N94-35882

OPTIMAL DISTURBANCE REJECTING CONTROL OF HYPERBOLIC SYSTEMS

Saroj K. Biswas
Department of Electrical Engineering
Temple University
Philadelphia, PA

N. U. Ahmed
Department of Electrical Engineering
University of Ottawa
Ottawa, Ontario, Canada

p. 6

ABSTRACT

Optimal regulation of hyperbolic systems in the presence of unknown disturbances is considered. Necessary conditions for determining the optimal control that tracks a desired trajectory in the presence of the worst possible perturbations are developed. The results also characterize the worst possible disturbance that the system will be able to tolerate before any degradation of the system performance. Numerical results on the control of a vibrating beam are presented.

I. INTRODUCTION

The H_∞ control problem for regulation of dynamical systems in the presence of perturbations has been a subject of considerable research in recent years [4,5,6,7,10,16]. Although the original formulation of the H_∞ method was in terms of frequency domain terms, extensions in state space terms leading to feedback control using Riccati type equations have been developed [4,7,10,16]. For the infinite dimensional systems, the H_∞ control has started to gain momentum. For a summary of recent results, see the survey paper [3]. Like its finite dimensional counterpart, the frequency domain approach [9,15] as well as the state space analysis [11,12,] in the presence of both bounded and unbounded perturbations has been considered in the literature. The problems pertinent to the H_∞ control design are: a) input-output stability, b) disturbance decoupling, and c) disturbance attenuation.

This paper is concerned with disturbance attenuation of hyperbolic systems in the presence of worst possible disturbances. We utilize the concepts of optimal control theory [1,8] for infinite dimensional systems for deriving the control law for optimum regulation of the system in the presence of worst possible disturbances. The method presented in this paper is a generalization of an H_∞-type method developed in [13,14] for finite dimensional systems. The ratio of disturbance energy to the energy of the controlled system is used as a measure of performance for disturbance attenuation. We present conditions for estimation of the largest perturbation that can be attenuated and the corresponding controller to attenuate this perturbation.

The paper is organized as follows: Section II introduces the H_∞ control problem. Necessary conditions for optimum disturbance attenuation are presented in section III followed by numerical results on control of a vibrating beam in section IV. Some concluding remarks are given in section V.

II. NOTATIONS AND PROBLEM STATEMENT

We shall use the following notations for abstract function spaces throughout the paper. Let H be a Hilbert space, and V a linear subspace of H carrying the structure of a reflexive Banach space with the injection $V \hookrightarrow H$ continuous. We identify H with its dual so that $V \hookrightarrow H \hookrightarrow V'$, where V' is the topological dual of V.

Suppose A be a bounded linear self adjoint operator $A \in \mathcal{L}(V, V')$ satisfying the conditions

$$\begin{aligned} &|\langle A\varphi, \psi\rangle| \le c\,\|\varphi\|_V\,\|\psi\|_V,\ c \ge 0,\ \varphi, \psi \in V \\ &\langle A\varphi, \varphi\rangle_{V',V} + \beta|\varphi|_H^2 \ge \alpha\|\varphi\|_V^2,\ \alpha > 0,\ \beta \in R,\ \varphi \in V \end{aligned} \tag{1}$$

Consider the hyperbolic system

$$\begin{aligned} &\frac{\partial^2 y}{\partial t^2} + Ay = Bu + Cv, \quad t \in I \equiv (0, T), \\ &y(0) = y_0, \qquad \frac{\partial y}{\partial t}(0) = y_0' \end{aligned} \tag{2}$$

where the operator A is as defined above. The control applied to the system is denoted $u \in \mathcal{U} \equiv L_2(I, H)$, and B is a bounded linear operator $B \in \mathcal{L}(H)$. Suppose the system is perturbed by a disturbance $v \in L_2(I, H)$ through the operator $C \in \mathcal{L}(H)$. The initial conditions $y_0 \in V$ and $y_0' \in H$ are also considered to be initial disturbances to the system.

With this introduction we now pose the control problem:

Given the perturbed system (2), find the control $u \in L_2(I, H)$ that keeps the state trajectory as close as possible to a desired trajectory in the presence of maximum possible additive disturbance $v \in L_2(I, H)$ and maximum possible initial disturbances $y_0 \in V$ and $y_0' \in H$.

