N94- 35875

Robust Control Design Techniques for Active Flutter Suppression

Hitay Özbay[1] and *Glen R. Bachmann*
Department of Electrical Engineering
The Ohio State University
Columbus, OH 43210

p. 13

Abstract

In this paper an active flutter suppression problem is studied for a thin airfoil in unsteady aerodynamics. The mathematical model of this system is infinite dimensional, because of Theodorsen's function which is irrational. Several second order approximations of Theodorsen's function are compared. A finite dimensional model is obtained from such an approximation. We use H^∞ control techniques to find a robustly stabilizing controller for active flutter suppression.

1 Introduction

In this paper an active flutter suppression problem is studied for a thin airfoil in unsteady aerodynamics. Because of the interaction between the structure and the flow, flutter (dynamic instability) occurs at a certain flow speed. Therefore, it is important to design active feedback controllers stabilizing the airfoil. A robustly stabilizing feedback compensator is obtained from the H^∞ control theory. This theory gives us the largest amount of uncertainty (due to neglected aerodynamics) which can be tolerated in the problem of active flutter suppression.

In general, mathematical models for airfoils in unsteady aerodynamics are linear time invariant infinite dimensional systems. The basic difficulty in such systems is

[1]e-mail: ozbay@ee.eng.ohio-state.edu

to compute the aerodynamic loads due to unsteady flow. The simplest models (in the frequency domain) for the unsteady aerodynamics contain Theodorsen's function as the infinite dimensional part. There are several techniques for designing feedback controllers directly from the infinite dimensional airfoil model see e.g. [1]. In this method the controller itself is infinite dimensional, and hence one has to approximate it in order to obtain an implementable finite dimensional controller. Another method is to approximate the infinite dimensional part of the system and design a finite dimensional controller from the finite dimensional approximate model. In this paper we consider the second method, and design a *robust controller*, which stabilizes not only the finite dimensional model, but also the infinite dimensional model. The main tool used here in the robust controller design is the H^∞ control theory.

In the next section we define a mathematical model for a thin airfoil. Several second order approximations for the Theodorsen's function are compared in Section 3. In Section 4 we present a robust stabilization algorithm for flutter suppression in the presence of unmodeled aerodynamics. Concluding remarks are made in the last section.

2 A mathematical model for the airfoil

We consider the following mathematical model (see e.g. [1], [2]), for a thin airfoil shown in Figure 1,

$$M_s\ddot{z}(t) + B_s\dot{z}(t) + K_s z(t) = \frac{1}{m_s}F(t) + Gu(t), \tag{1}$$

where $z(t) = [h(t), \alpha(t), \beta(t)]^T$, and $u(t)$ represents the control input.

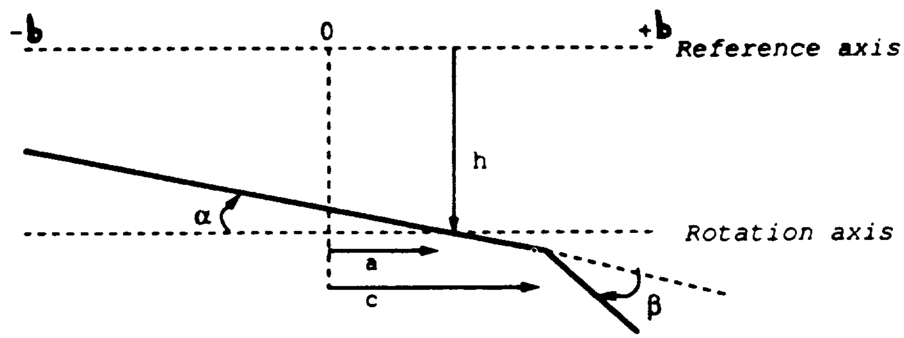

Figure 1: Thin airfoil

The matrices M_s, B_s, K_s and G are in the form

$$M_s = \begin{bmatrix} 1 & x_\alpha & x_\beta \\ x_\alpha & r_\alpha^2 & r_\beta^2 + x_\beta(c-a) \\ x_\beta & r_\beta^2 + x_\beta(c-a) & r_\beta^2 \end{bmatrix}, \quad B_s = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2r_\beta^2\zeta_\beta\omega_\beta \end{bmatrix},$$

$$K_s = \begin{bmatrix} \omega_h^2 & 0 & 0 \\ 0 & r_\alpha^2\omega_\alpha^2 & 0 \\ 0 & 0 & r_\beta^2\omega_\beta^2 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 0 \\ r_\beta^2\omega_\beta^2 \end{bmatrix},$$

where all the constants are related to the geometry and physical properties of the structure.

