LQG OPTIMAL COMPENSATOR TRANSFER FUNCTION FOR THE NASA LaRC CSI EVOLUTIONARY MODEL

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Abstract

Following the general form for LQG optimal compensators for flexible structures with collocated rate sensors we develop an explicit compensator transfer function for the NASA LaRC CSI Evolutionary model in the form:

$$\psi(i\omega) = g\, i\omega B_u^*(-M_b\omega^2 + T(i\omega) + i\gamma\omega B_u B_u^*)^{-1} B_u$$

where $T(i\omega)$ is a 48×48 positive definite matrix whose derivation is the main result of this report. The undamped mode frequencies can be expressed in terms of $T(i\omega)$ as the zeros of

$$\text{Det}\, |-\omega^2 M_b + T(i\omega)|$$

while "clamped-clamped" modes of the structure (with all nodes clamped) are the poles.

1. Introduction

In this paper we present an explicit compensator transfer function for the NASA LaRC Evolutionary Model [1], using the Continuum Model developed in [2]. In particular the notation follows closely that in [2]. The compensator is obtained upon specialization of the general development in [3].

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2. Compensator Transfer Function

The optimal compensator transfer function is given by (see [3]):

$$\frac{p}{\sqrt{\lambda}} \sqrt{d_a/d_r}\, B^*(p^2M + A + p\gamma BB^*)^{-1} B \, .$$

The main step is to calculate

$$\psi(p) = B^*(p^2M + A + p\gamma BB^*)^{-1} B \, .$$

We shall consider only the Continuum Model in [2] (case 3) in which the main bus, the tower and the appendages are flexible but the antenna is lumped.

Let

$$\psi(p)v = u$$

$$B^*(p^2M + A + p\gamma BB^*)^{-1} Bv = u \, .$$

Let

$$(p^2M + A + p\gamma BB^*)^{-1} Bv = \begin{vmatrix} f \\ b \end{vmatrix} \, .$$

Then

$$p^2M_0f + A_0f = 0 \tag{2.1}$$

where f is also subject to the "linkage conditions" (see [2]),

$$p^2M_bb + A_bf + p\gamma B_uB_u^*b = B_uv \tag{2.2}$$

and

$$u = B_u^*b \, .$$

We shall now specialize to $p = i\omega$, ω real.

To solve (2.1), we let

$$z = \begin{vmatrix} z_1 \\ \vdots \\ z_8 \end{vmatrix} ,$$

where

$$
\begin{aligned}
z_1 &= f(0,0,0)\\
z_2 &= f_x(0,0,0)\\
z_3 &= f_y(s_2, 0-, 0)\\
z_4 &= f_y(s_2, 0+, 0)\\
z_5 &= f_z(s_T, 0, 0+)\\
z_6 &= f_x(s_4+, 0, 0)\\
z_7 &= f_y(s_5, 0-, 0)\\
z_8 &= f_y(s_5, 0+, 0)\ .
\end{aligned}
$$

Let

$$
\mathcal{A}(-p^2) = \begin{vmatrix} 0 & I \\ A_2^{-1}(A_0 + p^2 M_0) & A_2^{-1} A_1 \end{vmatrix}
$$

if $p = i\omega$,

$$
\mathcal{A}(-p^2) = \mathcal{A}(\omega)
$$

in the notation of [2]. From now on, let $p = i\omega$. Let

$$
e^{\mathcal{A}(\omega)s} = \begin{vmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{vmatrix}
$$

and

$$
e^{\mathcal{A}_{S_2}(\omega)s} = \begin{vmatrix} P_{S_2,11}(s) & P_{S_2,12}(s) \\ P_{S_2,21}(s) & P_{S_2,22}(s) \end{vmatrix}
$$

and $\mathcal{A}_{S_T}$, $\mathcal{A}_{S_5}$ as before in [2]. Then

$$
\begin{vmatrix} f(s,0,0) \\ f_x(s,0,0) \end{vmatrix} = e^{\mathcal{A}(\omega)s} \begin{vmatrix} z_1 \\ z_2 \end{vmatrix}, \qquad 0 < s < s_2
$$

