DISTRIBUTED PARAMETER MODELING OF REPEATED TRUSS STRUCTURES

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Abstract

A new approach to find homogeneous models for beam-like repeated flexible structures is proposed which conceptually involves two steps.

Step one: Approximation of 3-D non-homogeneous model by a 1-D periodic beam model. The structure is modeled as a 3-D non-homogeneous continuum. The displacement field is approximated by Taylor series expansion. Then, the cross sectional mass and stiffness matrices are obtained by energy equivalence using their additive properties. Due to the repeated nature of the flexible bodies, the mass and stiffness matrices are also periodic. This procedure is systematic and requires less dynamics detail.

Step two: Homogenization from 1-D periodic beam model to 1-D homogeneous beam model. The periodic beam model is homogenized into an equivalent homogeneous beam model using the additive property of compliance along the generic axis. The major departure from previous approaches in literature is using compliance instead of stiffness in homogenization. An obvious justification is that the stiffness is additive at each cross section but not along the generic axis. The homogenized model preserves many properties of the original periodic model.

1   Introduction

As the number of repeated cells in a truss structure increases, the 3-D model can be approximated better and better by an equivalent 1-D model. The repeated structure then can be modeled as a homogeneous anisotropic continuum beam. The parameters of the continuum beam are functions of the element properties of the truss structure.

Finding the 1-D homogeneous anisotropic beam model from the reference model, the 3-D non-homogeneous anisotropic model, of the truss structure may be referred to as a homogenization process, of which there are many examples.

The approach presented here follows that of Noor's [1, 2] and Lions's [3]. See [3] for mathematical details of the homogenization process, where some results are taken by our paper for granted. The Noor's method is a direct averaging method, which justifies equivalence in the sense of equal kinetic energy and potential energy under the condition of equal nodal displacements and velocities. It imposes a kinematic assumption on the displacement field, then averages the stiffness and mass matrices (by FEM) over a repeated cell. Although the stiffness matrix is additive at each cross-section, it is not along the generic axis. Thus, this method always gives higher stiffness than it should be. This shortcoming will be overcome by our approach.

Our approach consists of two steps, as illustrated in Figure (2). The first step deals with the approximation of 3-D non-homogeneous model of a repeated structure by the 1-D periodic beam model. The second step then homogenized it to a 1-D homogeneous beam model. The 3-D non-homogeneous model is a collection of the Eulerian Equation of Motion of each element of the structure, and is referred to as a reference model for the successive approximation. By applying the Taylor series expansion and energy equivalence, a 1-D periodic beam model is found systematically. Solid beam is used to clarify the basic idea, then an extension from solid beam to non-solid structure(eg. lattice structure) is presented in section (5).
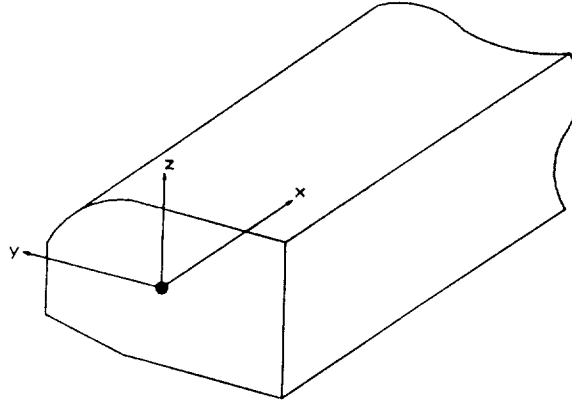

Figure 1: Anisotropic Beam

Consider a structure constructed by linear elastic anisotropic materials as shown in figure (1). The material coordinates attached have x-axis as the generic axis along the centroids, and y-z as the principal axes of the area inertia of cross sections. This choice of reference will be adopted throughout the paper.

Let the bounded open set $\Omega \subset R^3$ denote the space occupied by the structure and Γ the boundary of Ω. Let U, V, and W be the displacements in x, y and z direction, respectively, measured w.r.t. the natural state (undeformed position), observed from inertial coordinates, and represented in the material coordinates. The equation of motion [4, 5] is

$$
\begin{aligned}
\rho \ddot{U} &= \sigma_{xx,x} + \sigma_{yx,y} + \sigma_{zx,z} + f_x \\
\rho \ddot{V} &= \sigma_{xy,x} + \sigma_{yy,y} + \sigma_{zy,z} + f_y \qquad \text{in } \Omega \\
\rho \ddot{W} &= \sigma_{xz,x} + \sigma_{yz,y} + \sigma_{zz,z} + f_z
\end{aligned}
\tag{1}
$$

with well-posed initial and boundary conditions to render existence of unique solution, where

$$
\rho = \rho(x, y, z) \in L^\infty(\Omega)
$$

and f(x,y,z) is the external body force. The constitutive law is

$$
\sigma = C^0 e \tag{2}
$$

and

$$
\sigma = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{bmatrix}, \quad
e = \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} U_{,x} \\ V_{,y} \\ W_{,z} \\ V_{,z} + W_{,y} \\ U_{,z} + W_{,x} \\ U_{,y} + V_{,x} \end{bmatrix}
\tag{3}
$$

where σ denotes stresses ,e strains and C^0 a real symmetric positive-definite matrix with

$$
c_{ij}^0 = c_{ij}^0(x, y, z) \in L^\infty(\Omega)
$$

Here equation (1) is taken as the reference model of the beam. Our task is to approximate the 3-spatial-dimensional (3-D) equation (1) by a 1-spatial-dimensional (1-D) beam eq. to arbitrary accuracy of the displacement field. Instead of going through the term by term scrutinizing as in solid continuum mechanics, we provide a unified and systematic approach. This will insight the general pattern and properties of the 1-D beam eq.

The final goal is the capability of modeling repeated truss structure as a 1-D beam. The properties of repeated truss structure, though non-homogeneous (i.e. $\rho = \rho(x,y,z)$), are periodic along the generic axis x. A homogenization process then is needed as will be described in section 8.

2 Justification of Taylor's Expansion

Before applying the Taylor approximation in the next section, let's justify its applicability to our problem first. Let $\mathcal{H} = L^2(\Omega)$ be real valued Hilbert space and

$$\vec{U}(t,x,y,z) = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} U \\ V \\ W \end{bmatrix}$$

Let the equation of motion be written as, by equation (1),

$$\rho\ddot{\vec{U}} + \mathcal{L}\vec{U} = f \qquad in \; \Omega \tag{4}$$

with well-posed homogeneous boundary conditions specified by forces or displacements in Γ, and

$$u_i(0,x,y,z) \in \mathcal{V}, \quad \dot{u}_i(0,x,y,z) \in \mathcal{H} \quad where \mathcal{V} = \mathcal{H}^1(\Omega)$$

If all coefficients are in $L^\infty(\Omega)$ and the strain energy associated with $\mathcal{L}$ is positive definite, then there exists a unique solution

$$u_i \in L^2((0,T);\mathcal{V}) \quad with \quad u_i' \; and \; \dot{u}_i \; in \; L^2((0,T);\mathcal{H})$$