For a mathematical formulation of this control problem, we introduce a cost function:

$$J(u, v, y_0, y_0') = \frac{\frac{1}{2}s_1 \int_\Omega |y_0|^2\,dx + \frac{1}{2}s_2 \int_\Omega |y_0'|^2\,dx + \frac{1}{2}\int_{I\times\Omega} r_2|v|^2\,dx\,dt}{\frac{1}{2}q_1 \int_{I\times\Omega} |y - y^d|^2\,dx\,dt + \frac{1}{2}q_2 \int_{I\times\Omega} |y_t - y_t^d|^2\,dx\,dt + \frac{1}{2}\int_{I\times\Omega} r_1|u|^2\,dx\,dt} \tag{3}$$

where s_1, s_2, q_1, q_2, r_1 and r_2 are scalar weighting factors, and y^d and y_t^d are desired trajectories respectively. Then the disturbance rejecting control problem is equivalent to the minimax problem of finding a control u and a scalar λ^* so that

$$\lambda^* = \inf_{\substack{v\neq 0 \\ y_0\neq 0 \\ y_0'\neq 0}} \sup_{u\in\mathcal{U}} J(u, v, y_0, y_0') \tag{4}$$

subject to the dynamics (2). The quantity λ^* can be interpreted as the disturbance rejection capacity of the system. A larger λ^* implies a better controller in the sense that the system will be able to tolerate larger amount perturbations before degradation of the system performance. A small λ^* means that the system is too sensitive to disturbances; despite the effects of the control the state trajectory is not close to the desired trajectory even in the presence of a small amount of perturbations.

We shall assume that a solution of this minimax problem exists. In what follows, we shall derive a set of necessary conditions that must be satisfied by the optimal controller.

III. MAIN RESULTS

We first give a brief outline of derivation of the main results. The minimax problem introduced above is solved in two steps, with the first step being finding the supremum of J over u assuming that the perturbations v and y_0, y_0' are known, and the second finding the infimum of J over nonzero perturbations. The first step determines the optimal control that regulates the system, and the second step characterizes the worst possible perturbation that the controller will be able to attenuate before a serious degradation of the system performance.

Clearly, the problem of finding the supremum of J over u is equivalent to minimizing the denominator of J given in (3) for fixed v, y_0, and y_0' subject to the dynamics (2). This is a well known problem in infinite dimensional control theory for hyperbolic systems (see [1,2,8] for details). Theorems 1 and 2 presented below pertain to this problem. We omit the proofs for brevity.

THEOREM 1. *For a given $v \in L_2(I,H)$, $y_0 \in V$, and $y_0' \in H$, the system* (1) *has a unique solution $y \in L_2(I,V) \cap C(\bar{I},V)$, $y_t \in L_2(I,H) \cap C(\bar{I},H)$. Furthermore, the mapping $(y_0, y_0', u) \longrightarrow (y, y_t)$ is continuous from $V \times H \times L_2(I,H) \longrightarrow L_2(I,V) \times L_2(I,H)$.* ■

For convenience of presentation, we introduce two new variables $\varphi_1 \equiv y$ and $\varphi_2 \equiv y_t$, and rewrite the system (2) as a first order equation:

$$\begin{aligned} &\frac{\partial \varphi}{\partial t} + \mathcal{A}\varphi = \mathcal{B}u + \mathcal{C}v \\ &\varphi(0) = (y_0, y_0') \equiv \rho \end{aligned} \tag{5}$$

where

$$\mathcal{A} = \begin{bmatrix} 0 & -I \\ A & 0 \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} 0 \\ B \end{bmatrix} \quad \text{and} \quad \mathcal{C} = \begin{bmatrix} 0 \\ C \end{bmatrix}.$$

Using the above notations, we also rewrite the denominator of the cost function (3) as

$$J_1(u) = \frac{1}{2}\int_0^T \langle \varphi - \varphi^d, Q(\varphi - \varphi^d)\rangle_{H\times H}\, dt + \frac{1}{2}\int_0^T \langle u, R_1 u\rangle_H \, dt \tag{6}$$

where $Q = \operatorname{diag}(q_1, q_2)$ and $R_1 = r_1$. Let q_1 and q_2 be nonnegative, and r_1 strictly positive.