and

$$\begin{vmatrix} f(s_2, 0, 0) \\ f_x(s_2-, 0, 0) \end{vmatrix} = e^{\mathcal{A}(\omega)s_2} \begin{vmatrix} z_1 \\ z_2 \end{vmatrix}$$

$$\begin{vmatrix} f(s_2, s, 0) \\ f_y(s_2, s, 0) \end{vmatrix} = e^{\mathcal{A}_{S_2}(\omega)s} \begin{vmatrix} f(s_2,0,0) \\ z_3 \end{vmatrix}, \qquad -\ell_1, \le s \le 0$$

$$= e^{\mathcal{A}_{S_2}(\omega)s} \begin{vmatrix} f(s_2,0,0) \\ z_4 \end{vmatrix}, \qquad 0 \le s \le \ell_1 .$$

Let us now display the values at the nodes only. Then going along the main bus: by Linkage Condition (2) (in [2]):

$$f_x(s_2+, 0, 0) = f_x(s_2-, 0, 0) + A_2^{-1} A_{2,S_2}(z_3 - z_4) .$$

Hence

$$\begin{vmatrix} f(s_T, 0, 0) \\ f_x(s_T-, 0, 0) \end{vmatrix} = e^{\mathcal{A}(\omega)(s_T - s_2)} \begin{vmatrix} f(s_2,0,0) \\ f_x(s_2+, 0, 0) \end{vmatrix}$$

and

$$\begin{vmatrix} f(s_4, 0, 0) \\ f_x(s_4-, 0, 0) \end{vmatrix} = e^{\mathcal{A}(\omega)(s_4 - s_T)} \begin{vmatrix} f(s_T,0,0) \\ f_x(s_T+, 0, 0) \end{vmatrix} .$$

By Linkage Condition (1) (in [2]):

$$f_x(s_T+, 0, 0) = f_x(s_T-, 0, 0) - A_2^{-1}(L_{1,T} f(s_T,0,0) - A_{2,T} z_5)$$

$$\begin{vmatrix} f(s_5, 0, 0) \\ f_x(s_5-, 0, 0) \end{vmatrix} = e^{\mathcal{A}(\omega)(s_5 - s_4)} \begin{vmatrix} f(s_4,0,0) \\ z_6 \end{vmatrix}$$

and finally

$$\begin{vmatrix} f(L, 0, 0) \\ f_x(L, 0, 0) \end{vmatrix} = e^{\mathcal{A}(\omega)(L - s_5)} \begin{vmatrix} f(s_5, 0, 0) \\ f_x(s_5+, 0, 0) \end{vmatrix}$$

where by Linkage Condition (3) (in [2]):

$$f_x(s_5+, 0, 0) = f_x(s_5-, 0, 0) + A_{2,S_5}(z_7 - z_8) .$$

Next:

$$\begin{vmatrix} f(s_T, 0, L_T) \\ f_z(s_T, 0, L_T) \end{vmatrix} = e^{\mathcal{A}_{S_T}(\omega)L_T} \begin{vmatrix} f(s_T, 0, 0) \\ z_5 \end{vmatrix}$$

$$\begin{vmatrix} f(s_5, -\ell_2, 0) \\ f_y(s_5, -\ell_2, 0) \end{vmatrix} = e^{\mathcal{A}_{S_5}(\omega)(-\ell_2)} \begin{vmatrix} f(s_5, 0, 0) \\ z_7 \end{vmatrix}$$

$$\begin{vmatrix} f(s_5, +\ell_2, 0) \\ f_z(s_5, +\ell_2, 0) \end{vmatrix} = e^{\mathcal{A}_{S_5}(\omega)\ell_2} \begin{vmatrix} f(s_5, 0, 0) \\ z_8 \end{vmatrix} .$$

Hence we can calculate $f(\cdot, \cdot, \cdot)$ as

$$f = \mathcal{L}(\omega)z .$$

In particular we can calculate b in terms of z. Let

$$b = L_b(\omega)z$$

where $L_b(\omega)$ is a (matrix) function of ω. Let $b = \text{col.}\,(b_1, ..., b_8)$ and

$$L_b = \{L_{ij}\} , \qquad \text{each } L_{ij} \text{ being } 6\times 6 .$$

Then

$$b_1 = z_1 ; \qquad L_{11} = I , \quad L_{1i} = 0 , \quad i \neq 1$$

$$\begin{aligned} b_2 &= f(s_2, -\ell_1, 0) \\ &= P_{S_2,11}(-\ell_1) f(s_2, 0, 0) + P_{S_2,12}(-\ell_1)z_3 \\ &= P_{S_2,11}(-\ell_1)[P_{11}(s_2)z_1 + P_{12}(s_2)z_2] + P_{S_2,12}(-\ell_1)z_3 . \end{aligned}$$