Proposition 1

$$\exists \quad H_n X_n \to \vec{U} \qquad strongly \; in \; [L^2((0,T);\mathcal{V})]^3$$

with X_n the solution of $M_n\ddot{X}_n + A_n X_n = f_n$, where
$H_n = H_n(y,z) \quad X_n = X_n(t,x) \in [L^2((0,T);\tilde{V})]^n$

$$(H_n X_n)' \to \vec{U}' \qquad in \quad [L^2((0,T);\mathcal{H})]^3 = \mathcal{W}_1$$

$$H_n \dot{X}_n \to \dot{\vec{U}} \qquad in \quad \mathcal{W}_1$$

X_n' and $\dot{X}_n \in [L^2((0,T);\tilde{H})]^3 = \mathcal{W}_2$, with
$\tilde{H} = L^2(0,L) \quad \tilde{V} = \tilde{H}^1 \quad and$

$$M_n(x) = \int\int_A H_n^*(y,z)\rho(x,y,z)H_n(y,z)dy\,dz$$

$$C_n(x) = \int\int_A T_n^*(y,z)C_o(x,y,z)T_n(y,z)dy\,dz$$

$$f_n(t,x) = \int\int_A H_n^*(y,z)f(t,x,y,z)dy\,dz$$

$$[C_n\epsilon_X^n, \epsilon_\Psi^n]_{\mathcal{W}_2} = [A_n X_n, \Psi_n]_{\mathcal{W}_2}$$

Proof:

For each $u_i(t, x, y, z)$

$$
\begin{array}{llllll}
\exists & u_{im} & \longrightarrow & u_i & \text{strongly in} & L^2((0,T); \mathcal{V}) \\
& u'_{im} & \longrightarrow & u'_i & \text{strongly in} & L^2((0,T); \mathcal{H}) \\
& \dot{u}_{im} & \longrightarrow & \dot{u}_i & \text{strongly in} & L^2((0,T); \mathcal{H})
\end{array}
$$

where u_{im} defined in $(0, T)$ is analytic function in y and z. The strong convergence is guaranteed since analytic functions are dense in $L^2((0,T); \mathcal{V})$

Let

$$
\begin{array}{lll}
u^{(1)}(a, b) & = & \displaystyle\sum_{i=y,z} D_i u(a) \, b_i \\[2em]
u^{(2)}(a, b) & = & \displaystyle\sum_{i=y,z} \sum_{j=y,z} D_{i,j} u(a) \, b_i b_j \\[2em]
u^{(3)}(a, b) & = & \displaystyle\sum_{i=y,z} \sum_{j=y,z} \sum_{k=y,z} D_{i,j,k} u(a) \, b_i b_j b_k
\end{array}
\tag{5}
$$

$$
\vdots
$$

By Taylor's theorem

$$
\begin{array}{lll}
u_{im} & = & \displaystyle\sum_{k=0}^{n} \frac{1}{k!} \, u_{im}^{(k)}((t, x, 0, 0), (t, x, y, z)) + H.O.T. \\[2em]
& = & H_n(y, z) X_{imn}(t, x) + H.O.T.
\end{array}
\tag{6}
$$

By strong convergence of the Taylor series

$$
\begin{array}{llllll}
H_n X_{imn} & \longrightarrow & u_{im} & \text{strongly in} & L^2((0,T); \mathcal{V}) \\
(H_n X_{imn})' & \longrightarrow & u'_{im} & \text{strongly in} & L^2((0,T); \mathcal{H}) \\
H_n \dot{X}_{imn} & \longrightarrow & \dot{u}_{im} & \text{strongly in} & L^2((0,T); \mathcal{H})
\end{array}
$$

Therefore, in general, we have

$$
H_n X_n \longrightarrow \vec{U} \qquad \text{strongly in} \quad [L^2((0,T); \mathcal{V})]^3
\tag{7}
$$

where
$H_n = H_n(y, z)$ $X_n = X_n(t, x)$ and $X_n \in [L^2((0,T); \tilde{V})]^n$

$$
(H_n X_n)' \longrightarrow \vec{U}' \qquad \text{in} \quad \mathcal{W}_1
$$

$$
H_n \dot{X}_n \longrightarrow \dot{\vec{U}} \qquad \text{in} \quad \mathcal{W}_1
$$

Also,

$$T_n \epsilon_n \to \epsilon \quad \text{by} \quad (H_n X_n)' \to \vec{U}'$$

Therefore,

$$[\rho \ddot{\vec{U}} + \mathcal{L}\vec{U}, \Phi]_{\mathcal{W}_1} = [f, \Phi]_{\mathcal{W}_1} \qquad \forall \quad \Phi \in [L^2((0,T); \mathcal{V})]^3$$

$$\Rightarrow -[\rho \dot{\vec{U}}, \dot{\Phi}]_{\mathcal{W}_1} + [C^0 c_U, c_\Phi]_{\mathcal{W}_1} = [f, \Phi]_{\mathcal{W}_1}$$

or,

$$\lim_n \{ -[\rho H_n \dot{X}_n, H_n \dot{\Psi}_n]_{\mathcal{W}_1} + [C_o T_n \epsilon_X^n, T_n \epsilon_\Psi^n]_{\mathcal{W}_1} \} = [f, H_n \Psi_n]_{\mathcal{W}_1}$$

$$\lim_n \{ -[H_n^* \rho H_n \dot{X}_n, \dot{\Psi}_n]_{\mathcal{W}_2} + [T_n^* C_o T_n \epsilon_X^n, \epsilon_\Psi^n]_{\mathcal{W}_2} \} = [H_n^* f, \Psi_n]_{\mathcal{W}_2}$$

Using $M_n(x), C_n(x) and f_n(t,x)$ defined before, we have

$$\lim_n \{ -[M_n \dot{X}_n, \dot{\Psi}_n]_{\mathcal{W}_2} + [C_n \epsilon_X^n, \epsilon_\Psi^n]_{\mathcal{W}_2} \} = [f_n, \Psi_n]_{\mathcal{W}_2}$$

$$\lim_n [M_n \ddot{X}_n + A_n X_n, \Psi_n]_{\mathcal{W}_2} = [f_n, \Psi_n]_{\mathcal{W}_2} \tag{8}$$

Equations (5) and (6) imply

$$H_n X_n \to \vec{U} \quad \text{with} \quad X_n \text{ the solution of } M_n \ddot{X}_n + A_n X_n = f_n$$

This completes the proof and justifies the applicability of Taylor's expansion.

3   Taylor Series Approximation

Let cross sections with concentrated forces be taken as boundary sections (boundary points in beam equations) and local effects of applied forces be neglected. Assume the physical displacements **U, V and W** are analytic in y and z so that the Taylor series expansion is applicable. We apply the Taylor series expansion, using equations (5) and (6), at each cross section x and any time t to have

$$U(t,x,y,z) = U + \frac{\partial U}{\partial y} y + \frac{\partial U}{\partial z} z + \frac{\partial^2 U}{\partial y^2} \frac{y^2}{2} + \frac{\partial^2 U}{\partial z^2} \frac{z^2}{2} + \frac{\partial^2 U}{\partial y \partial z} yz + H.O.T. \tag{9}$$

where all terms on the RHS are evaluated at (t,x,0,0). Similar equations can be written down for V(t,x,y,z) and W(t,x,y,z).

