\partial \Sigma_p / \partial N = 3 \text{ cm}^3 \text{ sec}^{-1}$, one finds

$$N_c = 3 \times 10^{12} \text{ cm}^{-2} \left(\frac{\psi_{pc}}{60 \text{ kV}} \right)^{2/3}. \quad (44)$$

(I use 60 kV as a reference polar cap potential, because a similar value is associated with the threshold for substorm occurrence [see *Weimer et al.*, 1992; *Ahn et al.*, 1992].)

The third key result expresses the central potential drop ψ_c in terms of the polar cap potential drop, Q , and ionospheric parameters and is just a combination of (8) and (43):

$$\begin{aligned} \psi_c &= \frac{\alpha e}{\Gamma Q} (N_c^2 - N_o^2) = \frac{\alpha e N_c^2}{\Gamma Q} \\ &= \frac{5^{2/3}}{16} \left(\frac{\Gamma}{Q \alpha e} \right)^{1/3} \left(\frac{\partial \Sigma_p}{\partial N} \right)^{1/3} \left[\frac{\psi_{pc} (1 + \cos \theta_{pc})}{R_E \sin \theta_{pc}} \right]^{4/3}. \end{aligned} \quad (45)$$

Using the same constants as before

$$\psi_c = 3.4 \text{ kV} \left(\frac{\psi_{pc}}{60 \text{ kV}} \right)^{4/3}. \quad (46)$$

This result can be compared to the $N_o = 0$ version of (23), an equation based on a simple-minded fudging of the linear theory. One can easily check that (23) in the $N_o = 0$ limit gives (45) except that the constant factor $5^{2/3} \approx 0.183$ in (45) is replaced by $4^{-4/3} \approx 0.157$. The agreement is remarkably close.

A final key result is the auroral dissipation of power, integrated over the auroral zone. I will define a theoretically useful, if not immediately observationally relevant, excess dissipation power P as that due to the field $E_{\perp} = -\nabla_{\perp} \psi$ associated with the difference ($\psi = \psi_i - \psi_e$) between the ionospheric potential with and without auroras. So

$$P = 2R_E \sin \theta_{pc} \int dx d\phi \Sigma_p (\nabla_{\perp} \psi)^2 \quad (47)$$

where the factor of 2 counts both polar caps and the integrals over dx (distance across the aurora) and $d\phi$ extend over the auroral zone. To be definite, I will use for the x dependence of N and ψ the first term in (35), valid for $n_c \gg 1$, integrate in x over $-a^{-1} < x < a^{-1}$, assume that N and ψ depend on $\sin \phi$ as their dependence on ψ_{pc} would suggest, that is, $\psi \sim |\sin \phi|^{4/3}$ and $N \sim |\sin \phi|^{2/3}$, and integrate ϕ over the region $\sin \phi \leq 0$. The result is

$$P = \left(\frac{5}{64} \right) \frac{\Gamma(\frac{1}{2})\Gamma(\frac{11}{3})}{\Gamma(\frac{25}{6})} \left(\frac{\Gamma}{\alpha e} \right) \left(\frac{\partial \Sigma_p}{\partial N} \right)^2 \frac{\psi_{pc}^3 (1 + \cos \theta_{pc})^2}{R_E^2 \sin^2 \theta_{pc}} \quad (48)$$

$$P = 10^{17} \text{ ergs sec}^{-1} \left(\frac{\psi_{pc}}{60 \text{ kV}} \right)^3. \quad (49)$$

This is substantial dissipation, quite enough to influence substorm and tail processes as discussed in connection with (1) in section 1.

I am unaware of any studies in the literature which deal directly with the main results in (44), (45), and (46). But it is clear that the predicted numbers are, in an average sense, in reasonable agreement with auroral observations [e.g., *Evans*, 1974; *Reiff*, 1988; *Lindqvist and Marklund*, 1990; *Chiu et al.*, 1982].