The necessary and sufficient condition that $u_0 \in L_2(I,H)$ be optimal in the sense of minimization of the cost (6) is that

$$J_1'(u_0; u - u_0) \geq 0 \quad \text{for all } u \in \mathcal{U} \tag{7}$$

where $J_1'(u_0; u - u_0)$ is the Gateaux derivative of J_1 at $u_0 \in \mathcal{U}$ in the direction $u - u_0$. This is given in the next theorem:

THEOREM 2. *Consider the system* (5) *for fixed additive disturbance $v \in L_2(I,H)$, fixed initial disturbance $\rho \in V \times H$, and the desired trajectory $\varphi^d \in L_2(I,H) \times L_2(I,H)$. Then the optimal control $u_0 \in L_2(I,H)$ that minimizes the cost* (6) *is characterized by the solution of the two-point-boundary-value problem:*

$$\frac{\partial \varphi}{\partial t} + \mathcal{A}\varphi + \mathcal{B}R_1^{-1}\mathcal{B}^*\psi = \mathcal{C}v, \qquad \varphi(0) = \rho \tag{8}$$

$$-\frac{\partial \psi}{\partial t} + \mathcal{A}^*\psi = Q(\varphi - \varphi^d), \qquad \psi(T) = 0 \tag{9}$$

The optimal control u_0 is then given by

$$u_0 = -R_1^{-1}\mathcal{B}^*\psi. \quad ■ \tag{10}$$

At this point we return to the disturbance rejecting control problem introduced earlier. Clearly $J(u_0)$ is a function of v and ρ which are yet to be determined. We substitute $J_1(u_0)$ into the denominator of the cost function (3) leading to

$$J(v,\rho) = \frac{\frac{1}{2}\langle \rho, S\rho\rangle_H + \frac{1}{2}\int_I \langle v, R_2 v\rangle_H \, dt}{\frac{1}{2}\int_I \langle \varphi - \varphi^d, Q(\varphi - \varphi^d)\rangle_{H\times H}\, dt + \frac{1}{2}\int_I \langle \mathcal{B}R_1^{-1}\mathcal{B}^*\psi, \psi\rangle\, dt} \tag{11}$$

Then the disturbance rejecting control problem is equivalent to finding a scalar λ^* so that

$$\lambda^* = \inf_{\substack{v \neq 0 \\ \rho \neq 0}} J(v, \rho) \tag{12}$$

subject to the dynamics (8) - (9). Note that since λ^* is the optimal solution, we have $0 < \lambda^* \leq \lambda = J(v, \rho)$. Hence clearly, the following function

$$J_2(v, \rho) = \frac{1}{2}\langle \rho, S\rho \rangle_H + \frac{1}{2}\int_I \langle v, R_2 v \rangle_H \, dt - \frac{\lambda^*}{2}\int_I \langle \varphi - \varphi^d, Q(\varphi - \varphi^d) \rangle_{H \times H} \, dt - \frac{\lambda^*}{2}\int_I \langle BR_1^{-1}B^*\psi, \psi \rangle \, dt \tag{13}$$

is convex and nonnegative, and has a minimum at $J_2 = 0$. Thus the problem of finding the infimum indicated in (12) is equivalent to minimizing (13) subject to the system dynamics (8) - (9).

By virtue of Theorem 1, it is clear that for any $v \in L_2(I, H)$, $\rho \in V \times H$, and $\varphi^d \in L_2(I, H) \times L_2(I, H)$, the equations (8) - (9) have a unique solution $\varphi \in L_2(I, V) \times L_2(I, H)$ and $\psi \in L_2(I, V) \times L_2(I, H)$. In addition, the solution has a unique Gateaux derivative satisfying the following theorem:

THEOREM 3. *The solution* (φ, ψ) *of the two-point-boundary-value problem* (8) - (9) *corresponding to* $v \in L_2(I, H)$ *and* $\rho \in V \times H$ *has a unique Gateaux derivative at every* $v_0 \in L_2(I, H)$ *and* $\rho_0 \in V \times H$ *satisfying*

$$\frac{\partial \widehat{\varphi}}{\partial t} + \mathcal{A}\widehat{\varphi} + \mathcal{B}R_1^{-1}\mathcal{B}^*\widehat{\psi} = \mathcal{C}(v - v_0), \qquad \widehat{\varphi}(0) = \rho - \rho_0 \tag{14}$$

$$-\frac{\partial \widehat{\psi}}{\partial t} + \mathcal{A}^*\widehat{\psi} - Q\widehat{\varphi} = 0, \qquad \widehat{\psi}(T) = 0 \tag{15}$$

with $\widehat{\varphi} \in L_2(I, V) \times L_2(I, H)$ *and* $\widehat{\psi} \in L_2(I, V) \times L_2(I, H)$ ■