Hence

$$L_{21} = P_{S_2,11}(-\ell_1)P_{11}(s_2)$$

$$L_{22} = P_{S_2,11}(-\ell_1)P_{12}(s_2)$$

$$L_{23} = P_{S_2,12}(-\ell_1)$$

$$b_3 = f(s_2, +\ell_1, 0) = P_{S_2,11}(+\ell_1)(P_{11}(s_2)z_1 + P_{12}(s_2)z_2) + P_{S_2,12}(\ell_1)z_4$$

$$L_{31} = P_{S_2,11}(\ell_1)P_{11}(s_2)$$

$$L_{32} = P_{S_2,11}P_{12}(s_2)$$

$$L_{33} = 0$$

$$L_{34} = P_{S_2,12}(\ell_1)$$

$$b_4 = f(s_T, 0, L_T)$$

$$= P_{S_T,11}(L_T)[P_{11}(s_T - s_2)(P_{11}(s_2)z_1 + P_{12}(s_2)z_2)$$
$$+ P_{12}(s_T - s_2)(P_{21}(s_2)z_1 + P_{22}(s_2)z_2 + A_2^{-1}A_{2,S_2}(z_3 - z_4))]$$
$$+ P_{S_T,12}(L_T)z_5$$

$$L_{41} = P_{S_T,11}(L_T)[P_{11}(s_T - s_2)P_{11}(s_2) + P_{12}(s_T - s_2)P_{12}(s_2)]$$

$$L_{42} = P_{S_T,11}(L_T)[P_{11}(s_T - s_2)P_{12}(s_2) + P_{12}(s_T - s_2)P_{22}(s_2)]$$

$$L_{43} = P_{S_T,11}(L_T)(P_{12}(s_T - s_2)A_2^{-1}A_{2,S_2})$$

$$L_{44} = -P_{S_T,11}(L_T)P_{12}(s_T - s_2)A_2^{-1}A_{2,S_2}$$

$$L_{45} = P_{S_T,12}(L_T)$$

$$b_5 = f(s_4, 0, 0)$$

$$L_{51} = P_{11}(s_4 - s_T)P_{11}(s_T - s_2)P_{11}(s_2) + P_{11}(s_4 - s_T)P_{12}(s_T - s_2)P_{21}(s_2)$$
$$+ P_{12}(s_4 - s_T)P_{21}(s_T - s_2)P_{11}(s_2) + P_{12}(s_4 - s_T)P_{22}(s_T - s_2)P_{21}(s_2)$$
$$- P_{12}(s_4 - s_T)P_{22}(s_T - s_2)A_2^{-1}L_{1,T}\{P_{11}(s_T - s_2)P_{11}(s_2) + P_{12}(s_T - s_2)P_{21}(s_2)\}$$

$$L_{52} = P_{11}(s_4 - s_T)P_{11}(s_T - s_2)P_{12}(s_2) + P_{12}(s_T - s_2)P_{22}(s_2)$$
$$+ P_{12}(s_4 - s_T)P_{21}(s_T - s_2)P_{12}(s_2) + P_{12}(s_4 - s_T)P_{22}(s_T - s_2)P_{22}(s_2)$$
$$- P_{12}(s_4 - s_T)P_{22}(s_T - s_2)A_2^{-1}L_{1,T}\{P_{11}(s_T - s_2)P_{12}(s_2) + P_{12}(s_T - s_2)P_{22}(s_2)\}$$

$$L_{53} = P_{11}(s_4 - s_T)P_{12}(s_T - s_2)A_2^{-1}A_{2,S_2} + P_{12}(s_4 - s_T)P_{22}(s_T - s_2)A_2^{-1}A_{2,S_2}$$
$$- P_{12}(s_4 - s_T)P_{22}(s_T - s_2)A_2^{-1}L_{1,T}\{P_{12}(s_T - s_2)A_2^{-1}A_{2,S_2}\}$$