The displacement field can well be approximated by a few dominant terms for most physical beams. The generalized displacements of the beam eq. can be chosen by order of magnitude analysis. For example, for an 8-generalized-displacement (8-d) beam eq., we choose

$$X = \begin{bmatrix} u \\ v \\ w \\ \phi_1 \\ \phi_2 \\ \phi_3 \\ \bar{u} \\ \epsilon_{23} \end{bmatrix} = \begin{bmatrix} U \\ V \\ W \\ \frac{1}{2}\left(\frac{\partial W}{\partial y} - \frac{\partial V}{\partial z}\right) \\ \frac{\partial U}{\partial z} \\ -\frac{\partial U}{\partial y} \\ \frac{\partial^2 U}{\partial y \partial z} \\ \frac{1}{2}\left(\frac{\partial W}{\partial y} + \frac{\partial V}{\partial z}\right) \end{bmatrix}_{(t,x,0,0)} \tag{10}$$

We will call X the generalized displacements .

The approximation of the displacement field up to the specified accuracy then is, by equations (9) and (10),

$$
\begin{aligned}
U(t,x,y,z) &\doteq u - \phi_3 y + \phi_2 z + \bar{u} yz \\
V(t,x,y,z) &\doteq v + (\epsilon_{23} - \phi_1)z \\
W(t,x,y,z) &\doteq w + (\epsilon_{23} + \phi_1)y
\end{aligned}
\tag{11}
$$

or

$$\begin{bmatrix} U \\ V \\ W \end{bmatrix} \doteq HX \tag{12}$$

with cross sectional shape function H=H(y,z) found from equation (11). The equation (11) is known as kinematic assumption in Structural Dynamics, viewed as a polynomial approximation to the displacement field.

Since dynamic eq. of the beam is completely characterized by mass and constitutive properties, the approximation of 3-D eq. by the 1-D beam is equivalent to transforming the point properties to sectional properties, ie, from mass density ρ and constitutive matrix C^0 to mass matrix M and stiffness matrix C , respectively. The mass inertia (M) and stiffness (C) have additive property at any given cross-section; therefore we can find M and C by approximating the cross-sectional kinetic and potential energy, respectively. This additive property justifies the validity of domain extension from material-domain to structure-domain.

4 Energy Equivalence Method

We can find the sectional mass matrix by approximating the kinetic energy using equations (10) and (12). The kinetic energy of a piece of the beam (between any two cross sections) is

$$KE = \frac{1}{2}\int\int\int \rho(\dot{U}^2 + \dot{V}^2 + \dot{W}^2)dA\,dx$$

$$\doteq \frac{1}{2} \int \int \int \rho \dot{X}^T H^T H \dot{X} dA \, dx$$

$$= \frac{1}{2} \int \dot{X}^T M \dot{X} \, dx$$

It can be shown that, for anti-symmetric mass distribution,

$$M = \int \int \rho H^T H \, dA = \{m_{ij}(x)\}$$

$$= \begin{bmatrix}
m_{11} & \cdot & \cdot & \cdot & \cdot & \cdot & -m_{56} & \cdot \\
 & m_{11} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & m_{11} & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & & m_{55}+m_{66} & \cdot & \cdot & \cdot & m_{66}-m_{55} \\
 & & & & m_{55} & m_{56} & m_{57} & \cdot \\
 & & & & & m_{66} & m_{67} & \cdot \\
 & s & y & m & & & m_{77} & \cdot \\
 & & & & & & & m_{55}+m_{66}
\end{bmatrix} \quad (13)$$

The M in eq. (13) is the most general pattern of mass matrix for anti-symmetric anisotropic 8-d beams. The first (6 × 6) part of M is the general pattern of mass matrix for Timoshenko beams.

We can also find the sectional stiffness matrix C by approximating the potential energy of the beam. From equations (2) and (11), we have

$$e = \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{bmatrix} \doteq \begin{bmatrix} u' - \phi_3' y + \phi_2' z + \bar{u}' yz \\ 0 \\ 0 \\ 2\epsilon_{23} \\ \phi_2 + w' + (\bar{u} + \epsilon_{23}' + \phi_1')y \\ -\phi_3 + v' + (\bar{u} + \epsilon_{23}' - \phi_1')z \end{bmatrix} = T\epsilon \quad (14)$$

where T=T(y,z) is a (6 × 9) matrix ;

$$\epsilon = \begin{bmatrix} u' \\ v' - \phi_3 = \gamma_{12} \\ w' + \phi_2 = \gamma_{13} \\ \phi_1' \\ \phi_2' \\ \phi_3' \\ \bar{u}' \\ \bar{u} + \epsilon_{23}' \\ 2\epsilon_{23} = \gamma_{23} \end{bmatrix} = \begin{bmatrix} X' \\ \cdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -\phi_3 \\ \phi_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ \bar{u} \\ 2\epsilon_{23} \end{bmatrix}$$

$$= KX' + GX \quad (15)$$

$$G = \begin{bmatrix} 0_{3\times3} & L_1 & 0_{3\times2} \\ 0_{3\times3} & 0_{3\times3} & 0_{3\times2} \\ & & 0 \quad 0 \\ 0_{3\times3} & 0_{3\times3} & 1 \quad 0 \\ & & 0 \quad 2 \end{bmatrix}_{9\times8}$$

$$L_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} I_{8\times8} \\ 0_{1\times8} \end{bmatrix}_{9\times8}$$

The potential energy of a piece of the beam is

$$PE = \frac{1}{2} \int\int\int e^T C^0 e \, dA \, dx$$
$$= \frac{1}{2} \int\int\int \epsilon^T T^T C^0 T \epsilon \, dA \, dx$$
$$= \frac{1}{2} \int \epsilon^T C \epsilon \, dx$$

The stiffness matrix for anti-symmetric of c_{ij}^0 is

$$C = \int\int T^T C^0 T \, dA = \{c_{ij}(x)\}$$

$$= \begin{bmatrix}
c_{11} & c_{12} & c_{13} & \cdot & \cdot & \cdot & c_{17} & \cdot & c_{19} \\
 & c_{22} & c_{23} & \cdot & \cdot & \cdot & c_{27} & \cdot & c_{29} \\
 & & c_{33} & \cdot & \cdot & \cdot & c_{37} & \cdot & c_{39} \\
 & & & c_{44} & c_{45} & c_{46} & c_{47} & c_{48} & \cdot \\
 & & & & c_{55} & c_{56} & c_{57} & c_{58} & \cdot \\
 & & & & & c_{66} & c_{67} & c_{68} & \cdot \\
 s & y & m & & & & c_{77} & c_{78} & c_{79} \\
 & & & & & & & c_{88} & \cdot \\
 & & & & & & & & c_{99}
\end{bmatrix} \tag{16}$$

The stiffness matrix C in eq. (16) is the most general pattern for anti-symmetric anisotropic 8-d beams. The first (6×6) part is that of Timoshenko beams. Knowing the general patterns of M and C is very useful, especially in assuming the model structure in system identification. Since most of the truss structures ever built are at least cross-sectional anti-symmetric, we consider this case only hereafter.

5   Extension to Lattice Structures

For non-solid beams, we need to apply the concept of domain extension, from material-domain to structure-domain, so that the results in the above sections can

be applied. Let's take a rectangular lattice structure as an example. The space physically occupied by the structure material is called the material-domain. The smallest simple-connected rectangular space enclosing the structure is called the structure-domain(Ω), which includes the space not occupied by the structure material(Ω_e).