Necessary conditions of optimality for minimization of (13) are now derived with the help of the above results and the fact that the Gateaux derivative

$$J_2'(v_0, \rho_0; v - v_0, \rho - \rho_0) \geq 0 \tag{16}$$

for all $v \in L_2(I, H)$ and $\rho_0 \in V \times H$, where J_2' is the Gateaux derivative at v_0, ρ_0 in the direction $v - v_0$, $\rho - \rho_0$. We present the result in the following theorem:

THEOREM 4. *The worst additive disturbance* v_0 *and the worst initial disturbance* ρ_0 *that can be attenuated by the optimal control* u_0 *are characterized by simultaneous solution of the following equations:*

$$\frac{\partial \varphi}{\partial t} + \mathcal{A}\varphi + \mathcal{B}R_1^{-1}\mathcal{B}^*\psi = \mathcal{C}R_2^{-1}\mathcal{C}^*\xi, \tag{17}$$

$$-\frac{\partial \psi}{\partial t} + \mathcal{A}^*\psi = Q(\varphi - \varphi^d), \tag{18}$$

$$-\frac{\partial \xi}{\partial t} + \mathcal{A}^*\xi - Q\eta = \lambda^* Q(\varphi - \varphi^d) \tag{19}$$

$$\frac{\partial \eta}{\partial t} + \mathcal{A}\eta + \mathcal{B}R_1^{-1}\mathcal{B}^*\xi = \lambda^*\mathcal{B}R_1^{-1}\mathcal{B}^*\psi \tag{20}$$

with the boundary conditions

$$\begin{aligned} \varphi(0) &= \rho_0 & \quad \psi(T) &= 0 \\ \xi(0) &= S\rho_0 & \quad \xi(T) &= 0 \\ \eta(0) &= 0 & & \end{aligned} \tag{21}$$

The worst disturbances are given by

$$v_0 = R_2^{-1}C^*\xi$$
$$\rho_0 = S^{-1}\xi(0) \tag{22}$$

The optimal control that regulates the system in the presence of the worst disturbance is given by

$$u_0 = -R_1^{-1}\mathcal{B}^*\psi \tag{23}$$

PROOF: Taking the Gateaux derivative of the J_2 at $v_0 \in L_2(I,H)$ and $\rho_0 \in V \times H$, we have

$$0 \le J_2'(v_0,\rho_0; v-v_0, \rho-\rho_0) = \langle S\rho_0, \rho-\rho_0\rangle_H + \int_I \langle R_2 v_0, v-v_0\rangle\, dt - \lambda^* \int_I \langle \hat{\varphi}, Q(\varphi-\varphi^d)\rangle\, dt - \lambda^* \int_I \langle \hat{\psi}, \mathcal{B}R_1^{-1}\mathcal{B}\psi\rangle\, dt$$

The result follows from Theorem (3) and adjoint system (19) - (20). ■

It is worthwhile to mention here that equations (17) - (21) represent a two-point-boundary-value problem with λ^* as a parameter which is unknown. The smallest value of λ for which this TPBVP, i.e., (17) - (21) has a solution is the optimal λ^* or the disturbance rejection capacity of the system. The corresponding control u_0 is then obtained using (23) and the worst disturbance v_0 and ρ_0 that can be attenuated is given by (22).

IV. EXAMPLE

We consider the cantilever beam equation (normalized)

$$\frac{\partial^2 y}{\partial t^2} + \frac{\partial^4 y}{\partial x^4} = g(x)u(t) + h(x)v(t), \qquad x \in \Omega \equiv (0,1), \quad t > 0 \tag{24}$$

subject to boundary conditions

$$y(0,t) = 0, \quad \frac{\partial y}{\partial x}(0,t) = 0, \quad \frac{\partial^2 y}{\partial x^2}(1,t) = 0, \quad \frac{\partial^3 y}{\partial x^3}(1,t) = 0 \tag{25}$$

Define the operator A in $H = L_2(\Omega)$ by

$$A\phi = \frac{\partial^4 \phi}{\partial x^4}, \qquad D(A) = \left\{\phi : \phi \in H^4(\Omega), \phi(0) = 0, \frac{\partial \phi}{\partial x}(0) = 0, \frac{\partial^2 \phi}{\partial x^2}(1) = 0, \frac{\partial^3 \phi}{\partial x^3}(1) = 0\right\}$$

where $H^4(\Omega)$ is the Sobolev space of order four on Ω. For V we take $V = \{\phi \in H^2(\Omega), \phi(0) = 0, \frac{\partial \phi}{\partial x}(0) = 0\}$.

