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Semigroup Approximation and Robust Stabilization of Distributed Parameter Systems

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ABSTRACT

Theoretical results that enable rigorous statements of convergence and exponential stability of Galerkin approximations of LQR controls for infinite dimensional, or distributed parameter, systems have proliferated over the past ten years. In addition, extensive progress has been made over the same time period in the derivation of robust control design strategies for *finite dimensional systems*. However, the study of the convergence of robust finite dimensional controllers to robust controllers for infinite dimensional systems remains an active area of research. In this paper we consider a class of soft-constrained differential games evolving in a Hilbert space. Under certain conditions, a saddle point control can be given in feedback form in terms of a solution to a Riccati equation. By considering a related LQR problem, we can show a convergence result for finite dimensional approximations of this differential game. This yields a computational algorithm for the feedback gain that can be derived from similar strategies employed in infinite dimensional LQR control design problems. The approach described in this paper also inherits the additional properties of stability robustness common to game theoretic methods in finite dimensional analysis. These theoretical convergence and stability results are verified in several numerical experiments.

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(1) INTRODUCTION

During the past ten years, significant progress has been made in the derivation of convergence criteria for Galerkin approximations of linear quadratic regulator control problems in Hilbert spaces [Banks, Gibson79,Gibson91,Ito90...]. Usually, these methods synthesize classical results on the convergence of Galerkin approximations for elliptic, hyperbolic and parabolic partial differential equations with minimization strategies for convex cost functionals. During the past few years, however, researchers studying control theory have been increasingly interested in the derivation of control schemes that are *robust* with respect to uncertainty, either structured or unstructured, in the underlying model [McFarlane,Maciejowski].

One approach that has been employed with success in the development of *finite dimensional* controllers is the min-max, or soft-constrained differential game, formulation [Basar]. In this class of techniques the "best" controller is sought subject to a "worst case" disturbance. Extension of this approach to infinite dimensions has also been made recently *in principle* [Curtain]. However, the arguments regarding the convergence of the associated finite dimensional Riccati approximations can be more delicate than in the similar LQR minimization case [Attouch,Cavazzuti].

In this paper we show that, in certain cases, the solution to the min-max problem is equivalent to the solution of a related LQR minimization problem. In these cases the approximation theory for the LQR minimization problem [Gibson79,Gibson91,Ito90...] can be brought to bear, and the solution of the min-max problem can be approximated by a sequence of solutions to finite dimensional Riccati equations. Thus, this gives a computational method for obtaining a feedback control for a class of infinite dimensional problems which is both optimal and robust with respect to uncertainty.

(2) PROBLEM STATEMENT

Let H, U and W be real, separable Hilbert spaces and suppose that $B \in L(U, H)$ and $\Phi \in L(W, H)$ are bounded operators.

Consider the evolution equation on H

$$\dot{x}(t) = Ax(t) + Bu(t) + \Phi w(t)$$
$$x(0) = x_0 \in H \qquad is\ given \tag{2.1}$$

where A is the infinitesimal generator of a strongly continuous exponentially stable semigroup S(t) on H. In the following discussion, it is assumed that the initial point x(0) is fixed. Furthermore, it is assumed that one can define the observation from the state via the relationship

$$y(t) = Cx(t)$$
$$C \in L(H, H) \tag{2.2}$$

The optimal control problem to be considered is the infinite dimensional version of the so-called "soft-constrained differential game" as described, for example, in [Basar]. This problem can be stated by first defining the "disturbance-augmented" cost functional

$$J(u,w) = \frac{1}{2}\int_0^\infty \{\|y(t)\|_H^2 + (Nu(t), u(t))_U - \frac{1}{\gamma^2}(Mw(t), w(t))_W\}\, dt \quad (2.3)$$

where $\gamma \in \Re$ is a fixed positive constant, and $N \in L(U, U), M \in L(W, W)$.

Define the spaces $\overline{U} = L^2(O, \infty; U)$ and $\overline{W} = L^2(O, \infty; W)$. The differential game to be solved is to find

$$J_0 = \inf_{u \in \overline{U}} \sup_{w \in \overline{W}} J(u,w) \quad (2.4)$$

subject to dynamics governed by (2.1) - (2.2).

A solution (u^0, w^0) is called a saddle point of J(u,w) if and only if

$$J(u^0, w) \le J(u^0, w^0) \le J(u, w^0)$$
$$\forall (u, w) \in \overline{U} \times \overline{W} \quad (2.5)$$

Roughly speaking, the problem to be solved consists of two parts:

(P1) Find conditions that are applicable to a reasonably large class of problems for which there exists a unique saddle point solution

$$(u^0, w^0) \in \overline{U} \times \overline{W},$$

and such that the solution is given in feedback form.

(P2) Find a method for constructing a sequence of finite dimensional approximations whose feedback solutions converge to the solution of (2.4).

Next, we discuss conditions under which P1 and P2 can be solved.

(3.) CHARACTERIZATION OF A SOLUTION

We make the following assumptions (which we will show guarantee that there exists a unique saddle point value to the differential game outlined above):

$(H1)$ *There exists* $d_1 > 0$ *such that* $(Nu,u)_U \geq d_1 \|u\|_U^2$ *for all* $u \in U$

$(H2)$ *There exists* $d_2 > 0$ *such that* $(Mw,w)_W \geq d_2 \|w\|_W^2$ *for all* $w \in W$

$(H3)$ $$BN^{-1}B^* - \gamma^2 \Phi M^{-1} \Phi^* \geq 0$$

Conditions H1 and H2 are necessary to ensure invertibility of N and M. Condition H3 is necessary for the characterization of the saddle point in feedback form, and for the arguments for convergence of finite dimensional approximations. *While condition H3 is somewhat strong, the assumptions above are applicable to a wide variety of distributed parameter control problems and are extremely convenient for consideration of the convergence of finite dimensional Galerkin approximations.* (In a future manuscript we will report on structures application for which H3 is not satisfied, and on our efforts to relax this assumption).

Before stating the main existence and uniqueness result, we consider a reformulation of the state equation. Define the space $\bar{H} = L^2(0, \infty; H)$ and observe that a homogeneous (zero initial data) version of (2.1) is given by

$$\Lambda x = \bar{B}u + \bar{\Phi}w .$$

Here the operator Λ is defined on the domain

$$dom\ \Lambda = \{x \in \bar{H}\ \frac{dx}{dt} - Ax \in \bar{H}, x(0) = 0\}$$

by

$$\Lambda x(t) = \frac{dx}{dt} - Ax(t),$$

and $\bar{B} \in L(\bar{U}, \bar{H})$, and $\bar{\Phi} \in L(\bar{W}, \bar{H})$ are multiplication operators given by

$$(\bar{B}u)(t) = Bu(t)$$
$$(\bar{\Phi}w)(t) = \Phi w(t) .$$

We also define the multiplication operators $\bar{C} \in L(\bar{H}, \bar{H})$, $\bar{N} \in L(\bar{U}, \bar{U})$ and $\bar{M} \in L(\bar{W}, \bar{W})$ in the obvious way. In this paper we frequently will not distinguish between an operator (such as C) and the corresponding multiplication operator ($\bar{C}$) when it is clear from the context which operator is used.

With these definitions, the variation of constants form of the solution to (2.1) can be given by

$$x = \Lambda^{-1}\bar{B}u + \Lambda^{-1}\bar{\Phi}w + f \tag{3.1}$$

where $f = S(t)x_0$. Here, $\Lambda^{-1} \in L(\bar{H}, \bar{H})$ is defined by $\Lambda^{-1}x(t) = \int_o^t S(t-s)x(s)\,ds$. Hence, the differential game is to find

$$J_0 = \inf_{u \in \bar{U}} \sup_{w \in \bar{W}} J(u, w)$$

subject to (3.1).