A displacement field is assumed for the space not occupied by the structure material so that the displacement field on structure-domain is in $\mathcal{H}^1(\Omega)$. Therefore, the Taylor series expansion and energy equivalence method for calculating sectional properties can be applied directly. The sectional properties shall not be affected by the introducing of the displacement field in Ω_e, since both the kinetic energy and potential energy are zero in Ω_e. We can then pretend we are dealing with a solid flexible structure in regular shape.

6 Generalized Beam Equations

The governing eq. can be found from integrating by parts of potential energy.

$$
\begin{aligned}
2PE &= \int \epsilon^T C(x)\epsilon\, dx = \int \epsilon^T F\, dx \\
&= \int (KX' + GX)^T F\, dx \\
&= \int \{[-KX]^T F' + (GX)^T F\}\, dx + (KX)^T F\big|_0^\ell \\
&= \int X^T[-K^T F' + G^T F]\, dx + X^T (K^T F)\big|_0^\ell
\end{aligned}
$$

The dynamic eq. in force-acceleration form is

$$
M(x)\ddot{X} - K^T F' + G^T F = 0 \tag{17}
$$

or

$$
M(x)\ddot{X} = K^T F' - G^T F =
\begin{bmatrix}
N' \\
Q'_{12} \\
Q'_{13} \\
M'_t \\
M'_2 - Q_{13} \\
M'_3 + Q_{12} \\
M'_{23} - M_4 \\
M'_4 - 2Q_{23}
\end{bmatrix}
\tag{18}
$$

The above two equations are valid for beams which are nonhomogeneous along the generic axis.

From equation (15),

$$
F = C\epsilon = C(KX' + GX)
$$

we have, for a special case of homogeneous beams.

$$-K^T F' + G^T F = -K^T C K X'' - (K^T C G - G^T C K) X' + G^T C G X.$$

The dynamic eq. in mass-stiffness form is

$$M \ddot{X} - K^T C K X'' - (K^T C G - G^T C K) X' + G^T C G X = 0 \qquad (19)$$

where

$$M = \begin{bmatrix} m_{11} & \cdot & \cdot & \cdot & \cdot & \cdot & m_{17} & \cdot \\ & m_{22} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & m_{33} & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & m_{44} & \cdot & \cdot & \cdot & m_{48} \\ & & & & m_{55} & m_{56} & m_{57} & \cdot \\ s & y & m & & & m_{66} & m_{67} & \cdot \\ & & & & & & m_{77} & \cdot \\ & & & & & & & m_{88} \end{bmatrix} \qquad (20)$$

$$K^T C K = \begin{bmatrix} c_{11} & c_{12} & c_{13} & \cdot & \cdot & \cdot & c_{17} & \cdot \\ & c_{22} & c_{23} & \cdot & \cdot & \cdot & c_{27} & \cdot \\ & & c_{33} & \cdot & \cdot & \cdot & c_{37} & \cdot \\ & & & c_{44} & c_{45} & c_{46} & c_{47} & c_{48} \\ & & & & c_{55} & c_{56} & c_{57} & c_{58} \\ & & & & & c_{66} & c_{67} & c_{68} \\ s & y & m & & & & c_{77} & c_{78} \\ & & & & & & & c_{88} \end{bmatrix} \qquad (21)$$

$$K^T C G - G^T C K = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & c_{13} & -c_{12} & \cdot & 2c_{19} \\ & \cdot & \cdot & \cdot & c_{23} & -c_{22} & \cdot & 2c_{29} \\ & & \cdot & \cdot & c_{33} & -c_{23} & \cdot & 2c_{39} \\ & & & \cdot & \cdot & \cdot & c_{48} & \cdot \\ & & & & \cdot & \cdot & c_{58} - c_{37} & \cdot \\ & & & & \cdot & c_{68} + c_{27} & \cdot \\ s & k & e & w & & & \cdot & 2c_{79} - c_{88} \\ s & y & m & & & & & \cdot \end{bmatrix} \qquad (22)$$

$$G^T C G = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & c_{33} & -c_{23} & \cdot & 2c_{39} \\ & & & & & c_{22} & \cdot & -2c_{29} \\ s & y & m & & & & c_{88} & \cdot \\ & & & & & & & c_{99} \end{bmatrix} \qquad (23)$$

with m_{ij} from equation (13) and c_{ij} from equation (16) .

The end boundary conditions are either $X_i = 0$ or $F_i = 0$. There are 2^8 possible combinations, theoretically. For example:$X_b = 0$ for clamped end, $K^T F_b = (K^T C K)X_b' + K^T C G X_b = 0$ for free end.

If there are lumped masses, then the conditions become :

(a) for interior points

$$M_b \ddot{X}_b - (K^T C K) \, \Delta X_b' = 0$$

where

$$\Delta X_b = X(b^+) - X(b^-)$$

(b) for exterior points

$$M_b \ddot{X}_b + sig[(K^T C K)X_b' + (K^T C G)X_b] = 0$$

where

$$sig = \begin{cases} 1, & \text{for positive ends (ends with positive outward normal);} \\ -1, & \text{for negative ends.} \end{cases}$$

and

$$K^T C G = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & c_{13} & -c_{12} & \cdot & 2c_{19} \\ \cdot & \cdot & \cdot & \cdot & c_{23} & -c_{22} & \cdot & 2c_{29} \\ \cdot & \cdot & \cdot & \cdot & c_{33} & -c_{23} & \cdot & 2c_{39} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_{48} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_{58} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_{68} & \cdot \\ \cdot & \cdot & \cdot & \cdot & c_{37} & -c_{27} & c_{78} & 2c_{79} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_{88} & \cdot \end{bmatrix} \tag{24}$$

7 Timoshenko Beams

The Timoshenko beam eq. is obtained by deleting the last two generalized displacements (ie., $\bar{u}$ and ϵ_{23} in X) in equation (19), to have

$$M_t \ddot{X}_t - C_t X_t'' - A_1 X_t' + A_0 X_t = 0 \tag{25}$$

where

$$M_t = \begin{bmatrix} m_{11} & \cdot & \cdot & \cdot & \cdot & \cdot \\ & m_{22} & \cdot & \cdot & \cdot & \cdot \\ & & m_{33} & \cdot & \cdot & \cdot \\ & & & m_{44} & \cdot & \cdot \\ s & y & m & & m_{55} & m_{56} \\ & & & & m_{56} & m_{66} \end{bmatrix}$$

$$C_t = \begin{bmatrix} c_{11} & c_{12} & c_{13} & \cdot & \cdot & \cdot \\ & c_{22} & c_{23} & \cdot & \cdot & \cdot \\ & & c_{33} & \cdot & \cdot & \cdot \\ & & & c_{44} & c_{45} & c_{46} \\ s & y & m & & c_{55} & c_{56} \\ & & & & & c_{66} \end{bmatrix} = \begin{bmatrix} c_1 & 0_{3\times3} \\ 0_{3\times3} & c_2 \end{bmatrix}$$

$$L = \begin{bmatrix} \cdot & \cdot & \cdot & c_{13} & -c_{12} \\ \cdot & \cdot & \cdot & c_{23} & -c_{22} \\ \cdot & \cdot & \cdot & c_{33} & -c_{23} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} = \begin{bmatrix} 0_{3\times3} & c_1 L_1 \\ 0_{3\times3} & 0_{3\times3} \end{bmatrix}$$

$$A_1 = L - L^T$$

$$A_0 = \begin{bmatrix} 0_{3\times3} & \cdot & \cdot & & \cdot \\ & & \cdot & & \cdot \\ & & & c_{33} & -c_{23} \\ sym & & & & c_{22} \end{bmatrix} = \begin{bmatrix} 0_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & L_1^T C_1 L_1 \end{bmatrix}$$

The force boundary conditions are

$$M_b \ddot{X}_b - C_t \, \Delta X_b' - L \, \Delta X_b = F_b \tag{26}$$

and the geometric boundary conditions are X_b = specified value.