In order to apply Laplace transform techniques, we will assume that $z(t) = 0$ for $t \leq 0$. This corresponds to the indicial problem (see e.g. [1], and [3]). Aeroelastic loads are represented by $F(t) = [P(t), M_\alpha(t), M_\beta(t)]^T$. We can represent $F(t)$ as

$$F(t) = M_a\ddot{z}(t) + B_a\dot{z}(t) + K_a z(t) + F_c(t) \tag{2}$$

where $F_c(t)$ is the "circulatory" part of $F(t)$. The matrices M_a, B_a and K_a can be computed in terms of the problem data [9] [8]

$$M_a = -\rho b^2 \begin{bmatrix} \pi & -\pi b a & -T_1 b \\ -a\pi b & \pi b^2(\frac{1}{8} + a^2) & -(T_7 + (c-a)T_1)b^2 \\ -T_1 b & -(T_7 + (c-a)T_1)b^2 & -T_3 b^2/\pi \end{bmatrix}$$

$$B_a = -\rho b^2 V \begin{bmatrix} 0 & \pi & -T_4 \\ 0 & \pi(0.5-a)b & (T_1 - T_8 - (c-a)T_4 + 0.5T_{11})b \\ 0 & (T_4(a-0.5) - T_1 - 2T_9)b & -T_4T_{11}b/2\pi \end{bmatrix}$$

$$K_a = -\rho b^2 V^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & T_4 + T_{10} \\ 0 & 0 & (T_5 - T_4T_{10})/\pi \end{bmatrix}.$$

where T_i's are Theodorsen's constants, see e.g. [9].

Using Theodorsen's formulation, $F_c(t)$ can be expressed in the frequency domain as (see e.g. [9] pp. 395–396, or [8] pp. 26–28)

$$\hat{F}_c(s) = C(s)(B_{c1} + sB_{c2})\hat{z}(s) \tag{3}$$

where s is the Laplace transform variable, $\hat{}$ represents the Laplace transformed version of a time signal, $C(j\omega)$ is the Theodorsen's function, and B_{c1}, B_{c2} are constant matrices given by $B_{c1} = b_1 c_1$ and $B_{c2} = b_1 c_2$ where $b_1 = \rho V b \begin{bmatrix} -2\pi \\ 2\pi b(a+0.5) \\ T_{12} b \end{bmatrix}$, $c_1 = V\,[\,0 \quad 1 \quad T_{10}/\pi\,]$, and $c_2 = [\,1 \quad b(0.5-a) \quad bT_{11}/2\pi\,]$.

Suppose that the measured output for feedback is

$$y(t) := c_1 z(t) + c_2 \dot{z}(t).$$

Then, taking the Laplace transforms of (1) and (2), and then using (3) we obtain a transfer function from u to y, denoted by $P(s)$:

$$\frac{\hat{y}(s)}{\hat{u}(s)} = P(s) = \frac{C_o(sI-A)^{-1}B_o}{1 - C_o(sI-A)^{-1}B_1\, C(s)} \tag{4}$$

where $C(s)$ is the Theodorsen's function, and

$$A = \begin{bmatrix} 0_{3\times 3} & I_{3\times 3} \\ (M_s - M_a)^{-1}(K_a - K_s) & (M_s - M_a)^{-1}(B_a - B_s) \end{bmatrix}$$

$$C_o = [c_1 \quad c_2]\,, \quad B_1 = \begin{bmatrix} 0_{3\times 1} \\ b_1 \end{bmatrix}\,, \quad B_o = \begin{bmatrix} 0_{3\times 1} \\ (M_s - M_a)^{-1} G \end{bmatrix}.$$

Note that the plant can be seen as a feedback system whose feedback path consists of the aerodynamics represented by Theodorsen's function, as shown below.