Theorem (3.1) : Suppose that conditions (H1), (H2) and (H3) hold. Then there exists a unique

$$(u^0, w^0) \in \bar{U} \times \bar{W}$$

such that

$$J_0 = \inf_{u \in \bar{U}} \sup_{w \in \bar{W}} J(u, w) = J(u^0, w^0)$$

Moreover, (u^0, w^0) is given by

$$w^0(t) = \gamma^2 M^{-1} \Phi^* \Pi x^0(t) \tag{3.2}$$

$$u^0(t) = -N^{-1} B^* \Pi x^0(t) \tag{3.3}$$

where

$$x^0 = \Lambda^{-1}Bu^0 + \Lambda^{-1}\Phi w^0 + f \quad .$$

Here Π is a positive definite solution of the algebraic Riccati equation

$$(\Pi x, Ay)_H + (Ax, \Pi y)_H + (C^* Cx, y)_H - (\Pi\Omega\Omega\Pi x, y)_H = 0 \tag{3.4}$$

for any $x, y \in D(A) \subset H$ where

$$\Omega \equiv [BN^{-1}B^* - \gamma^2 \Phi M^{-1} \Phi^*]^{1/2} \geq 0$$

Observe that (H3) guarantees that

$$\Omega = [BN^{-1}B^* - \gamma^2 \Phi M^{-1} \Phi^*]^{1/2}$$

is well defined. Since Equations (3.2), (3.3), and (3.4) relate the saddle point to the algebraic Riccati equation, we can apply known methods and techniques for approximation of Riccati equations to this problem.

The proof of Theorem (3.1) is developed through a sequence of lemmas. (We are using notation and definitions from [Zeidler]).

First, recall that for each fixed (u,w), the G-derivative of J(u,w) with respect to u, denoted by J_u (u,w), is defined by

$$(J_u(u,w), x)_{\bar{U}} = \lim_{t\to 0} \frac{J(u+tx, w) - J(u,w)}{t} \tag{3.5}$$

Similarly, the G-derivative J_w(u,w) of J(u,w) with respect to w is given by

$$(J_w(u,w), y)_{\bar{W}} = \lim_{t\to 0} \frac{J(u, w+ty) - J(u,w)}{t} \tag{3.6}$$

for any $y \in W$. Also,

$$(J_u(u,w), -J_w(u,w)) \in \bar{U} \times \bar{W}$$

and for any $(x, y) \in U \times W$, we have

$$((J_u(u,w), -J_w(u,w)), (x,y))_{\bar{U}\times\bar{W}} = (J_u(u,w), x)_{\bar{U}} + (-J_w(u,w), y)_{\bar{W}} \quad .$$

Recall also that if X is a Hilbert space, then an operator Γ (which may not be linear) from X to X is called strongly monotone if there is a fixed positive number d such that

$$(\Gamma x_1 - \Gamma x_2, x_1 - x_2)_X \geq d\|x_1 - x_2\|_X^2$$

for any $x_1, x_2 \in X$. The following lemma gives some nice properties about G-derivatives of J(u,w).

Lemma 3.1: If (H_1) and (H_2) hold, then the following are true:

i) for fixed $w \in \bar{W}$, $J_u(u,w)$ is strongly monotone on J.

ii) for fixed $u \in \bar{U}$, and sufficiently small γ^2, $-J_w$ is strongly monotone on $\bar{W}$.

iii) for sufficiently small γ^2, $(J_u, -J_w)$ is strongly monotone on $\bar{U} \times \bar{W}$.

Proof: By direct computation using (2.3) we have

$$J(u,w) = \frac{1}{2}(Du,u)_{\bar{U}} + (u, Gw + Ff)_{\bar{U}} + Q(w,f) \tag{3.7}$$

where

$$D = N + B^*(\Lambda^{-1})^* C^* C\Lambda^{-1}B$$

$$G = B^*(\Lambda^{-1})^* C^* C\Lambda^{-1}\Phi$$

$$F = B^*(\Lambda^{-1})^* C^* C$$

$$Q(w,f) = (C\Lambda^{-1}\Phi w, f)_{\bar{H}} - \frac{1}{2\gamma^2}(Mw,w)_{\bar{W}} + \frac{1}{2}(C\Lambda^{-1}\Phi w, C\Lambda^{-1}\Phi w)_{\bar{H}} + \frac{1}{2}(Cf, Cf)_{\bar{H}}$$

According to (3.5), we obtain

$$J_u(u,w) = Du + Gw + Ff \tag{3.8}$$

Therefore, for any $u_1, u_2 \in \bar{U}$, and fixed $w \in \bar{W}$, it follows from (H1) that

$$(J_u(u_1,w) - J_u(u_2,w), u_1 - u_2)_{\bar{U}} = (D(u_1 - u_2), u_1 - u_2)_{\bar{U}}$$

$$\geq (N(u_1 - u_2), u_1 - u_2)_{\bar{U}} \geq d_1 \|u_1 - u_2\|^2_{\bar{U}}$$

Hence, i) is true. Similarly, we can write

$$J(u,w) = -\frac{1}{2}(D'w,w)_{\bar{W}} + (w, G^*u + F'f)_{\bar{W}} + Q'(u,f) \tag{3.9}$$

where

$$D' = \frac{1}{\gamma^2}M - \Phi^*(\Lambda^{-1})^* C^* C\Lambda^{-1}\Phi$$

$$G^* = \textit{the adjoint of } G = \Phi^*(\Lambda^{-1})^* C^* C\Lambda^{-1}B$$

$$F' = \Phi^*(\Lambda^{-1})^* C^* C$$

$$Q'(u,f) = \frac{1}{2}((N + B^*(\Lambda^{-1})^* C^* C\Lambda^{-1}B)u, u)_{\bar{U}} + \frac{1}{2}(Cf, Cf)_{\bar{H}} + (C\Lambda^{-1}Bu, Cf)_{\bar{H}}$$

Hence, from (3.6), we have

$$J_w(u,w) = -D'w + G^* u + F'f \tag{3.10}$$

For any $w_1, w_2 \in \overline{W}$ and fixed $u \in \overline{U}$, it follows from (H2) that

$$(-J_w(u,w_1) + J_w(u,w_2), w_1 - w_2)_{\overline{W}} = (D'(w_1 - w_2), w_1 - w_2)_{\overline{W}}$$

$$= \frac{1}{\gamma^2}(M(w_1 - w_2), w_1 - w_2)_{\overline{W}} - (C\Lambda^{-1}\Phi(w_1 - w_2), C\Lambda^{-1}\Phi(w_1 - w_2))_{\overline{H}}$$

$$\geq \left(\frac{d_2}{\gamma^2} - \|C\Lambda^{-1}\Phi\|^2\right)\|w_1 - w_2\|^2_{\overline{W}}$$

Therefore, for sufficiently small γ^2, there is a positive constant d so that

$$(-J_w(u,w_1) + J_w(u,w_2), w_1 - w_2)_{\overline{W}} \geq d\|w_1 - w_2\|^2_{\overline{W}}$$

Thus, ii) is verified. Finally, for any $(u_1, w_1), (u_2, w_2) \in \overline{U} \times \overline{W}$ and sufficiently small γ^2, it follows that ,

$$((J_u(u_1,w_1) - J_w(u_1,w_1)) - (J_u(u_2,w_2) - J_w(u_2,w_2)), (u_1,w_1) - (u_2,w_2))_{\overline{U}\times\overline{W}}$$

$$= (J_u(u_1,w_1) - J_u(u_2,w_2), u_1 - u_2)_{\overline{U}} + (-J_w(u_1,w_1) + J_w(u_2,w_2), w_1 - w_2)_{\overline{W}}$$

$$= (D(u_1 - u_2) + G(w_1 - w_2), u_1 - u_2)_{\overline{U}} + (D'(w_1 - w_2) - G^*(u_1 - u_2), w_1 - w_2)_{\overline{W}}$$

$$= (D(u_1 - u_2), u_1 - u_2)_{\overline{U}} + (D'(u_1 - u_2), w_1 - w_2)_{\overline{W}}$$

$$+ (G(w_1 - w_2), u_1 - u_2)_{\overline{U}} - (G^*(u_1 - u_2), w_1 - w_2)_{\overline{W}}$$

$$= (D(u_1 - u_2), u_1 - u_2)_{\overline{U}} + (D'(w_1 - w_2), w_1 - w_2)_{\overline{W}}$$

$$\geq d_1\|u_1 - u_2\|^2_{\overline{U}} + d\|w_1 - w_2\|^2_{\overline{W}}$$

Hence, this lemma is completed.