8 Multi-scale Averaging Method

For Periodic Beam-like structure, M and C are periodic in x with period ℓ.

$$M = M(x), \quad C = C(x), \quad \text{K and G are constant matrices}$$

The equation of motion of Timoshenko beam from equation (17) in section 6 can be rewritten as

$$M\ddot{X} - (K^T C K X' + K^T C G X)' + G^T C K X' + G^T C G X = f \tag{27}$$

Let

$$B_1 = K^T C K \quad B_2 = K^T C G \quad B_3 = G^T C G$$

We have

$$M\ddot{X} - (B_1 X')' - (B_2 X)' + B_2^T X' + B_3 X = f \tag{28}$$

Let

$$AX = -(B_1 X')' - (B_2 X)' + B_2^T X' + B_3 X \tag{29}$$

We will consider the following structure

$$M\ddot{X} + AX = f \tag{30}$$

$$X(0) = 0 \quad, \quad K^T C K X' + K^T C G X = B_1 X' + B_2 X = 0 \quad, \text{at} \quad x = L$$

$$A = A^*, \quad A > 0$$

Let $s = \frac{x}{\ell} \in R$, $C^\ell(x) = C(\frac{x}{\ell}) = C(s)$, $B_1^\ell(x) = B_1(s)$,
$B_2^\ell(x) = B_2(s)$, $B_3^\ell(x) = B_3(s)$
For $X_i^\ell(t,x) = X_i(t,x,\frac{x}{\ell}) = X_i(t,x,s)$

$$X_i' = \frac{\partial X_i}{\partial x} + \ell^{-1}\frac{\partial X_i}{\partial s} \tag{31}$$

We have

$$\begin{aligned}
AX_i &= -(B_1 X_i')' - (B_2 X_i)' + B_2^T X_i' + B_3 X_i \\
&= \ell^{-2} A_0 X_i + \ell^{-1} A_1 X_i + \ell^0 A_2 X_i
\end{aligned} \tag{32}$$

where

$$A_0 X_i = -\frac{\partial}{\partial s}\left(B_1 \frac{\partial X_i}{\partial s}\right) \tag{33}$$

$$A_1 X_i = -\frac{dB_1}{ds}\frac{\partial X_i}{\partial x} - 2B_1 \frac{\partial^2 X_i}{\partial x \partial s} - (B_2 - B_2^T)\frac{\partial X_i}{\partial s} - \frac{dB_2}{ds} X_i \tag{34}$$

$$A_2 X_i = -B_1 \frac{\partial^2 X_i}{\partial x^2} - (B_2 - B_2^T)\frac{\partial X_i}{\partial x} + B_3 X_i \tag{35}$$

Let [3, 6]

$$\begin{aligned}
X^\ell(t,x) &= X_0 + \ell X_1 + \ell^2 X_2 + \ell^3 X_3 + \cdots \\
X_i &= X_i(t,x,s) \quad i = 0,1,2,\cdots \quad \text{periodic in } s
\end{aligned}$$

$$\begin{aligned}
f &= AX + M\ddot{X} \\
&= \ell^{-2}(A_0 X_0 + \ell A_0 X_1 + \ell^2 A_0 X_2 + \ell^3 A_0 X_3 + \cdots) \\
&+ \ell^{-1}(A_1 X_0 + \ell A_1 X_1 + \ell^2 A_1 X_2 + \ell^3 A_1 X_3 + \cdots) \\
&+ (A_2 X_0 + \ell A_2 X_1 + \ell^2 A_2 X_2 + \ell^3 A_2 X_3 + \cdots) \\
&+ M(\ddot{X}_0 + \ell \ddot{X}_1 + \ell^2 \ddot{X}_2 + \ell^3 \ddot{X}_3) \\
&= \ell^{-2}(A_0 X_0) + \ell^{-1}(A_1 X_0 + A_0 X_1) \\
&+ (A_2 X_0 + A_1 X_1 + A_0 X_2 + M\ddot{X}_0) \\
&+ \ell(A_2 X_1 + A_1 X_2 + A_0 X_3 + M\ddot{X}_1) \\
&+ \ell^2(\cdots)
\end{aligned}$$

The above is valid for all ℓ. Thus, we need

$$A_0 X_0 = 0 \tag{36}$$

$$A_1 X_0 + A_0 X_1 = 0 \tag{37}$$

$$M \ddot{X}_0 + A_2 X_0 + A_1 X_1 + A_0 X_2 = f \tag{38}$$

$$M \ddot{X}_1 + A_2 X_1 + A_1 X_2 + A_0 X_3 = 0 \tag{}$$

$$\vdots$$

Proposition 2

$$X_0 = X_0(t, x) \quad (not \quad depend \quad on \quad s) \tag{39}$$

Proposition 3

$$X_1(t, x, s) = -Y_1 \frac{\partial X_0}{\partial x} - Y_2 X_0 + \tilde{X}_1(t, x) \tag{40}$$

with

$$A_0 Y_1 = -\frac{dB_1}{ds} \tag{41}$$

$$A_0 Y_2 = -\frac{dB_2}{ds} \tag{42}$$

Y_1 and Y_2 are periodic in s since X_1 is. Moreover, Y_1 and Y_2 can be independent of t and x and unique up to a constant additive. Note that solution of Y_1 and Y_2 are guaranteed since

$$\int_{|S|} \frac{dB_1}{ds} ds = 0$$

$$\int_{|S|} \frac{dB_2}{ds} ds = 0$$

Proposition 4

$$
\begin{aligned}
f(x) &= -\frac{1}{|S|} \int_{|S|} (B_1 - B_1 \frac{dY_1}{ds}) ds \, \frac{\partial^2 X_0}{\partial x^2} \\
&\quad - \frac{1}{|S|} \int_{|S|} (B_2 - B_2^T - B_1 \frac{dY_2}{ds} + B_2^T \frac{dY_1}{ds}) ds \, \frac{\partial X_0}{\partial x} \\
&\quad + \frac{1}{|S|} \int_{|S|} (B_3 - B_2^T \frac{dY_2}{ds}) ds X_0 + \frac{1}{|S|} \int_{|S|} M(s) ds \, \ddot{X}_0 \\
&= A_h X_0 + M_h \ddot{X}_0
\end{aligned} \tag{43}
$$

The above is the homogenized eq. found by the multi-scale perturbation method.