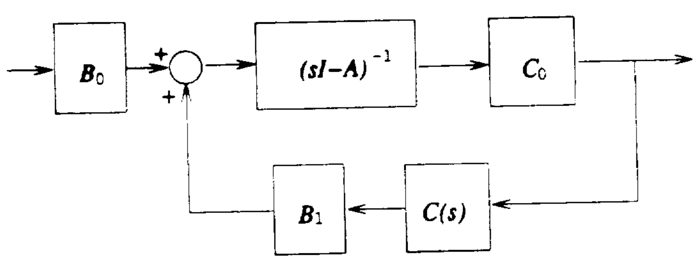

Figure 2: Structure of the plant

The function $C(s)$ is irrational, and in practice it is approximated by a low order rational function, say $C_a(s)$. This leads to an approximate model for the plant to be controlled

$$P_a(s) = \frac{C_o(sI-A)^{-1}B_o}{1 - C_o(sI-A)^{-1}B_1\, C_a(s)}.$$

In Section 4 we will see that a rational feedback controller K_a, which stabilizes P_a, stabilizes the original infinite dimensional model P if and only if the H^∞ norm of the "weighted" closed loop transfer function is less than the inverse of the L^∞ approximation error

$$\|C - C_a\|_\infty := \sup_\omega |C(j\omega) - C_a(j\omega)|.$$

Therefore, we have a better chance of stabilizing P by a rational K_a if we can make the L^∞ error in Theodorsen's function approximation. In the next section we compare the L^∞ errors of several second order approximates of the Theodorsen's function.

3 L^∞ Approximation of the Theodorsen's function

As mentioned above, the Theodorsen's function, $C(s)$ which appears in the feedback path of the plant model, is infinite dimensional. For controller design (synthesis) and simulation (analysis) purposes we would like to use a finite dimensional approximate $C_a(s)$ instead of the exact irrational $C(s)$, which is given by (see e.g. [9])

$$C(j\omega) = \mathrm{Re}[C(j\omega)] + j\mathrm{Im}[C(j\omega)] \tag{5}$$

where

$$\mathrm{Re}[C(j\omega)] = \frac{J_1(\omega)(J_1(\omega) + Y_0(\omega)) + Y_1(\omega)(Y_1(\omega) - J_0(\omega))}{(J_1(\omega) + J_0(\omega))^2 + (Y_1(\omega) - J_0(\omega))^2},$$

$$\mathrm{Im}[C(j\omega)] = \frac{(Y_1(\omega)Y_0(\omega) + J_1(\omega)J_0(\omega))}{(J_1(\omega) + J_0(\omega))^2 + (Y_1(\omega) - J_0(\omega))^2}.$$

(J_0, J_1, Y_0, Y_1 are the Bessel functions). Several second order approximations of (5) can be found in the literature, see for example [8]. These approximations are in the form

$$C_a(s) = \frac{(1 + \tau_1 s)(1 + \tau_2 s)}{(1 + \tau_3 s)(1 + \tau_4 s)} \tag{6}$$

where $\tau_1, \tau_2, \tau_3, \tau_4$ are positive real constants to be chosen. For example, the following sets of numerical values are proposed by R. Jones, W. P. Jones and R. L. Moore respectively

$$\tau_1 = 18.6 \ , \quad \tau_2 = 1.97 \ , \quad \tau_3 = 21.98 \ , \quad \tau_4 = 3.33 \tag{7}$$

$$\tau_1 = 20.62 \ , \quad \tau_2 = 1.85 \ , \quad \tau_3 = 24.39 \ , \quad \tau_4 = 3.125 \tag{8}$$

$$\tau_1 = 10.61 \ , \quad \tau_2 = 1.774 \ , \quad \tau_3 = 13.51 \ , \quad \tau_4 = 2.744 \ . \tag{9}$$

For each of these sets of numbers the error function $|C(j\omega) - C_a(j\omega)|$ is plotted in Figure 3.