The existence and uniqueness of a saddle point of J(u,w) are illustrated in the following lemma.

Lemma 3.2: Assume that (H1) and (H2) hold. There exists $\theta_0 > 0$ *such that if* $0 < \gamma^2 \le \theta_0$ *, then J(u,w) has a unique saddle point.*

Proof: Clearly J(u,w) is continuous with respect to u and w. Further, it follows directly from (3.7) and (3.9) that for sufficiently small γ^2,

i) $u \to J(u, w)$ is convex for each fixed $w \in \overline{W}$,

ii) $w \to -J(u, w)$ is convex for each fixed $u \in \overline{U}$,

iii) $J(u, w) \to \infty$ as $\|u\| \to \infty$ for each fixed $w \in \overline{W}$,

iv) $-J(u, w) \to \infty$ as $\|w\| \to \infty$ for each fixed $u \in \overline{U}$.

The result follows from Theorem I.1 in [Bensoussan].

The following result characterizes the saddle point.

Corollary 3.1: (u^0, w^0) *is the saddle point of J(u,w) if and only if* $J_u(u^0, w^0) = J_w(u^0, w^0) = 0$.

Proof: See [Zeidler], p. 467.

The next result gives a further characterization of the saddle point.

Lemma 3.3: Let

$$u^0(t) = -N^{-1}B^*(\Lambda^{-1})^* C^* C x^0(t) \tag{3.11}$$

$$w^0(t) = \gamma^2 M^{-1}\Phi^*(\Lambda^{-1})^* C^* C x^0(t) \tag{3.12}$$

where $x^0(t) = \Lambda^{-1}Bu^0(t) + \Lambda^{-1}\Phi w^0(t) + f$. *Then* (u^0, w^0) *is the unique saddle point of J(u,w).*

Proof: From corollary 3.1, we only need to check that

$$J_u(u^0, w^0) = J_w(u^0, w^0) = 0$$

where u^0, w^0 are given by (3.11) and (3.12). In fact, applying (3.8), we have

$$\begin{aligned} J_u(u^0, w^0) &= (N + B^*(\Lambda^{-1})^* C^* C \Lambda B)(-N^{-1}B^*(\Lambda^{-1})^* C^* C x^0) \\ &+ (B^*(\Lambda^{-1})^* C^* C \Lambda^{-1}\Phi)(\gamma^2 M^{-1}\Phi^*(\Lambda^{-1})^* C^* C x^0) + B^*(\Lambda^{-1})^* C^* C f \\ &= B^*(\Lambda^{-1})^* C^* C(-x^0 + \Lambda^{-1}Bu^0 + \Lambda^{-1}\Phi w^0 + f) = 0 \end{aligned}$$

Similarly, from (3.10) we have

$$J_w(u^0, w^0) = -(\frac{1}{\gamma^2}M + (-\Phi^* (\Lambda^{-1})^* C^* C\Lambda^{-1}\Phi)) (\gamma^2 M^{-1}\Phi^* (\Lambda^{-1})^* C^* Cx^0)$$

$$+ (\Phi^* (\Lambda^{-1})^* C^* C\Lambda^{-1}B) (-N^{-1}B^* (\Lambda^{-1})^* C^* Cx^0) + \Phi^* (\Lambda^{-1})^* C^* Cf$$

$$= \Phi^* (\Lambda^{-1})^* C^* C(-x^0 + \Lambda^{-1}\Phi w^0 + \Lambda^{-1}Bu^0 + f) = 0 .$$

Hence, (u^0,w^0) is the unique saddle point of $J(u,w)$.

The following lemma characterizes the saddle point in terms of the solution to an algebraic Riccati equation. We note that H3 has not been used until now.

Lemma 3.4: Assume that (H1)-(H3) hold. Then the unique saddle point (u^0, w^0) *of J can be expressed as*

$$u^0(t) = -N^{-1}B^* \Pi x^0(t)$$

$$w^0(t) = \gamma^2 M^{-1}\Phi^* \Pi x^0(t)$$

where $x^0(t) = (\Lambda^{-1}Bu^0)(t) + (\Lambda^{-1}\Phi w^0)(t) + f(t)$ *, and* Π *is the unique solution of the Riccati equation*

$$(\Pi x, Ay)_{\bar{H}} + (Ax, \Pi y)_{\bar{H}} + (C^* Cx, y)_{\bar{H}} - (\Pi\Omega^2\Pi x, y)_{\bar{H}} = 0$$

for any $x, y \in D(A) \subsetneq H$ *and* $\Omega = (BN^{-1}B^* - \gamma^2\Phi M^{-1}\Phi^*)^{1/2}$.

Proof: This result follows from [Bensoussan].

Therefore, the proof of Theorem 3.1 is completed.

(4.) CONVERGENCE OF GALERKIN APPROXIMATIONS

Perhaps one of the most attractive features of the method described in this paper is that the convergence of the Galerkin approximations of the saddle point solution to the differential game is guaranteed by the rich collection of Galerkin approximation results available from infinite dimensional LQR minimization formulations. This is because the solution of the infinite dimensional LQR minimization problem

$$\inf_{v \in H} \int_0^\infty \{|Cz(t)|_H^2 + |v(t)|_H^2\}\, dt \tag{4.1}$$

subject to the evolution equation in H

$$\begin{aligned} \dot{z}(t) &= Az(t) + \Omega v(t) \\ z(0) &= z_0 \in H \end{aligned} \tag{4.2}$$

is characterized by the same algebraic Riccati equation that solves the differential game of the preceding section, namely

$$(\Pi x, Ay)_H + (Ax, \Pi y)_H + (C^* Cx, y)_H - (\Pi \Omega \Omega \Pi x, y)_H = 0 \tag{4.3}$$

With this observation, the idea is to construct finite dimensional versions of (4.1) - (4.3) and then to apply known convergence results such as those found in [Ito] and [Kappel, Salamon]. To proceed, let $\{H_n\}_{n=1}^{\infty}$ be a family of finite dimensional subspaces of H satisfying

$$H = \overline{\bigcup_{n=1}^{\infty} H_n}$$

We assume that there are operators $A_n \in L(H_n, H_n)$, $\Omega_n \in L(H_n, H_n)$, $C_n \in L(H_n, H_n)$ and that P_n is the orthogonal projection from H to H_n.