$$L_{54} = -P_{11}(s_4 - s_T)P_{12}(s_T - s_2)A_2^{-1}A_{2,S_2}$$
$$+ P_{12}(s_4 - s_T)P_{22}(s_T - s_2)A_2^{-1}L_{1,T}\{P_{12}(s_T - s_2)A_2^{-1}A_{2,S_2}\}$$

$$L_{55} = P_{12}(s_4 - s_T)P_{22}(s_T - s_2)A_2^{-1}A_{2,T}$$

$$b_6 = f(s_5, -\ell_2, 0) = P_{S_5,11}(-\ell_2)(P_{11}(s_5 - s_4)b_5 + P_{12}(s_5 - s_4)z_6) + P_{S_5,11}(-\ell_2)z_7$$

$$L_{6i} = P_{S_5,11}(-\ell_2)P_{11}(s_4 - s_5)b_{5i}\,, \qquad i \le 5$$

$$L_{66} = P_{S_5,11}(-\ell_2)P_{12}(s_4 - s_5)$$

$$L_{67} = P_{S_5,12}(-\ell_2)$$

$$b_7 = f(s_5, +\ell_2, 0)$$

$$L_{7i} = P_{S_5,11}(\ell_2)P_{11}(s_5 - s_4)L_{5i}\,, \qquad i \le 5$$

$$L_{76} = P_{S_5,11}(\ell_2)P_{12}(s_5 - s_4)$$

$$L_{77} = 0$$

$$L_{78} = P_{S_5,12}(\ell_2)$$

$$b_8 = f(L, 0, 0)$$

$$L_{8i} = [P_{11}(L - s_5)P_{11}(s_5 - s_4) + P_{12}(L - s_5)P_{21}(s_5 - s_4)]L_{5i}, \qquad i \le 5$$

$$L_{86} = P_{11}(L - s_5)P_{12}(s_5 - s_4) + P_{12}(L - s_5)P_{22}(s_5 - s_4)$$

$$L_{87} = P_{12}(L - s_5)A_2^{-1}A_{2,s_5}$$

$$L_{88} = -P_{12}(L - s_5)A_2^{-1}A_{2,s_5}$$

Hence

$$L_b(\omega) = \begin{vmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ L_{21} & L_{22} & L_{23} & 0 & 0 & 0 & 0 & 0 \\ L_{31} & L_{32} & L_{33} & L_{34} & 0 & 0 & 0 & 0 \\ L_{41} & L_{42} & L_{43} & L_{44} & L_{45} & 0 & 0 & 0 \\ L_{51} & L_{52} & L_{53} & L_{54} & L_{55} & 0 & 0 & 0 \\ L_{61} & L_{62} & L_{63} & L_{64} & L_{65} & L_{66} & L_{67} & 0 \\ L_{71} & L_{72} & L_{73} & L_{74} & L_{75} & L_{76} & 0 & L_{78} \\ L_{81} & L_{82} & L_{83} & L_{84} & L_{85} & L_{86} & L_{87} & L_{88} \end{vmatrix}$$

Suppose ω is a transmission zero of $\psi(i\omega)$:

$$\psi(i\omega)v = 0, \qquad v \neq 0,$$

then in the notation

$$x = \begin{vmatrix} f \\ b \end{vmatrix}$$

we have

$$A_0 f = \omega^2 M_0 f$$

$$A_b f = B_u v + \omega^2 M_b b; \qquad B_u^* b = 0$$

$$L_A(\omega)z - \omega^2 M_b L_b(\omega)z = B_u v.$$

Suppose for some z:

$$(L_A(\omega) - \omega^2 M_b L_b(\omega))z = 0.$$

Then

$$-\omega^2 M_b b + A_b f = 0$$

$$-\omega^2 M_0 f + A_0 f = 0$$

and hence

$$\omega \text{ is an undamped structure mode}$$

which would imply that

$$B_u^* b \neq 0$$

which is a contradiction. Hence

$$f = \mathcal{L}(\omega) z$$

where

$$B_u^* L_b(\omega) z = 0 , \qquad z \neq 0$$

and conversely. Hence the transmission zeros of $\psi(i\omega)$ are precisely the zeros of