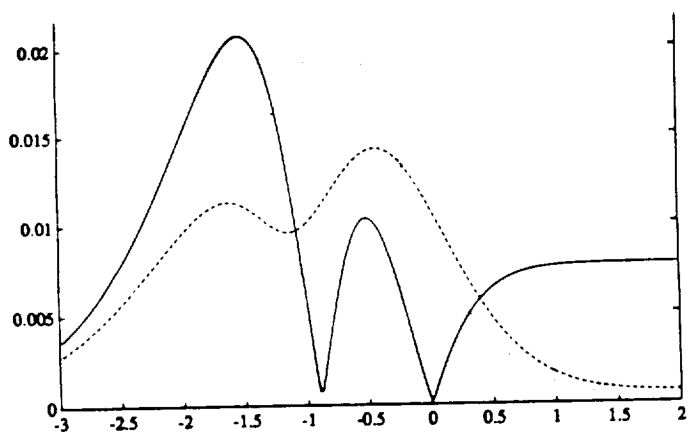

Figure 3: Error function for R. Jones, W. P. Jones and R. L. Moore approximates of the Theodorsen's function .

As we can see from this figure, R. Jones's approximation is the best one (in the L^∞ norm) among the three second order approximates listed above. In different norms, other approximations may be better than the one which is best in L^∞ norm. But since we are going to use H^∞ control techniques (in order to guarantee the robustness of the controllers derived from the approximate plant), we will need an error bound in the L^∞ norm. Below we will show that it is possible to improve the L^∞ error of the R. Jones approximation by fine tuning the values of $\tau_1, \ldots, \tau_4$.

We found that the values of

$$\tau_1 = 18.57 \ , \quad \tau_2 = 2.057 \ , \quad \tau_3 = 21.93 \ , \quad \tau_4 = 3.446 \tag{10}$$

give a function $C_a(j\omega)$ whose magnitude is "close" to being a Chebyshev approximation for the magnitude of $C(j\omega)$, (i.e. the error function $|\,|C_a(j\omega)| - |C(j\omega)|\,|$, shown in Figure 4, "nearly" satisfies the necessary and sufficient conditions for $|C_a(j\omega)|$ to be a Chebyshev approximation of $|C(j\omega)|$).

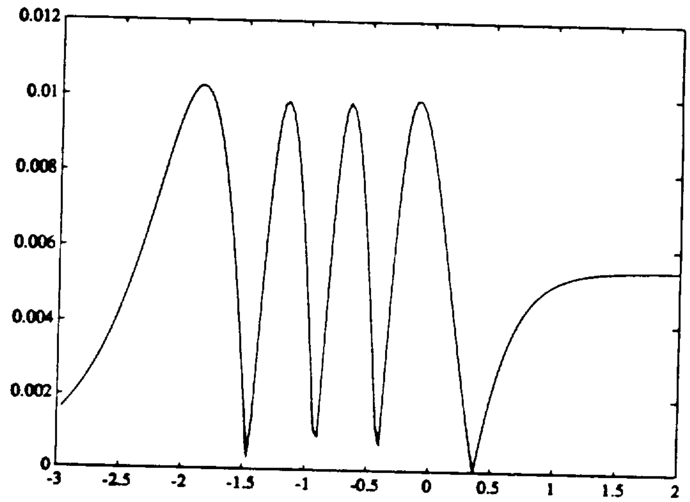

Figure 4: $\Big|\,|C_a(j\omega)| - |C(j\omega)|\,\Big|$ versus $\log(\omega)$

We have obtained the above values for τ_i's by slightly modifying the approximation scheme proposed in [11]. We would like to determine if this choice for C_a is a "good" L^∞ approximate of C. For this purpose we first point out the following relationship between the L^∞ error and the error in magnitude and phase functions:

Lemma: Let $\phi(\omega)$ and $\phi_a(\omega)$ denote the phase of $C(j\omega)$ and $C_a(j\omega)$ respectively, i.e.

$$C(j\omega) = |C(j\omega)|e^{j\phi(\omega)} \qquad C_a(j\omega) = |C_a(j\omega)|e^{j\phi_a(\omega)}.$$

Then we have

$$|C(j\omega) - C_a(j\omega)| \leq \Big|\ |C(j\omega)| - |C_a(j\omega)|\ \Big| + |C(j\omega)|\ \Big|\phi(\omega) - \phi_a(\omega)\Big|. \tag{11}$$