Proof:

From eq.(38), to render the sol. of X_2, we need

$$\int_{|S|} (A_2 X_0 + A_1 X_1 + M \ddot{X}_0) ds = \int_{|S|} f(x) ds = |S| f(x) \tag{44}$$

$$A_2 X_0 = -B_1 \frac{\partial^2 X_0}{\partial x^2} - (B_2 - B_2^T) \frac{\partial X_0}{\partial x} + B_3 X_0$$

$$A_1 X_1 = -\frac{dB_1}{ds} \frac{\partial X_1}{\partial x} - 2B_1 \frac{\partial^2 X_1}{\partial x \partial s} - (B_2 - B_2^T) \frac{\partial X_1}{\partial s} - \frac{dB_2}{ds} X_1$$

$$= -\frac{dB_1}{ds} \left(-Y_1 \frac{\partial^2 X_0}{\partial x^2} - Y_2 \frac{\partial X_0}{\partial x} + \frac{\partial \tilde{X}_1}{\partial x} \right)$$

$$- 2B_1 \left(-\frac{dY_1}{ds} \frac{\partial^2 X_0}{\partial x^2} - \frac{dY_2}{ds} \frac{\partial X_0}{\partial x} \right)$$

$$- (B_2 - B_2^T) \left(-\frac{dY_1}{ds} \frac{\partial X_0}{\partial x} - \frac{dY_2}{ds} X_0 \right)$$

$$- \frac{dB_2}{ds} \left(-Y_1 \frac{\partial X_0}{\partial x} - Y_2 X_0 + \tilde{X}_1(x) \right)$$

(Note that terms associated with $\frac{\partial \tilde{X}_1}{\partial x}$ and $\tilde{X}_1(x)$ are 0 after integration.)

$$= \left(2B_1 \frac{dY_1}{ds} + \frac{dB_1}{ds} Y_1 \right) \frac{\partial^2 X_0}{\partial x^2}$$

$$+ \left(\frac{dB_1}{ds} Y_2 + 2B_1 \frac{dY_2}{ds} + (B_2 - B_2^T) \frac{dY_1}{ds} + \frac{dB_2}{ds} Y_1 \right) \frac{\partial X_0}{\partial x}$$

$$+ \left((B_2 - B_2^T) \frac{dY_2}{ds} + \frac{dB_2}{ds} Y_2 \right) X_0$$

$$= (B_1 \frac{dY_1}{ds}) \frac{\partial^2 X_0}{\partial x^2} + (B_1 \frac{dY_2}{ds} - B_2^T \frac{dY_1}{ds}) \frac{\partial X_0}{\partial x} + (-B_2^T \frac{dY_2}{ds}) X_0$$

Collecting terms together, we have eq.(43) immediately.

9   Properties of Homogenized Operator

$$M_h = \frac{1}{|S|} \int_{|S|} M(s) ds \tag{45}$$

$$M(s) > 0, \quad M^*(s) = M(s) \quad \forall s$$

$$[M_h X, X] = \frac{1}{|S|} \int_{|S|} [M(s) X, X] ds > 0 \quad \forall X \neq 0$$

$$\Rightarrow \quad M_h > 0$$

$$M_h^* = \frac{1}{|S|} \int_{|S|} M^*(s)ds = \frac{1}{|S|} \int_{|S|} M(s)ds = M_h$$

$$\Rightarrow \quad M_h \quad \text{self} - \text{adjoint}$$

Consider the special case where $X(t,x) = X_h(t)$

$$KE_c = \int_{|S|} [M_h \dot{X}_h, \dot{X}_h]ds = |S|[M_h \dot{X}_h, \dot{X}_h]$$

$$\doteq KE_d = [M_d \dot{X}_d, \dot{X}_d]$$

Let

$$\dot{X}_d \doteq T_m \dot{X}_h$$

Then

$$M_h = \frac{1}{|S|}(T_m^T M_d T_m) \tag{46}$$

Therefore, equivalent mass matrix found by Noor's method agrees with that by homogenization theory, though not the case in the stiffness matrix.

$$\begin{aligned}
\bar{B}_1 &= \frac{1}{|S|} \int_{|S|} (B_1 - B_1 \frac{dY_1}{ds})ds \\
&= \frac{1}{|S|} \int_{|S|} [B_1 \frac{d}{ds}(Is - Y_1), \frac{d}{ds}(Is)]ds \tag{47}
\end{aligned}$$

By eq.(41)

$$A_0 Y_1 = -\frac{dB_1}{ds}$$

or

$$\int_{|S|} [-\frac{d}{ds}(B_1 \frac{dY_1}{ds}) + \frac{dB_1}{ds}, \Psi]ds = 0 \quad \forall \Psi \in \{H^1 : \text{ periodic}\}$$

$$\Rightarrow \int_{|S|} [B_1 \frac{dY_1}{ds} - B_1, \frac{d\Psi}{ds}]ds = 0$$

$$\Rightarrow \int_{|S|} [B_1 \frac{d}{ds}(Y_1 - Is), \frac{\partial \Psi}{\partial s}]ds = 0 \tag{48}$$

Take $\Psi = Y_1$
from eq.(47) and (48)

$$\bar{B}_1 = \frac{1}{|S|} \int_{|S|} [B_1 \frac{d}{ds}(Is - Y_1), \frac{d}{ds}(Is - Y_1)]ds \tag{49}$$

which is positive-definite and self-adjoint.

$$\begin{aligned}
\bar{B}_2 &= \frac{1}{|S|} \int_{|S|} [(B_2 - B_1 \frac{dY_2}{ds}) - B_2^T(I - \frac{dY_1}{ds})]ds \tag{50} \\
&= \frac{1}{|S|} \int_{|S|} \left([B_1^{-1}(B_2 - B_1 \frac{dY_2}{ds}), B_1(I - \frac{dY_1}{ds})] - B_1^{-1}[B_1(I - \frac{dY_1}{ds}), (B_2 - B_1 \frac{dY_2}{ds})] \right) ds
\end{aligned}$$

$$\bar{B}_3 = \frac{1}{|S|} \int_{|S|} (B_3 - B_2^T \frac{dY_2}{ds}) ds$$

by

$$\int_{|S|} [B_1 \frac{dY_2}{ds} - B_2) \cdot \frac{d\Psi}{ds}] ds = 0 \quad \forall \Psi \in \{H^1 : \text{periodic}\}$$

Take $\Psi = Y_2$, we have

$$\int_{|S|} [(B_1 \frac{dY_2}{ds} - B_2) \cdot \frac{dY_2}{ds}] = 0$$

or

$$\int_{|S|} B_1^{-1}[(B_1 \frac{dY_2}{ds} - B_2) \cdot B_1 \frac{dY_2}{ds}] = 0$$

$$\int_{|S|} B_1^{-1}[(B_1 \frac{dY_2}{ds} - B_2) \cdot (B_1 \frac{dY_2}{ds} - B_2)] = \int_{|S|} [-B_2^T \frac{dY_2}{ds} + B_2^T B_1^{-1} B_2] ds$$

Therefore,

$$\bar{B}_3 = \frac{1}{|S|} \int_{|S|} \left((B_3 - B_2^T B_1^{-1} B_2) + B_1^{-1}[(B_1 \frac{dY_2}{ds} - B_2), (B_1 \frac{dY_2}{ds} - B_2)] \right) ds \quad (51)$$