With these finite dimensional operators and spaces, one can consider the following LQR minimization problem:

$$\inf_{v \in H_n} \int_0^\infty \{|C_n z_n(t)|_{H_n}^2 + |v(t)|_{H_n}^2\}\, dt \tag{4.1$_n$}$$

subject to the evolution equation in H_n

$$\begin{aligned} \dot{Z}_n(t) &= A_n Z_n t + \Omega_n v(t) \\ Z_n(0) &= P_n Z_0 \end{aligned} \tag{4.2$_n$}$$

The optimal feedback gain for this problem is characterized by a solution to the following algebraic Riccati equation:

$$A_n^* \Pi + \Pi A_n + C_n^* C_n - \Pi \Omega_n \Omega_n \Pi = 0 \tag{4.3$_n$}$$

If Π_n and Π are the minimal nonnegative solutions of $(4.3)_n$ and (4.3) respectively, then we may appeal to results in the distributed parameter control literature ([Banks & Kunisch], [Gibson91], [Ito87,Ito89], [Kappel, Salamon]) for conditions under which $\Pi_n P_n \to \Pi$. The following conditions can be found in [Ito87].

(H4) For each $x \in H$, $S_n(t) P_n x \to S(t) x$ and $S_n^*(t) P_n x \to S^*(t) x$, and the convergence is uniform in t on bounded subintervals of [0,t].

Here, $S_n(t) = exp(tA_n)$ is the semigroup generated by A_n. Note that at t=0, this condition implies that $P_n x \to x$ for all $x \in H$.

(H5) For each $x \in H$, $\Omega_n P_n x \to \Omega x$, $C_n P_n x \to Cx$ and $C_n^* P_n x \to C^* x$.

(H6) **The family of pairs** (A_n, Ω_n) **and** (A_n, C_n) are uniformly stabilizable and uniformly detectable, respectively. In other words,

(i) there exists a sequence of operators $K_n \in L(H_n, H)$ such that

$$sup\|K_n\| \le \infty$$
$$\left\|e^{(A_n - \Omega_n K_n)t} P_n\right\| \le M_1 e^{-\omega_1 t} \qquad t \ge 0$$

for some positive constants M_1 and ω_1, and

(ii)there exists a sequence of operators $G_n \in L(H_n, H)$ such that

$$sup\|G_n\| \le \infty$$
$$\left\|e^{(A_n - G_n C_n)t} P_n\right\| \le M_2 e^{-\omega_2 t} \qquad t \ge 0$$

for some positive constants M_2>1 and ω_2.

The following result is found in [Ito].

Theorem 2 : Under the assumptions (H4)-(H6), the unique nonnegative solution Π_n *of* $(4.3)_n$ *converges strongly to the nonnegative solution* Π *of (4.3); that is,*

$$\lim_{n \to \infty} \|\Pi x - \Pi_n P_n x\| \to 0$$

The point of all this, of course, is that $(4.3)_n$ is finite dimensional, and so we can solve for Π_n numerically. This is done in the next section for specific examples.

5. NUMERICAL EXAMPLES

In this section we report on some numerical results for a simple example involving the heat equation in a rod. Consider the equation

$$y_t(t,x) = a y_{xx}(t,x) + b(x)u(t) + \phi(x)w(t) \qquad 0 \le x \le 1$$
$$y(0,x) = y_0(x) \tag{5.1}$$

with Neumann boundary conditions

$$y_x(t,0) = 0 = y_x(t,1) \tag{5.2}$$

In addition, consider the LQR cost functional

$$J_0(u) = \frac{1}{2}\int_0^\infty \{\int_0^1 q y(t,x)^2 dx + N|u(t)|^2\} dt \tag{5.3}$$

and the disturbance augmented cost functional

$$J_1(u) = \frac{1}{2}\int_0^\infty \{\int_0^1 q y(t,x)^2 dx + N|u(t)|^2 - \frac{1}{\gamma^2} M|w(t)|^2\} dt \tag{5.4}$$

We set $U = \Re$, $W = \Re$, and consider the following two problems:

$$\inf_{u \in \overline{U}} J_0(u) \tag{5.5}$$

subject to dynamics governed by (5.1) with $\phi \equiv 0$, and

$$\inf_{u \in \overline{U}} \sup_{w \in \overline{W}} J_1(u,w) \tag{5.6}$$

subject to the dynamics governed by (5.1).

In our basic numerical experiment, we implement the LQR feed back controller (from (5.5)) in the presence of a disturbance, and then do the same for the game theoretic controller (from (5.6)). We then compare the performance of the two controllers in the presence of disturbance. Before giving some numerical results, we briefly discuss how this problem is reformulated within the framework developed earlier.

First, set $H = L^2(0,1)$ and define the operators $B \in L(U,H)$, $\Phi \in L(W,H)$ by $Bu = b(x)u$, and $\Phi w = \phi(x)w$. In addition, define the operator A on the domain

$$dom\ A = \{y \in H^2(0,1);\quad y'(0) = y'(1) = 0\} \tag{5.7}$$

by $Ay = ay''$.

Next we introduce Galerkin approximations based on finite dimensional spaces H^n with linear spline ("hat" functions) shape functions. This leads to the following finite dimensional version of (5.1):

$$\dot{\alpha}(t) = A^n \alpha(t) + B^n u(t) + \Phi^n w(t) \qquad (5.8)$$

$$\alpha(0) = \alpha_0$$

where $\alpha(t) \in \Re^n$, and A^n, B^n, Φ^n are $n \times n$, $n \times 1$ and $n \times 1$ dimensional matrices, respectively. We use Π_0 to represent the solution of the finite-dimensional algebraic Riccati equation associated with the LQR cost functional

$$J_0^n(u) = \int_0^\infty \{q\alpha(t)^T \alpha(t) + N|u(t)|^2\} dt$$

and Π_1 to represent the solution of the Riccati equation associated with the game theory cost functional

$$J_1^n(u, w) = \int_0^\infty \{q\alpha(t)^T \alpha(t) + N|u(t)|^2 - \frac{1}{\gamma^2} M|w(t)^2|\} dt \quad .$$

In the figures below we plot the approximation to $y(t,x)$ for several different problems.

As data for these examples, we used a=1, N=1, M=1, q=10, $\gamma = 0.5$, $b(x) \equiv 0.25$, $\phi(x) \equiv -0.35$ and $y_o(x)=10x$.

In Figure 1 we plot the open loop solution for the problem

$$\dot{\alpha}(t) = A^n \alpha(t) \quad .$$

In Figure 2 we plot the LQR closed loop solution (no disturbance) for the problem

$$\dot{\alpha}(t) = (A^n - B^n N^{-1} B^{nT} \Pi_0) \alpha(t) \quad .$$

In Figure 3 we plot the LQR closed loop solution (with disturbance term) for the problem

$$\dot{\alpha}(t) = (A^n - B^n N^{-1} B^{nT} \Pi_0) \alpha(t) + \Phi^n w(t) \quad .$$

In Figure 4 we plot the game theory closed loop solution (with disturbance term) for the problem

$$\dot{\alpha}(t) = (A^n - B^n N^{-1} B^{nT} \Pi_1) \alpha(t) + \Phi^n w(t) \quad .$$

While figures (1) - (4) describe the qualitative nature of the transient response in each of the four cases, figures (5)-(8) illustrate an important difference in the examples by taking a cross-section in space at x=2/3. The basic observation to be made is that the game theory controller improves performance (in the sense of driving the state to the zero equilibrium position) in the presence of disturbances. We have performed several such experiments with various parameters (including Dirichlet boundary data) and observed qualitatively the same behavior. We are currently applying this method to systems involving elastic structures. Preliminary results indicate that an LQR controlled system may, even worse than performing poorly, become destabilized in the presence of disturbance. These results will be reported in a future manuscript.