Proof: By definition following equalities hold

$$\begin{aligned}
|C(j\omega) - C_a(j\omega)| &= \Big| \, |C(j\omega)|e^{j\phi(\omega)} - |C_a(j\omega)|e^{j\phi_a(\omega)} \, \Big| \\
&= \Big| \, |C(j\omega)|e^{j(\phi(\omega)-\phi_a(\omega))} - |C(j\omega)| + |C(j\omega)| - |C_a(j\omega)| \, \Big| \\
&= \Big| \, |C(j\omega)|(e^{j(\phi(\omega)-\phi_a(\omega))} - 1) + (|C(j\omega)| - |C_a(j\omega)|) \, \Big| .
\end{aligned}$$

On the other hand, for any $\theta \in [-\pi \ , \ \pi]$ we have

$$|e^{j\theta} - 1| \leq |\theta|.$$

Hence we conclude that

$$|C(j\omega) - C_a(j\omega)| \leq \Big| \, |C(j\omega)| - |C_a(j\omega)| \, \Big| + |C(j\omega)| \, \Big|\phi(\omega) - \phi_a(\omega)\Big|. \quad \square$$

This lemma says that the Chebyshev approximation error[2] for the magnitude function plus the corresponding "normalized" phase error is an upper bound for the overall L^∞ error. We also deduce from this lemma that in order to get a good L^∞ error bound we may try to develop an approximation scheme such that whenever the magnitude error is large, the normalized phase error is small and vice versa. However, if we obtain C_a from the Chebyshev approximate of $|C(j\omega)|$, this automatically fixes the normalized phase error function, which does not necessarily satisfy the above mentioned nice property. However, we will see from the following numerical example that this property is satisfied for the second order approximation we have proposed by (10). For C_a determined from (10), the two terms in the right hand side of (11), as well as the function in the left hand side of (11), are shown in Figure 5. It is quite surprising that the normalized phase error function alternates with the magnitude error function,

[2]What we mean by Chebyshev approximation for the magnitude function is the following: Suppose $|C(j\omega)|$ is known, and we want to approximate the real valued function in the L^∞ norm by a function $|C_a(j\omega)|$; the problem is to find a real rational $C_a(s)$ (whose order is fixed) achieving the smallest Chebyshev error

$$\sup_\omega \Big| |C(j\omega)| - |C_a(j\omega)| \Big|.$$

In the text we use the term L^∞ approximation for the approximation of the complex valued function $C(j\omega)$, and we use the term Chebyshev approximation for the approximation of the real valued function $|C(j\omega)|$.

i.e. whenever the first term is large the second term is small and vice versa. Also interesting is the fact that for this choice of C_a the function $|C(j\omega) - C_a(j\omega)|$ is an envelope of the two functions appearing in the right hand side of (11).

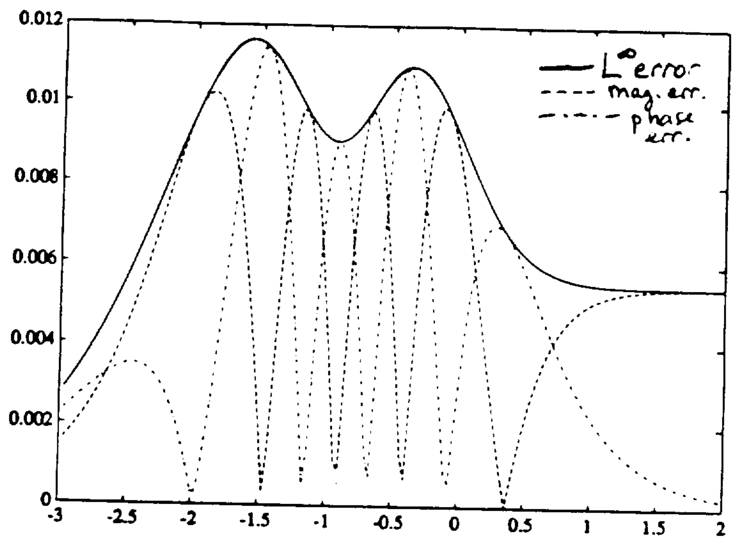

Figure 5: Relations between the L^∞ error function and the magnitude error and normalized phase error functions.