Let $L_2 = L_2(0, L)$

$$[A_h X_0, X_0]_{L_2} = \frac{1}{|S|} \int_{|S|} [B_1^{-1} V, V]_{L_2} ds$$
$$+ \frac{1}{|S|} \int_{|S|} [(B_3 - B_2^T B_1^{-1} B_2) X_0, X_0] ds \quad (52)$$

where

$$V = B_1 \frac{d}{ds}(Is - Y_1)\frac{\partial X_0}{\partial x} + (B_1 \frac{dY_2}{ds} - B_2) X_0$$
$$A_h = A_h^* \qquad A_h > 0$$

Proposition 5

$$f(x) = -\bar{B}_1 \frac{\partial^2 X_0}{\partial x^2} - \bar{B}_2 \frac{\partial X_0}{\partial x}$$
$$+ \bar{B}_3 X_0 + M_h \ddot{X}_0$$
$$= A_h X_0 + M_h \ddot{X}_0 \quad (53)$$

where $\bar{B}_1$ $\bar{B}_2$ $\bar{B}_3$ M_h are constant matrices given in eq.(49) (50) (51) (45), respectively. Also, A_h and M_h are self-adjoint and positive-definite.

All the above procedures are formal. In general, B's are not differentiable and the differential eq. should be interpreted in the weak sense. Note that it can be shown that the homogenized operators do not depend on B.C.

10 Calculation of Operator's Parameter

By eq.(41)

$$A_0 Y_1 = -\frac{dB_1}{ds}$$

$$\Rightarrow -\frac{d}{ds}(B_1 \frac{dY_1}{ds}) = -\frac{dB_1}{ds} \quad (Y_1 \text{ function of s only})$$

$$\Rightarrow B_1 \frac{dY_1}{ds} = B_1 - C_1 \quad (B_1 > 0 \quad C_1 \text{a constant matrix})$$

$$\Rightarrow \frac{dY_1}{ds} = I - B_1^{-1} C_1$$

$$(Y_1 \text{ periodic}) \Rightarrow \int_{|S|} \frac{dY_1}{ds} ds = 0$$

Therefore,

$$Y_1 \text{ periodic} \Rightarrow \int_{|S|} (I - B_1^{-1} C_1) ds = 0$$

$$C_1 = (\frac{1}{|S|} \int_{|S|} B_1^{-1} ds)^{-1} \tag{54}$$

By eq.(43), therefore,

$$\bar{B}_1 = \frac{1}{|S|} \int_{|S|} (B_1 - B_1 \frac{dY_1}{ds}) ds$$

$$= \frac{1}{|S|} \int_{|S|} C_1 ds$$

$$= C_1$$

$$= (\frac{1}{|S|} \int_{|S|} B_1^{-1} ds)^{-1} \tag{55}$$

$$\bar{B}_1 = \bar{B}_1^*, \qquad \bar{B}_1 > 0, \qquad \bar{B}_1 \leq \frac{1}{|S|} \int_{|S|} B_1 ds$$

From eq.(42)

$$\frac{dY_2}{ds} = B_1^{-1} B_2 - B_1^{-1} C_2$$

$$\int_{|S|} (B_1^{-1} B_2 - B_1^{-1} C_2) ds = 0$$

$$C_2 = (\int_{|2|} B_1^{-1} ds)^{-1} \int_{|S|} B_1^{-1} B_2 ds$$

$$= C_1 \frac{1}{|S|} \int_{|S|} B_1^{-1} B_2 ds$$

Using the above results and eq.(43)

$$
\begin{aligned}
\bar{B}_2 &= \frac{1}{|S|} \int_{|S|} (B_2 - B_2^T - B_1 \frac{dY_2}{ds} + B_2^T \frac{dY_1}{ds}) ds \\
&= \frac{1}{|S|} \int_{|S|} [B_2 - B_2^T - B_1(B_1^{-1}B_2 - B_1^{-1}C_2) \\
&\quad + B_2^T(I - B_1^{-1}C_1)] ds \\
&= \frac{1}{|S|} \int_{|S|} (C_2 - B_2^T B_1^{-1}C_1) ds \\
&= C_2 - \frac{1}{|S|} \int_{|S|} B_2^T B_1^{-1} ds C_1 \\
&= C_1 \frac{1}{|S|} \int_{|S|} B_1^{-1} B_2 ds - \frac{1}{|S|} \int_{|S|} B_2^T B_1^{-1} ds C_1 \quad (56)
\end{aligned}
$$

$$
\bar{B}_2^* = -\bar{B}_2
$$

By (43)

$$
\begin{aligned}
\bar{B}_3 &= \frac{1}{|S|} \int_{|S|} (B_3 - B_2^T \frac{dY_2}{ds}) ds \\
&= \frac{1}{|S|} \int_{|S|} [B_3 - B_2^T(B_1^{-1}B_2 - B_1^{-1}C_2)] ds \\
&= \frac{1}{|S|} \int_{|S|} (B_3 - B_2^T B_1^{-1} B_2) ds \\
&\quad + \frac{1}{|S|} \int_{|S|} [B_2^T B_1^{-1} C_1 \frac{1}{|S|} \int_{|S|} B_1^{-1} B_2 ds] ds \\
&= \frac{1}{|S|} \int_{|S|} (B_3 - B_2^T B_1^{-1} B_2) ds \\
&\quad + (\frac{1}{|S|} \int_{|S|} B_2^T B_1^{-1} ds) C_1 (\frac{1}{|S|} \int_{|S|} B_1^{-1} B_2 ds) \quad (57)
\end{aligned}
$$

Proposition 6

$$
\bar{B}_3^* = \bar{B}_3 \quad \text{and} \quad \bar{B}_3 \geq 0
$$

Proof:

$$
C > 0 \quad \Rightarrow \quad [C(KZ_1 + GZ_2), (KZ_1 + GZ_2] \geq 0 \quad \forall \ Z_1, Z_2
$$

$$
\Rightarrow \left[\begin{pmatrix} K^T C K & K^T C G \\ G^T C K & G^T C G \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}, \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \right] \geq 0 \quad \forall \ Z_1, Z_2
$$

$$
\Rightarrow \begin{pmatrix} B_1 & B_2 \\ B_2^T & B_3 \end{pmatrix} \geq 0, \quad \forall s, \quad \forall \ Z_1, Z_2
$$

60

Take $Z_1 = -B_1^{-1} B_2 Z_2$

$$\left[\begin{pmatrix} B_1 & B_2 \\ B_2^T & B_3 \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}, \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \right] = \left[(B_3 - B_2^T B_1^{-1} B_2) Z_2, Z_2 \right]$$

$$\Rightarrow B_3 - B_2^T B_1^{-1} B_2 \geq 0 \quad \forall \quad s \quad QED.$$

Proposition 7 *The homogenized eq. is*

$$\begin{aligned}
f(x) &= -\bar{B}_1 \frac{\partial^2 X_0}{\partial x^2} - \bar{B}_2 \frac{\partial X_0}{\partial x} \\
&+ \bar{B}_3 X_0 + M_h \ddot{X}_0 \\
&= A_h X_0 + M_h \ddot{X}_0
\end{aligned} \tag{58}$$

where $\bar{B}_1$ $\bar{B}_2$ $\bar{B}_3$ can be calculated directly from the original B's parameters as in eq.(55), (56) and (57) , respectively. Also, A_h is self-adjoint and semi-positive definite.