$$\text{Det } |B_u B_u^* L_b(\omega)| = 0 .$$

These values of ω are then the "clamped-clamped" modes of the articulated structure:

$$\left.\begin{aligned} -\omega^2 M_0 f + A_0 f &= 0 , \qquad f \neq 0 \\ B_u^* b &= 0 . \end{aligned}\right\}$$

and a subset of these corresponding to $b = 0$ or $L_b(\omega) z = 0$ are the clamped-clamped modes of the unloaded structure (every node is clamped). Let us consider first ω such that

$$\text{Det } |L_b(\omega)| \neq 0$$

so that we can invert $L_b(\omega)$. We have:

$$b_1 = z_1$$

$$b_2 - L_{21} b_1 = L_{22} z_2 + L_{23} z_3$$

$$b_3 - L_{31} b_1 = L_{32} z_2 + L_{34} z_4$$

$$b_4 - L_{41} b_1 = L_{42} z_2 + L_{43}(z_3 - z_4) + L_{45} z_5$$

where

$$z_5 = L_{55}^{-1}(b_5 - L_{51}z_1 - L_{52}z_2 - L_{53}z_3 - L_{54}z_4)$$

These four equations can be solved for z_1, z_2, z_3, z_4, in terms of b_1, ..., b_5, and then z_5 can be expressed in terms of b_5, b_4, b_3, b_2, b_1. Next

$$b_6 - P_{S_5,11}(-\ell_2)P_{11}(s_5 - s_4)b_5 = \tilde{b}_6 = L_{66}z_6 + L_{67}z_7$$

$$b_7 - P_{S_5,11}(\ell_2)P_{11}(s_5 - s_4)b_5 = \tilde{b}_7 = L_{76}z_6 + L_{78}z_8$$

$$b_8 - (P_{11}(L - s_5)P_{11}(s_5 - s_4) + P_{12}(L - s_5)P_{21}(s_5 - s_4))b_5$$
$$= \tilde{b}_8 = L_{86}z_6 + L_{87}z_7 - L_{87}z_8 .$$

These three equations can be solved for z_6, z_7, z_8:

$$L_{87}L_{67}^{-1}(\tilde{b}_6 - L_{66}z_6) - L_{87}L_{78}^{-1}(\tilde{b}_7 - L_{76}z_6) = \tilde{b}_8 - L_{86}z_6$$

$$L_{87}L_{67}^{-1}\tilde{b}_6 - L_{87}L_{78}^{-1}\tilde{b}_7 - \tilde{b}_8 = (L_{87}L_{67}^{-1}L_{66} - L_{87}L_{78}^{-1}L_{76} - L_{86})z_6 .$$

Hence

$$z_6 = (L_{87}L_{67}^{-1}L_{66} - L_{87}L_{78}^{-1}L_{76} - L_{86})^{-1}(L_{87}L_{67}^{-1}\tilde{b}_6 - L_{87}L_{78}^{-1}\tilde{b}_7 - \tilde{b}_8)$$

$$z_7 = L_{67}^{-1}(\tilde{b}_6 - L_{66}z_6)$$

$$z_8 = L_{78}^{-1}(\tilde{b}_7 - L_{76}z_6) .$$

Hence

$$z = L_b(\omega)^{-1}b , \qquad \text{Det } |L_b(\omega)| \neq 0 .$$

Next let

$$A_b f = L_A z .$$

Let us determine L_A. Now from the form of $A_b f$, it is convenient to break up L_A as

$$L_A = L_{A_1} + L_{A_2}$$

where

$$L_{A_1} z = \begin{vmatrix} -L_1 b_1 \\ -L_{1,s_2} b_2 \\ L_{1,s_2} b_3 \\ L_{1,T} b_4 \\ 0 \\ -L_{1,s_5} b_6 \\ L_{1,s_5} b_7 \\ L_1 b_8 \end{vmatrix}$$

and $\{b_i\}$ have been determined in terms of z_i. Next let

$$L_{A_2} z = h$$

so that

$$h_1 = -A_{21} z_2$$

$$h_2 = -A_{2,s_2} P_{s_2,21}(-\ell_1)[P_{12}(s_2)z_1 + P_{12}(s_2)z_2] - A_{2,s_2} P_{s_2,22}(-\ell_1) z_3$$