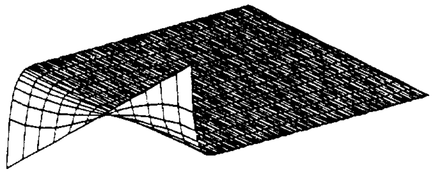

Figure (1) Open Loop Response

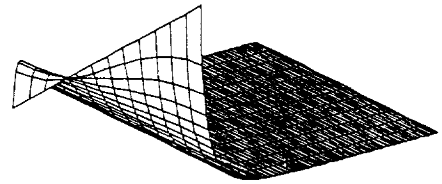

Figure (2) LQR Closed Loop Response
No Disturbance

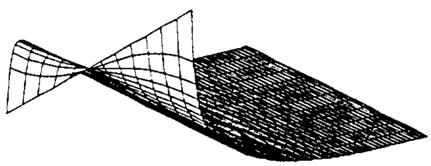

Figure (3) LQR Closed Loop Response
With Disturbance

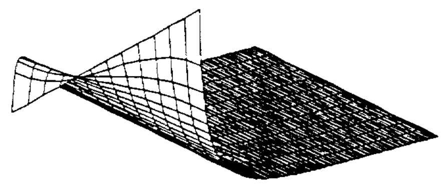

Figure (4) Game Theoretic Closed Loop
Response with Disturbance

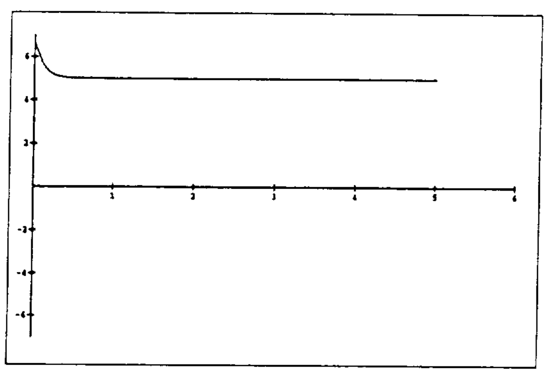

Figure (5) Open Loop Response, x=.75

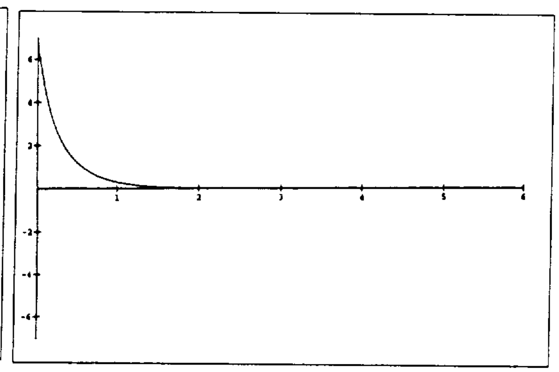

Figure (6) LQR Closed Loop Response, x=.75
Nó Disturbance

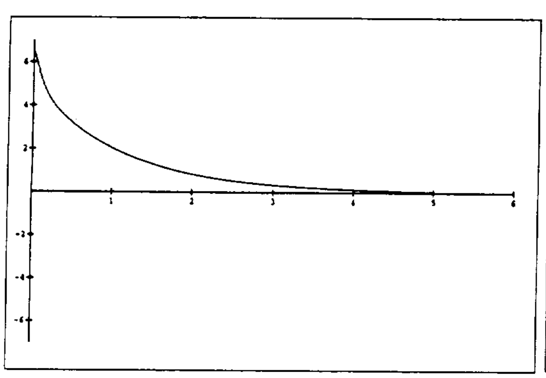

Figure (7) LQR Closed Loop Response, x=.75
With Disturbance

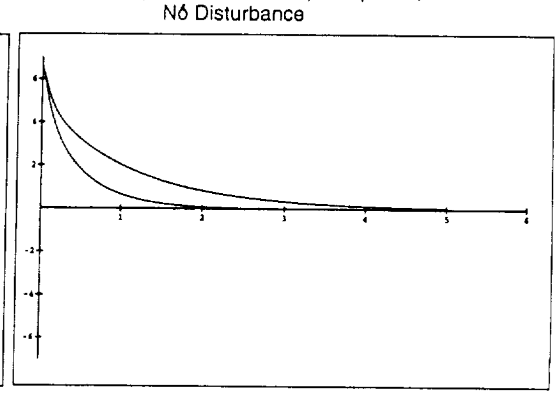

Figure (8) Game / LQR Closed Loop Response, x=.75
With Disturbance

One of the key motivations for the work in this paper is the applicability of the approach to uncertain distributed parameter systems. A second numerical example outlined in this section has been based on examples in [Rosen], but with the introduction of a region of the structure in which the control influence is uncertain. Again, the problem is to find

$$\inf_{u \in \overline{U}} \sup_{w \in \overline{W}} \int_0^{\infty} \{(Qy(t), y(t))_H + N(u(t), u(t))_U - \frac{1}{\gamma^2}(w(t), w(t))_W\} \, dt$$

where

$$\begin{aligned} H &= L^2(0,1) & \overline{H} &= L^2(0,\infty;H) \\ U &= L^2(0,1) & \overline{U} &= L^2(0,\infty;H) \\ W &= \Re & \overline{W} &= L^2(0,\infty;\Re) \end{aligned}$$

The min-max problem stated above is subject to the evolution equation

$$\frac{\partial}{\partial t} y(t,x) = a \frac{\partial^2}{\partial x^2} y(t,x) + bu(t,x) + \phi(x) w(t)$$

$$\begin{aligned} y(t,0) &= y(t,1) = 0 \quad & for \quad & t \geq 0 \\ y(0,\eta) &= y^0(\eta) \quad & for \quad & 0 \leq \eta \leq 1 \end{aligned}$$

where the operator ϕ is defined by

$$\phi(x) w(t) = \begin{cases} \dfrac{w(t)}{\beta_2 - \beta_1} & \beta_1 < x < \beta_2 \\ 0 & otherwise \end{cases}$$

and where a=.25, b=1.0, N=.01, β_1=.49, and β_2=.51. Motivated by [Rosen], the operator Q is defined to be simply the projection onto the first three open loop modes

$$(Qf)(x) = \sum_{i=1}^{3} (f, e_i) e_i(x)$$

where $e_i(x) = \sqrt{2} \sin(i\pi x)$.

The operator ϕ represents the "spatially structured" disturbance. In actual applications, the disturbance could be due to sensor dynamics or structured parametric uncertainty. In either case, the task is to design a finite dimensional controller that is robust with respect to the class of disturbances that can be input by ϕ. Figures (9) through (12) depict the transient response of the heat equation in the rod where

$$\theta = \frac{1}{\gamma^2}$$

is defined to be $\theta = 0.5$ in this example. The system is clearly exponentially stable, as predicted by the theory, despite the introduction of disturbance. From [Curtain] and the discussion earlier in this paper, one can conclude that the disturbance attenuation of the closed loop transfer function from disturbance to input and output is bounded by

$$\|T_{cl}\|_{\infty} < \gamma$$

where

$$\theta = \frac{1}{\gamma^2}$$

Even stronger conclusions can be obtained for this particular problem by noting that the entire heat equation, including disturbance, can be cast in terms of Hilbert-Schmidt operators as described in [Rosen]. The Hilbert-Schmidt norm of the difference between the approximating Riccati equation solutions and the actual Riccati equation solution converges to zero. This is demonstrated graphically in figures (13) through (15) which show the kernels used to represent the Riccati operators. The kernels clearly converge as the level of discretization increases. Furthermore, for the small value of θ=.0001 selected, the Riccati operators should be quite close to the LQR approximations. This is, in fact, the case, as can be concluded by comparing figures (13) - (15) with figures (4.1b), (4.1c) and (4.1d) of [Rosen]. The Riccati kernels for θ=.5 are depicted in figures (16)-(19).