Proof:
This is a similar result to the Proposition (5). Here we use the numerically calculable equations to prove it. The equations for $\bar{B}$'s have been shown already and the self-adjoint and positiveness are as follows:

$$\begin{aligned}
[A_h X_0, X_0] &= [-C_1 \frac{\partial^2 X_0}{\partial x^2}, X_0] \\
&+ \left[[-C_1 \frac{1}{|S|} \int_{|S|} B_1^{-1} B_2 ds + \frac{1}{|S|} \int_{|S|} B_2^T B_1^{-1} ds C_1] \frac{\partial X_0}{\partial x}, X_0 \right] \\
&+ \left[[\frac{1}{|S|} \int_{|S|} (B_3 - B_2^T B_1^{-1} B_2) ds + \frac{1}{|S|} \int_{|S|} B_2^T B_1^{-1} ds C_1 \frac{1}{|S|} \int_{|S|} B_1^{-1} B_2 ds] X_0, X_0 \right] \\
&= \left[C_1 \left(\frac{\partial X_0}{\partial x} + \frac{1}{|S|} \int_{|S|} B_1^{-1} B_2 ds X_0 \right), \left(\frac{\partial X_0}{\partial x} + \frac{1}{|S|} \int_{|S|} B_1^{-1} B_2 ds X_0 \right) \right] \\
&+ [\frac{1}{|S|} \int_{|S|} (B_3 - B_2^T B_1^{-1} B_2 ds X_0, X_0] \geq 0, \quad \forall \quad X_0
\end{aligned} \tag{59}$$

and

$$\begin{aligned}
[A_h X_0, Y_0] &= [-\bar{B}_1 \frac{\partial^2 X_0}{\partial x^2} - \bar{B}_2 \frac{\partial X_0}{\partial x} + \bar{B}_3 X_0, Y_0] \\
&= [X_0, -\bar{B}_1^* \frac{\partial^2 Y_0}{\partial x^2} + \bar{B}_2^* \frac{\partial Y_0}{\partial x} + \bar{B}_3^* Y_0] \\
&= [X_0, -\bar{B}_1 \frac{\partial^2 Y_0}{\partial x^2} - \bar{B}_2 \frac{\partial Y_0}{\partial x} + \bar{B}_3 Y_0] \\
&= [X_0, A_h^* Y_0] \quad \text{self} - \text{adjoint}
\end{aligned} \tag{60}$$

11 Conclusion

The differences between our approach and the direct averaging method [1, 2] are two fold.

- We know that the stiffness is additive at each cross section and the compliance is additive along the generic axis. The energy equivalence method averages the stiffness over one repeated cell and thus violates the additivity principle. Our approach intrinsically follows the additivity principle.

- Our approach finds the periodic governing equation first then homogenizes it. Namely, we replace the real structure by a 1-D periodic one, then average the four matrices to replace it again by a homogeneous beam. The direct averaging method averages the properties then finds the governing equation. Namely, the method averages two matrices for replacing the real structure by a homogeneous beam, then finds its governing equation.

These are the major reasons why our approach is more accurate than any previous.

References

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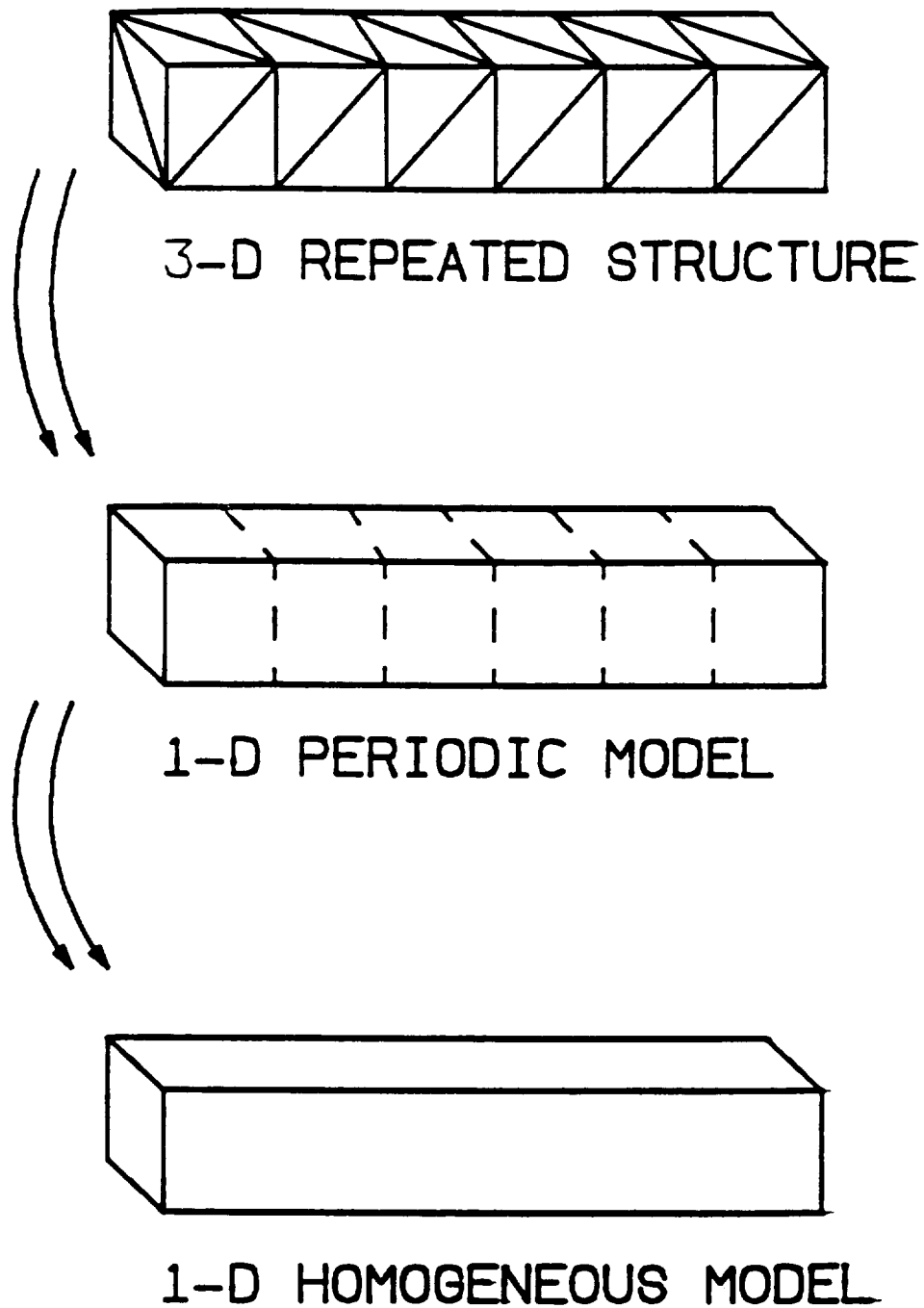
3-D REPEATED STRUCTURE

1-D PERIODIC MODEL

1-D HOMOGENEOUS MODEL

Figure 2: Schematics of Approximation