We assume that the desired state of the controlled system is the zero state, and that there is no initial disturbance. We compute the disturbance rejection capacity of the system using Theorem 4. Table I shows that a tighter regulation (i.e., higher Q) is possible only if less disturbance is allowed to be attenuated. It is intuitively correct to say that a better regulation of the state trajectory can be achieved if the disturbance amplitude is small. Similarly a cheaper control allows more disturbance accommodation by the controller as shown in Table II. Stated in a different way, this means that attenuation of larger amplitude disturbances will require more control energy.

TABLE I

Q	r_1	r_2	S	λ^*
1	1	1	10	0.5522
10	1	1	10	0.2130
20	1	1	10	0.1521
50	1	1	10	0.0922
100	1	1	10	0.0618

TABLE II

Q	r_1	r_2	S	λ^*
20	1	1	10	0.1521
20	2	1	10	0.1065
20	5	1	10	0.0637
20	10	1	10	0.0420

V. CONCLUSION

We present an H_∞-like control method for hyperbolic systems. Necessary conditions in the form of a two-point-boundary-value problem for determining the optimum controller and the worst exogenous input that can be attenuated by the optimum controller have been derived. The results are related to the H_∞ control problem in the sense that the H_∞ norm is given by the inverse of square root of λ^* [14]. The disturbance rejection capacity has been computed for a cantilever beam. Further research needs to be done to develop state feedback and output feedback controllers, and to extend the method to the infinite horizon control problems.

VI. REFERENCES

[1] Ahmed N.U. and K.L. Teo, *Optimal Control of Distributed parameter Systems*, North Holland, New York, 1981.

[2] Biswas S.K. and N.U. Ahmed, "Optimal Control of Large Space Structures Governed by a Coupled System of Ordinary and Partial Differential Equations", *Mathematics of Control, Signals, and Systems*, Vol. 2, pp. 1 - 16, 1989.

[3] Curtain R., "H_∞ Control for Distributed Parameter Systems: A Survey", *Proc. IEEE CDC*, pp. 22 - 26, 1990.

[4] Doyle John, K. Glover, P. Khargonekar, and B. Francis, "State- Space Solutions to Standard H_2 and H_∞ Control Problems", *IEEE Trans. on Automatic Control*, Vol. AC 34, pp. 831 - 847, 1989.

[5] Francis Bruce, *A Course in H_∞ Control theory*, Springer Verlag, Berlin, 1987.

[6] Francis B. and John Doyle, "Linear Control Theory with an H_∞ Optimality Criterion", *SIAM Jour. on Control and Optimization*, Vol. 25, pp. 815 - 844, 1987.

[7] Khargonekar P., I. Petersen, and M. Rotea, "H_∞ Optimal Control with State Feedback", *IEEE Trans. on Automatic Control*, Vol. AC 33, pp. 786 - 788, 1988.

[8] Lions J.L., *Optimal Control of Systems Governed by Partial Differential Equations*, Springer Verlag, Berlin, 1971.

[9] Ozbay H., "H_∞ Control of Distributed Systems: A Skew Toeplitz Approach", *Thesis*, University of Minnesota, June, 1989.

[10] Petersen Ian, "Disturbance Attenuation and H_∞ Optimization: A Design Method Based on the Algebraic Riccati Equation", *IEEE Trans. on Automatic Control*, Vol. AC 32, pp. 427 - 429, 1987.

[11] Pritchard A.J. and D. Salamon, "The Linear Quadratic Optimal Control Problem for Infinite Dimensional Systems with Bounded and Unbounded Operators", *SIAM Journal on Control and Optimization*, Vol. 25, pp. 121 - 144, 1987.

[12] Pritchard A.J. and S. Townley, "Robustness of Linear Systems", *Journal of Differential Equations*, Vol. 77, pp. 254 - 286, 1989.

[13] Subrahmanyam M.B., "Worst-Case Design of Optimal Controllers over a Finite Horizon", *Technical report, Naval Air Development center, NADC 91005-60*, Warminster, PA, January, 1991.

[14] Subrahmanyam M.B., *Optimal Control with a Worst Case Performance Criterion and Applications*, Lecture Notes in Control and Information Sciences, No. 145, Springer Verlag, New York, 1990.

[15] Wexler D., "On Frequency Domain Stability for Evolution Equations in Hilbert Space via the Algebraic Riccati equation", *SIAM Journal on Mathematical Analysis*, Vol. 11, pp. 969 - 983, 1980.

[16] Zhou K. and P.P. Khargonekar, "An Algebraic Riccati Equation Approach to H_∞ Optimization", *System and Control Letters*, Vol. 11, pp. 85 - 91, 1988.