$$h_3 = A_{2,s_2} P_{s_2,21}(\ell_1)[P_{11}(s_2)z_1 + P_{12}(s_2)z_2] + A_{2,s_2} P_{s_2,22}(-\ell_1) z_3$$

$$\begin{aligned} h_4 = A_{2,s_T}\{ & P_{s_T,22}(L_T) z_5 \\ & + P_{s_T,21}(L_T)[P_{11}(s_T - s_2)(P_{11}(s_2)z_1 + P_{12}(s_2)z_2) \\ & \qquad + P_{12}(s_T - s_2)(P_{21}(s_2)z_1 + P_{22}(s_2)z_2 + A_2^{-1} A_{2,s_2}(z_3 - z_4))]\} \end{aligned}$$

$$\begin{aligned} h_5 = A_2 \Big[& P_{21}(s_4 - s_T)[P_{11}(s_T - s_2)(P_{11}(s_2)z_1 + P_{12}(s_2)z_2) \\ & \qquad + P_{12}(s_T - s_2)(P_{21}(s_2)z_1 + P_{22}(s_2)z_2 + A_2^{-1} A_{2,s_2}(z_3 - z_4))] \\ & + P_{22}(s_4 - s_T)\{P_{21}(s_T - s_2)(P_{11}(s_2)z_1 + P_{12}(s_2)z_2) \\ & \qquad + P_{22}(s_T - s_2)(P_{21}(s_2)z_1 + P_{22}(s_2)z_2 + A_2^{-1} A_{2,s_2}(z_3 - z_4)) \\ & \qquad - A_2^{-1} L_{1,T}[P_{11}(s_T - s_2)(P_{11}(s_2)z_1 + P_{12}(s_2)z_2) \\ & \qquad\qquad + P_{12}(s_T - s_2)(P_{21}(s_2)z_1 + P_{22}(s_2)z_2 + A_2^{-1} A_{2,s_2}(z_3 - z_4))] \\ & \qquad + A_2^{-1} A_{2,T} z_5\} \Big] \end{aligned}$$

$$h_6 = -A_{2,s_5}\{P_{s_5,21}(-\ell_2)(P_{11}(s_5 - s_4)b_5 + P_{12}(s_5 - s_4)z_6) + P_{s_5,22}(-\ell_2)z_7\}$$

$$h_7 = A_{2,s_5}\{P_{s_5,21}(\ell_2)(P_{11}(s_5 - s_4)b_5 + P_{12}(s_5 - s_4)z_6 + P_{s_5,22}(\ell_2)z_8\}$$

$$h_8 = A_2\{P_{21}(L - s_5)[P_{11}(s_5 - s_4)b_5 + P_{12}(s_5 - s_4)z_6] + P_{22}(L - s_5)[P_{12}(s_5 - s_4)b_5 + P_{22}(s_5 - s_4)z_6 + A_2^{-1}A_{2,s_5}(z_7 - z_8)]\} .$$

Hence finally, in terms of L_A and L_b, (2.2) becomes

$$(-\omega^2 M_b L_b + L_A + \gamma i\omega B_u B_u^* L_b)z = B_u v .$$

For ω such that $L_b(\omega)$ is nonsingular we can write:

$$u = B_u^* b = B_u^* L_b(-\omega^2 M_b L_b + L_A + \gamma i\omega B_u B_u^* L_b)^{-1} B_u v$$

$$= B_u^*(-\omega^2 M_b + L_A L_b^{-1} + \gamma i\omega B_u B_u^*)^{-1} B_u v .$$