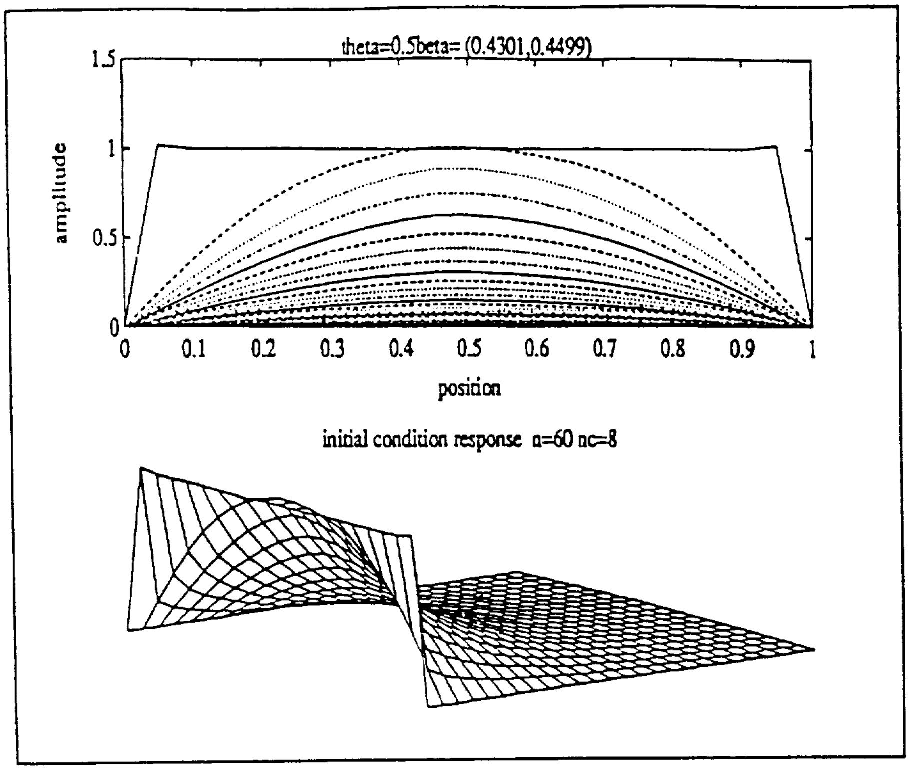

Figure (9) Heat Equation Transient Response, N_c=8

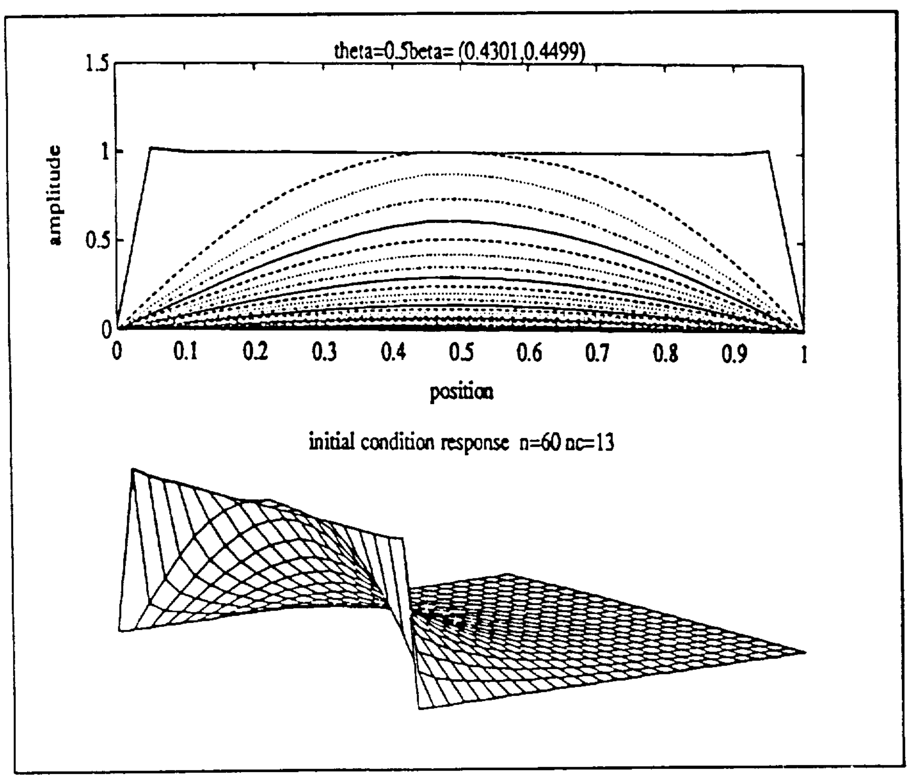

Figure (10) Heat Equation Transient Response, N_c=13

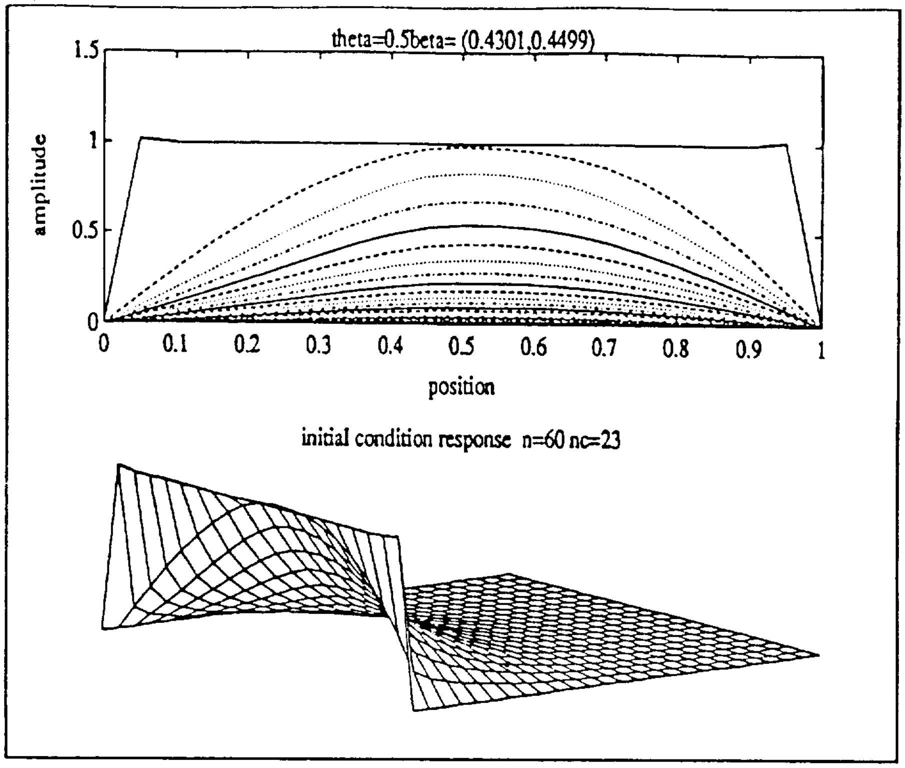

Figure (11) Heat Equation Transient Response, N_c=23

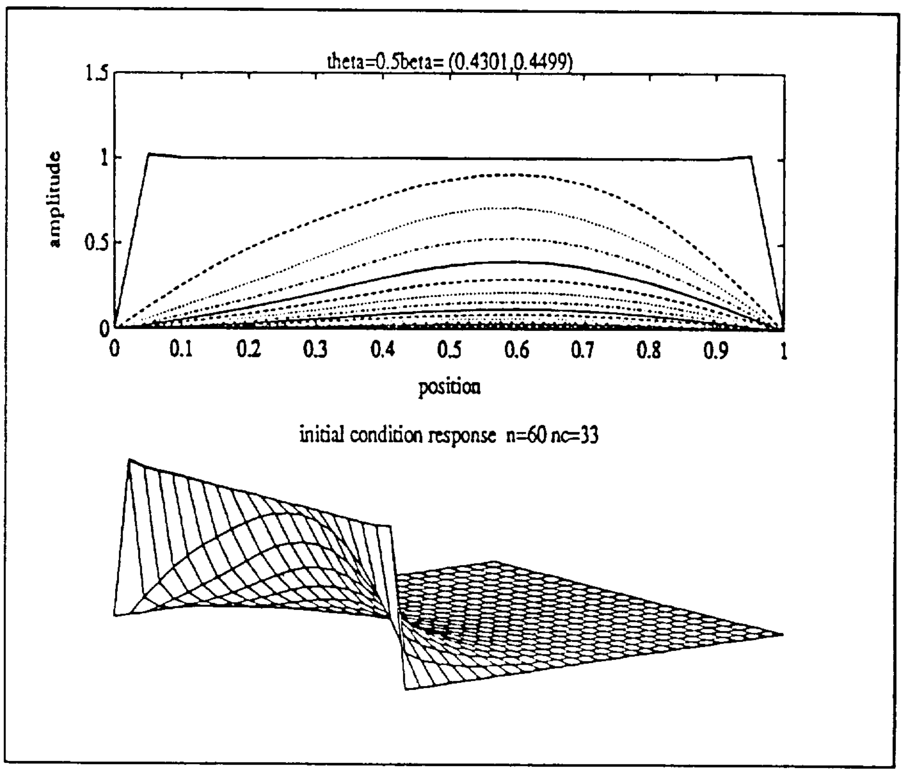