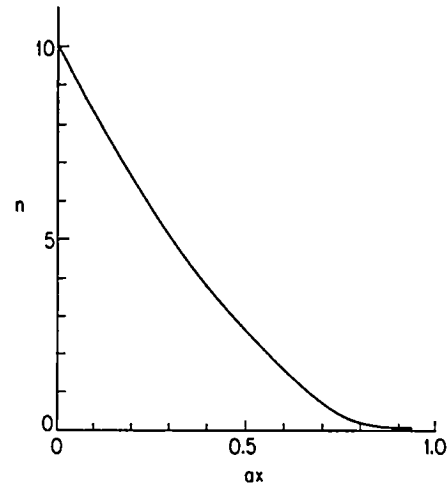


Fig. 1. Scaled density $n = N/N_o$ versus scaled length ax (for a , see (36)), for $n_c = 10$.

3. CONCLUDING REMARKS

My main results are (36), giving the inverted-V scale length a^{-1} as $\sqrt{20}\Lambda_c$, where Λ_c is the linear scale length (see (4)), based on the central Pedersen conductivity, and (43)–(49), giving N_c , ψ_c , and the dissipated power P . The scale length is rather bigger than one would have estimated from the linear theory in section 2, and so the linear profiles are not very good approximations. However, the ad hoc equation (23), which grafts certain elements of the nonlinear theory on to the linear analysis, gives a value for ψ_c in good agreement with the full nonlinear analysis. This analysis gives values for N_c , ψ_c , and P (with no arbitrary parameters), which are in decent agreement with observed values, and yields scaling laws for these quantities in their dependence on ψ_c which can be experimentally studied.

One cannot expect fully realistic spatial profiles of ψ and N near the center of the auroral zone, because my fundamental hypothesis for the magnetospheric potential ψ_e (equations (11) and (12)) yields cusps in ψ and N at $\theta = \theta_{pc}$ or $x = 0$. These are illustrated in Figures 1 and 2, in which N and ψ are plotted for a typical value of $n_c = 10$. In principle, N and ψ should approach $x = 0$ with zero slope, since these are symmetric around $x = 0$. There is no reason to distrust these profiles when the density and potential have dropped by, say, a factor of 2, and the dissipated power P is an integral over the profiles which is insensitive to the cusp behavior. The cusps will, of course, be removed by a smoother choice of ψ_e , but precisely how to make this choice is not presently clear.

Having analytic and fully parametrized expressions for, for example, the dissipated power P (see (48)) will be very useful for theoretical investigations of tail transport and dynamics. As mentioned earlier, this power is comparable to that lost in cross-tail drift, and can have a fundamental impact on the picture of disturbed tail phenomena. Investigations to be reported later are now underway on this subject.

It is, of course, important to know whether the predictions of this paper concerning the relation between ψ_{pc} and various auroral phenomena are borne out by data. The

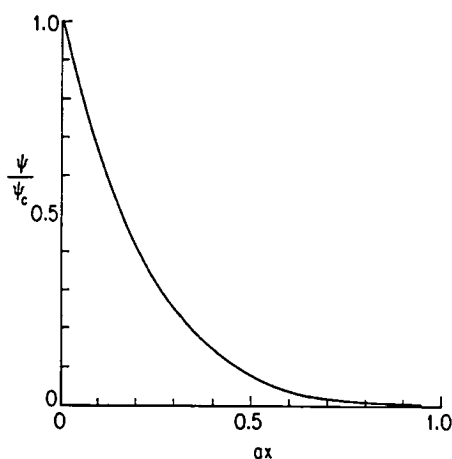


Fig. 2. Ionosphere-equatorial potential difference ψ , scaled by the central value ψ_c (for which, see (45) and (46)), for $n_c = 10$.

material necessary for such a study, in the form of convection field data (yielding ψ_{pc}), field-aligned current data, and field-aligned potential drops, all exists and needs to be correlated. The intervals for which all these data are simultaneously available is by no means negligible. A study of this sort would be of great interest quite aside from the present theoretical considerations.

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