Here

B_u is 48×8

B_u^* is 8×48

and

$$(-\omega^2 M_b + L_A L_b^{-1} + \gamma i\omega B_u B_u^*)^{-1} \quad \text{is} \quad 48 \times 48$$

and is conveniently broken up into 6×6 blocks, denoted

$$D = \{D_{ij}\} , \qquad i, j = 1,\ldots,8 .$$

We can now calculate

$$B_u^* D B_u .$$

Now

$$B_u u = \begin{vmatrix} B_{u_1} u \\ \vdots \\ B_{u_8} u \end{vmatrix}$$

where each B_{u_i} is 6×8, and letting

$$u = \begin{vmatrix} u_1 \\ u_2 \\ \vdots \\ u_8 \end{vmatrix}$$

we have

$$B_{u_1} u = \text{col} [\, 0,\ u_1,\ u_2,\ 0,\ 0,\ 0 \,]$$

$$B_{u_2} = B_{u_3} = 0$$

$$B_{u_4} u = \text{col} [\, u_3,\ u_4,\ 0,\ 0,\ 0,\ 0 \,]$$

$$B_{u_5} u = \text{col} [\, 0,\ u_5,\ u_6,\ 0,\ 0,\ 0 \,]$$

$$B_{u_6} = B_{u_7} = 0$$

$$B_{u_8} u = \text{col} [\, u_7,\ u_8,\ 0,\ -40u_8,\ 40u_7,\ 0 \,] .$$

Hence letting

$$b = \begin{vmatrix} b_1 \\ \vdots \\ b_8 \end{vmatrix}$$

$$B_u^* b = \sum_1^8 B_{u_i}^* b_i = B_{u_1}^* b_1 + B_{u_4}^* b_4 + B_{u_5}^* b_5 + B_{u_8}^* b_8$$

and writing

$$h = \begin{vmatrix} u \\ \text{v} \\ w \\ \phi_1 \\ \phi_2 \\ \phi_3 \end{vmatrix} .$$

$$B_{u_1}^* h = \text{col} [\, \text{v},\ \text{w},\ 0,\ 0,\ 0,\ 0,\ 0,\ 0 \,]$$

$$B_{u_4}^* h = \text{col} [\, 0,\ 0,\ u,\ \text{v},\ 0,\ 0,\ 0,\ 0 \,]$$

$$B_{u_5}^* h = \text{col} [\, 0,\ 0,\ 0,\ 0,\ \text{v},\ w,\ 0,\ 0 \,]$$

$$B_{u_8}^* h = \text{col} [\, 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ u+40\phi_2,\ \text{v}-40\phi_1 \,] .$$

Hence

$$(B_u^* D B_u)\mathrm{v} = B_{u_1}^* \sum_1^8 D_{1j} B_{u_j}\mathrm{v} + B_{u_4}^* \sum_1^8 D_{4j} B_{u_j}\mathrm{v} + B_{u_5}^* \sum_1^8 D_{5j} B_{u_j}\mathrm{v} + B_{u_8}^* \sum_1^8 D_{8j} B_{u_j}\mathrm{v}$$

$$\begin{aligned} &= B_{u_1}^*(D_{11}B_{u_1} + D_{14}B_{u_4} + D_{15}B_{u_5} + D_{18}B_{u_8})\mathrm{v} \\ &\quad + B_{u_4}^*(D_{41}B_{u_1} + D_{44}B_{u_4} + D_{45}B_{u_5} + D_{48}B_{u_8})\mathrm{v} \\ &\quad + B_{u_5}^*(D_{51}B_{u_1} + D_{54}B_{u_4} + D_{55}B_{u_5} + D_{58}B_{u_8})\mathrm{v} \\ &\quad + B_{u_8}^*(D_{81}B_{u_1} + D_{84}B_{u_4} + D_{85}B_{u_5} + D_{88}B_{u_8})\mathrm{v} \,. \end{aligned}$$

This shows in particular that we do not need to calculate all the D_{ij}. Also the controls at location S_i, $i = 1, 4, 5, 8$ are given by

$$B_{u_i}^*(D_{i1}B_{u_1} + D_{i4}B_{u_4} + D_{i5}B_{u_5} + D_{i8}B_{u_8})\mathrm{v} = B_{u_i}^* D_{ii} B_{u_i}\mathrm{v} + \sum_{j \neq i} B_{u_i}^* D_{ij} B_{u_j}\mathrm{v} \,,$$

where the first term involves only the sensors at locations S_i, and the summation represents the coupling to sensors at other locations. Also

$$B_{u_1}^* D_{11} B_{u_1}\mathrm{v} = \begin{vmatrix} D_{11,22}\mathrm{v}_1 + D_{11,23}\mathrm{v}_2 \\ D_{11,32}\mathrm{v}_1 + D_{11,33}\mathrm{v}_2 \\ 0 \\ \vdots \\ 0 \end{vmatrix} \qquad \mathrm{v} = \begin{vmatrix} \mathrm{v}_1 \\ \mathrm{v}_2 \\ \vdots \\ \mathrm{v}_8 \end{vmatrix}$$