Figure (12) Heat Equation Transient Response, N_c=33

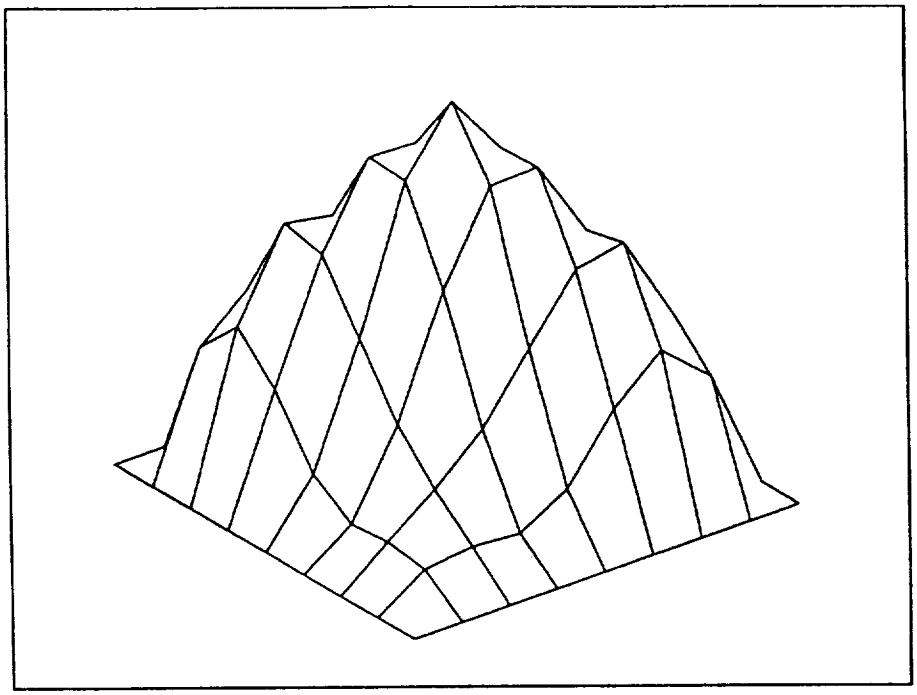

Figure (13) Heat Equation Riccati Kernel, N=8, θ=.0001

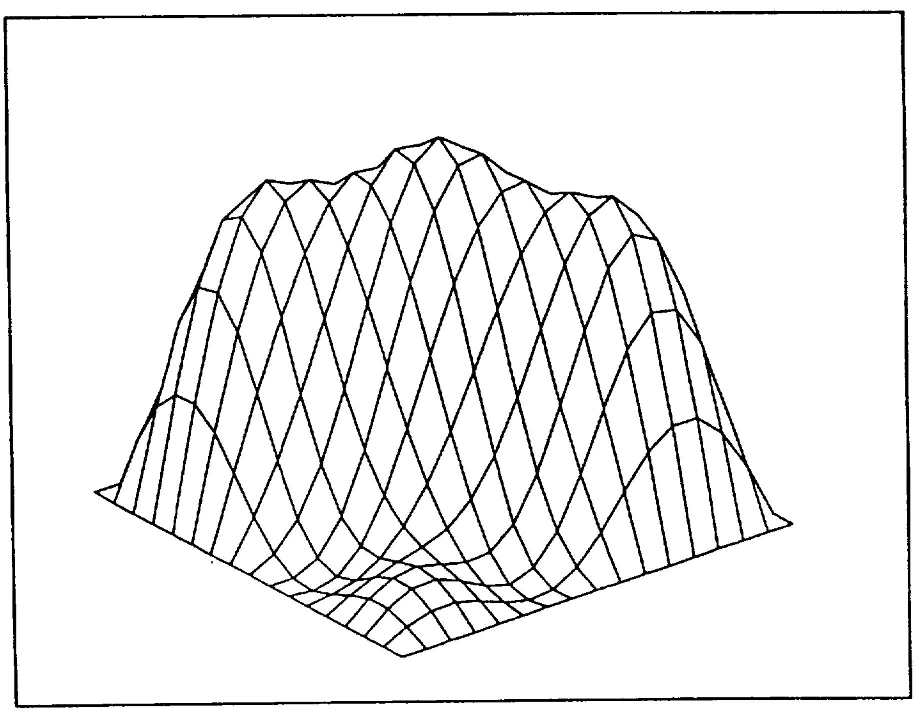

Figure (14) Heat Equation Riccati Kernel, N=8, θ=.0001

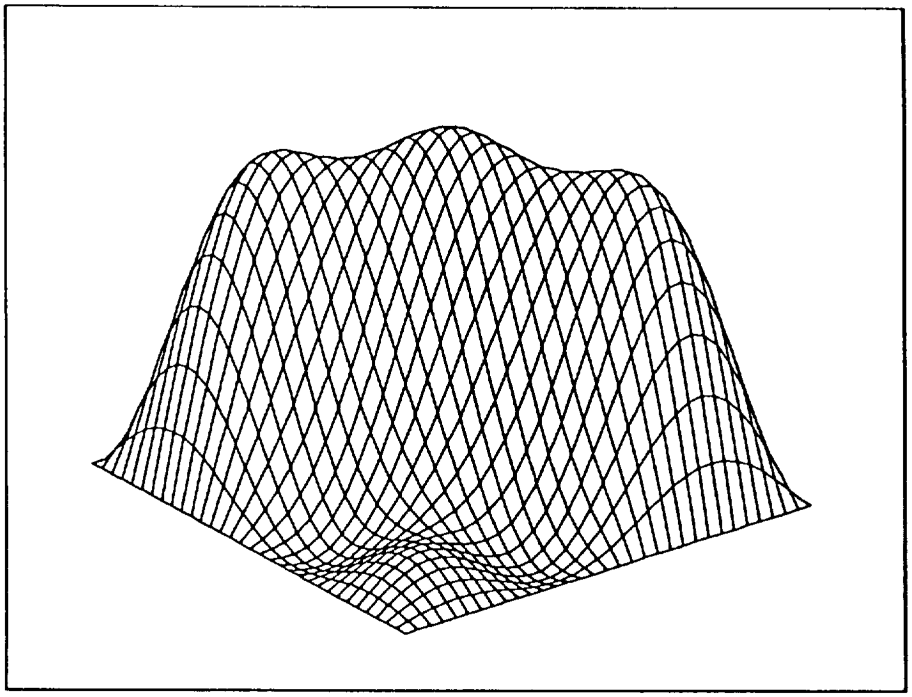