The above observations can be generalized as follows:

Conjecture:

Let $F(s) \in H^\infty$ be a minimum phase and positive real function (possibly irrational), and let $F_a(s) \in H^\infty$ be a fixed order real rational function approximating $F(s)$. If $|F_a(j\omega)|$ is the best Chebyshev approximation of $|F(j\omega)|$, then $F_a(s)$ is the best L^∞ approximate of $F(s)$. $\Box$

4 Active flutter suppression

Let us consider the thin airfoil model obtained in Section 2. When flutter occurs the plant $P(s)$ is unstable, and we would like to design a feedback controller stabilizing the closed loop system, shown in Figure 6. In our design we will use C_a given by the numerical values in (10). This gives us an approximate plant model P_a. A robustly stabilizing finite dimensional controller $K_a(s)$ will be obtained from P_a, and it will be shown that under a certain condition, this controller stabilizes the original infinite dimensional airfoil model, with a certain robustness level.

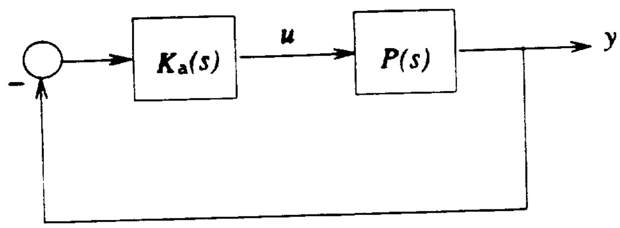

Figure 6: Feedback control system

Consider the approximate plant

$$P_a(s) = \frac{C_o(sI-A)^{-1}B_o}{1 - C_o(sI-A)^{-1}B_1\, C_a(s)}.$$

We can find rational transfer functions $N_1, N_2, M \in H^\infty$ such that

$$C_o(sI-A)^{-1}B_o = \frac{N_o(s)}{M(s)} \qquad \text{and} \qquad C_o(sI-A)^{-1}B_1 = \frac{N_1(s)}{M(s)}.$$

Therefore we can express P and P_a in the form

$$P_a(s) = \frac{N_o(s)}{M(s) - N_1(s)C_a(s)} \qquad\qquad P(s) = \frac{N_o(s)}{M(s) - N_1(s)C(s)}.$$

Thus, P and P_a differ in their denominator, in the sense that

$$P(s) = \frac{N_p(s)}{M_p(s)} \quad \text{and} \quad P_a(s) = \frac{N_p(s)}{M_{pa}(s)}$$

where $N_p(s) = N_o(s)$, $M_p(s) = M(s) - N_1(s)C(s)$, $M_{pa}(s) = M_p(s) + \Delta_M(s)$, and

$$\Delta_M(s) = N_1(s)(C(s) - C_a(s)).$$

Let ϵ_a be an upper bound of the L^∞ approximation error for the Theodorsen's function, i.e.

$$\|C - C_a\|_\infty < \epsilon_a,$$

(note that for the choice of C_a given by (10) we can choose $\epsilon_a = 0.012$). Then from [4], [5] and [12] we can deduce that a controller K_a stabilizing P_a and achieving an H^∞ performance

$$\gamma(K_a) = \|N_1 M_{pa}^{-1}(1 + P_a K_a)^{-1}\|_\infty \tag{12}$$

stabilizes the infinite dimensional plant P if[3]

$$\gamma(K_a) \le \frac{1}{\epsilon_a}. \tag{13}$$

One proves this as follows: A controller K_a stabilizes all plants of the form $P_a = \frac{N_p}{M_{pa} - \Delta_M}$ if the roots of

$$1 + K_a(s)\frac{N_p(s)}{M_{pa}(s) - \Delta_M(s)} = 0$$

are in the left half plane. This condition is satisfied if

$$L_\Delta := 1 - \Delta_M M_{pa}^{-1}(1 + P_a K_a)^{-1}$$

is invertible in H^∞. Since $\|\Delta_M\|_\infty \le \epsilon_a |N_1(j\omega)|$ and K_a stabilizes P_a (meaning that $(1 + P_a K_a)^{-1} \in H^\infty$), a sufficient condition for L_Δ to be invertible in H^∞ is (13).