where

$$D_{11} = \{D_{11,ij}\} \,, \qquad i, j = 1, \ldots, 6 \,.$$

Similarly

$$B_{u_4}^* D_{44} B_{u_4} = \begin{vmatrix} 0 \\ 0 \\ D_{44,11}\mathrm{v}_3 + D_{44,12}\mathrm{v}_4 \\ D_{44,21}\mathrm{v}_3 + D_{44,22}\mathrm{v}_4 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

$$B_{u_5}^* D_{55} B_{u_5} = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ D_{55,22}v_5 + D_{55,23}v_6 \\ D_{55,32}v_5 + D_{55,33}v_6 \\ 0 \\ 0 \end{vmatrix}$$

$$B_{u_8}^* D_{88} B_{u_8} = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ (D_{88,11}+40D_{88,15})v_7 + (D_{88,12}-40D_{88,14})v_8 \\ \quad + 40\{(D_{88,51}+40D_{88,55})v_7 + (D_{88,52}-40D_{88,54})v_8\} \\ (D_{88,21}+40D_{88,25})v_7 + (D_{88,22}-40D_{88,24})v_8 \\ \quad + 40\{(D_{88,41}+40D_{88,45})v_7 + (D_{88,42}-40D_{88,44})v_8\} \end{vmatrix}$$

It must be noted that in terms of feedback

$$\begin{aligned} v_1 &= \dot{v}(0) \\ v_2 &= \dot{w}(0) \\ v_3 &= \dot{u}(S_T) + 100\dot{\phi}_2(S_T) \\ v_4 &= \dot{v}(S_T) - 100\dot{\phi}_1(S_T) \\ v_5 &= \dot{v}(S_4) \\ v_6 &= \dot{w}(S_4) \\ v_7 &= \dot{u}(L) + 40\dot{\phi}_2(L) \\ v_8 &= \dot{v}(L) - 40\dot{\phi}_1(L) , \end{aligned}$$

the dot denoting derviative.

Let

$$T(i\omega) = L_A(\omega)L_b(\omega)^{-1}$$

so that the poles of $T(i\omega)$ are the zeros of $L_b(\omega)$. We shall show that $T(i\omega)$ is self-adjoint but not nonnegative definite! Let b_1, b_2 be arbitrary real, and let

$$L_b(\omega)^{-1} b_1 = z_1$$

$$L_b(\omega)^{-1} b_2 = z_2$$

$$f_1 = \mathcal{L}(\omega) z_1$$

$$f_2 = \mathcal{L}(\omega) z_2$$

$$x_1 = \begin{vmatrix} f_1 \\ b_1 \end{vmatrix}; \qquad x_2 = \begin{vmatrix} f_2 \\ b_2 \end{vmatrix}.$$

Then

$$[Ax_1, x_2] = [A_0 f_1, f_2] + [A_b f_1, b_2] = \omega^2 [M_0 f_1, f_2] + [A_b f_1, b_2].$$

Similarly

$$[Ax_2, x_1] = [A_b f_2, b_1] + \omega^2 [M_0 f_2, f_1].$$

Since

$$[Ax_1, x_2] = [Ax_2, x_1]$$

we have

$$[A_b f_2, b_1] = [A_b f_1, b_2]$$

or,

$$[A_b f_1, b_2] = \left[L_A(\omega)L_b(\omega)^{-1} b_1,\ b_2\right] = [A_b f_2, b_1] = \left[L_A(\omega)L_b(\omega)^{-1} b_2,\ b_1\right].$$

Hence $T(i\omega)$ is self-adjoint. Next

$$[T(i\omega)b_1,\ b_1] = [A_b f_1,\ b_1] = [Ax_1, x_1] - \omega^2 [M_0 f_1, f_1].$$

Hence $T(i\omega)$ is not nonnegative definite. Note finally that the structure modes frequencies can be expressed:

$$\text{Det}\ |-\omega^2 M_b + T(i\omega)| = 0.$$

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