In fact, if K_a stabilizes P_a and satisfies (13), then it stabilizes P with a certain robustness level, see e.g. [7]. The controller K_a^{opt}, which minimizes $\gamma(K_a)$ over all controllers stabilizing P_a, has the best chance of satisfying (13). Note also that we increase our chances of satisfying (13) by decreasing ϵ_a.

An interesting question about the stabilization of P by K_a is: How much can we increase ϵ_a so that

$$\gamma_a := \inf_{K_a \text{ stabilizing } P_a} \gamma(K_a) \le \frac{1}{\epsilon_a} \ ?$$

[3] We would like to point out that the perturbation in the plant is in the denominator only, so the term $\gamma(K_a)$ is slightly different than the one in [5] and [12], where both numerator and denominator perturbations are considered.

The answer to this question gives the largest L^∞ error we can tolerate in approximating $C(j\omega)$ so that we can still find an active feedback controller stabilizing the original plant. The problem of minimizing $\gamma(K_a)$ over all controllers K_a stabilizing P_a is a special case of a one block H^∞ optimal control problem, and can be solved easily by finding the singular values and vectors of a Hankel whose symbol is a rational function, or by using the Nevanlinna Pick interpolation, see e.g. [4] and references therein.

5 Concluding remarks

An active controller design method is illustrated for a thin airfoil. The model P for this system is infinite dimensional. By approximating the infinite dimensional part of the plant we have obtained a finite dimensional approximate model P_a. We have illustrated that using a Chebyshev approximation for the *magnitude* function $|C(j\omega)|$ we obtain a finite dimensional approximate of $C(j\omega)$ which is nearly optimal in the L^∞ norm.

A finite dimensional controller K_a^{opt} can be obtained by solving the one block H^∞ problem posed in Section 4. In the H^∞ problem formulation we used the finite dimensional approximate model P_a. We have shown that if the H^∞ optimal performance γ_a is less than the inverse of the L^∞ approximation error of the Theodorsen's function, ϵ_a, then the controller K_a stabilizes not only the finite dimensional model P_a, but also the original infinite dimensional model P.

References

[1] A. V. Balakrishnan, "Active control of airfoils in unsteady aerodynamics," *Applied Math. and Optimization*, **4** (1978), pp. 171–195.

[2] A. V. Balakrishnan and J. W. Edwards, "Calculation of the transient motion of elastic airfoils forced by control surface motion and gusts," NASA Technical Memorandum 81351, August 1980.

[3] R. L. Bisplinghoff, H. Ashley and R. Halfman, *Aeroelasticity*, Addison-Wesley, Cambridge, 1955.

[4] J. C. Doyle, B. A. Francis and A. R. Tannenbaum, *Feedback Control Theory*, Macmillan New York, 1992.

[5] T. T. Georgiou and M. C. Smith, "Optimal robustness in the gap metric," *IEEE Trans. Automat. Control*, **35** (1990), pp. 673–686.

[6] P. P. Khargonekar and M. A. Rotea, "Mixed H_2/H_∞ control: a convex optimization approach," *IEEE Transactions on Automatic Control*, **36** (1991), pp. 824–837.

[7] D. McFarlane, K. Glover and M. Vidyasagar, "Reduced order controller design using coprime factor model reduction," *IEEE Transactions on Automatic Control* **35** (1990), pp. 369–373.

[8] R. L. Moore, *Aeroservoelastic Stability Analysis of an Airplane with a Control Augmentation System*, PhD dissertation, The Ohio State University, 1978.

[9] R. H. Scanlan and R. Rosenbaum, *Introduction to the Study of Aircraft Vibration and Flutter*, Macmillan, New York, 1951.

[10] M. A. Snyder, *Chebyshev Methods in Numerical Approximation*, Prentice Hall, Englewood Cliffs NJ, 1966.

[11] Vajta, M., "New model reduction technique for a class of parabolic partial differential equations," Proc. of the IEEE International Conf. on Systems Engineering, Dayton OH, 1991, pp. 311–315.

[12] M. Vidyasagar and H. Kimura, "Robust controllers for uncertain linear multivariable systems," *Automatica*, **22** (1986), pp. 85–94.

PARAMETER ESTIMATION