Figure (15) Heat Equation Riccati Kernel, N=8, θ=.0001

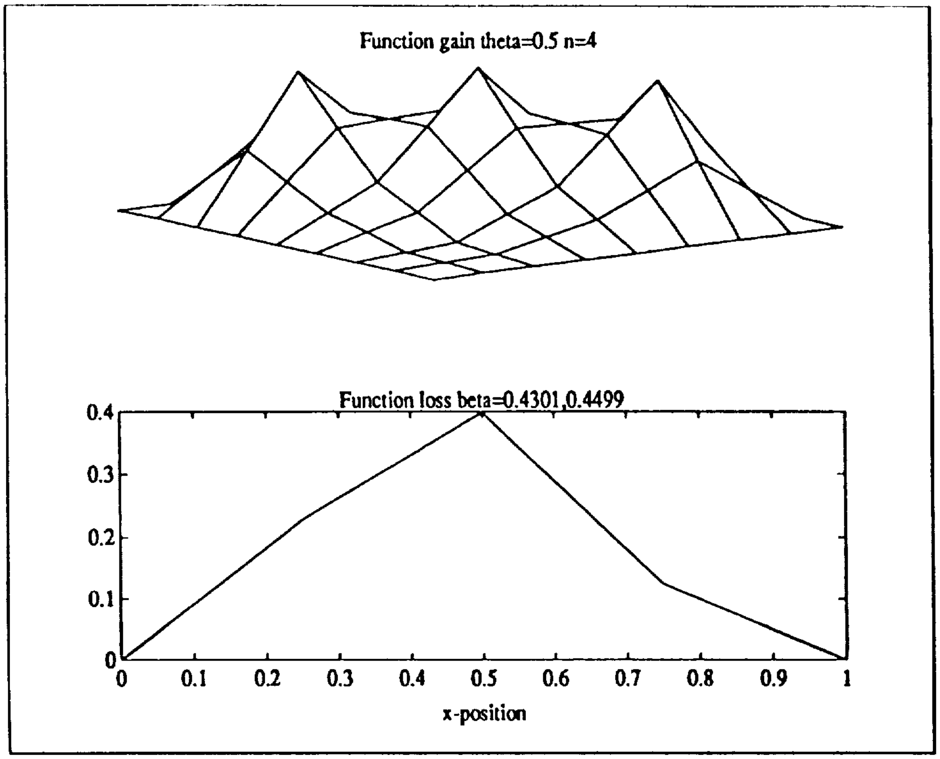

Figure (16) Heat Equation Riccati Kernel, N=8, θ=.5

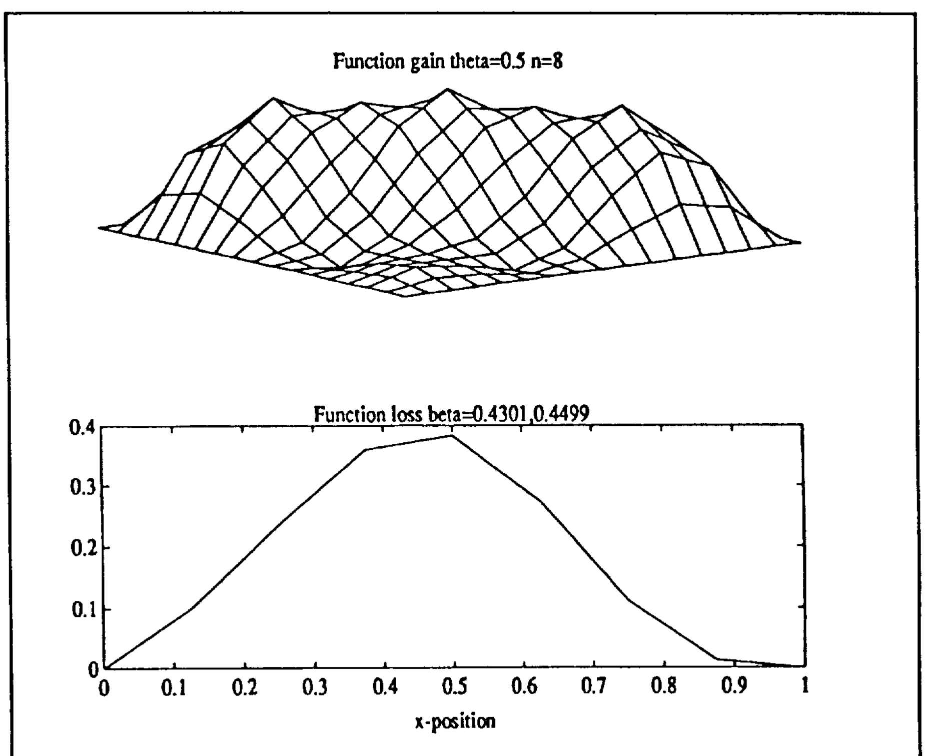

Figure (17) Heat Equation Riccati Kernel, N=16, θ=.5

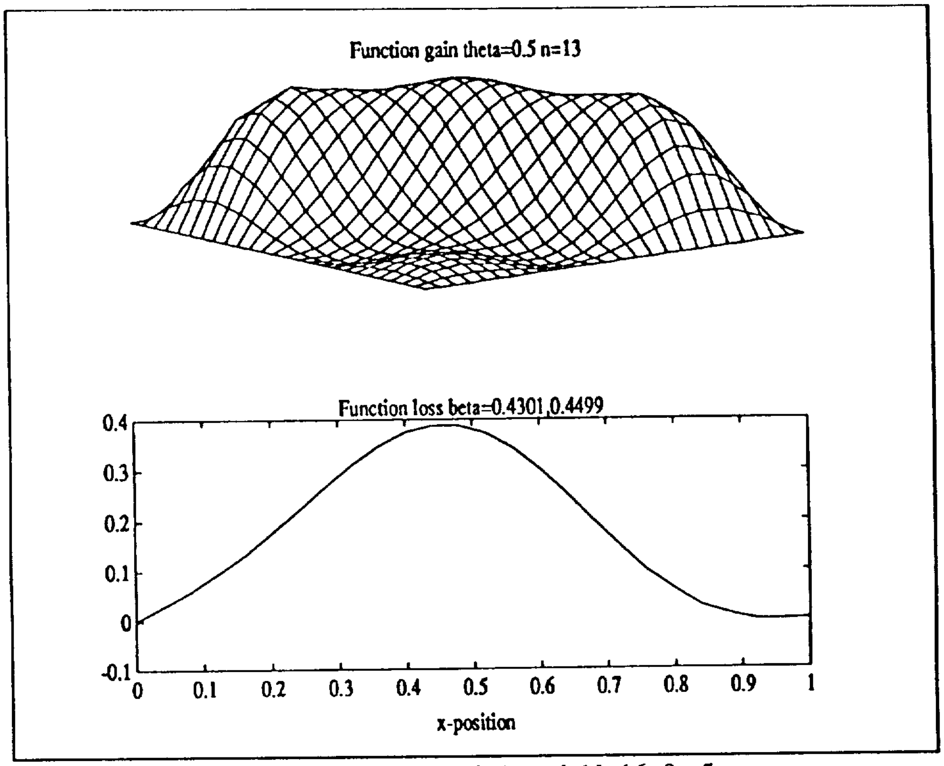

Figure (18) Heat Equation Riccati Kernel, N=16, θ=.5

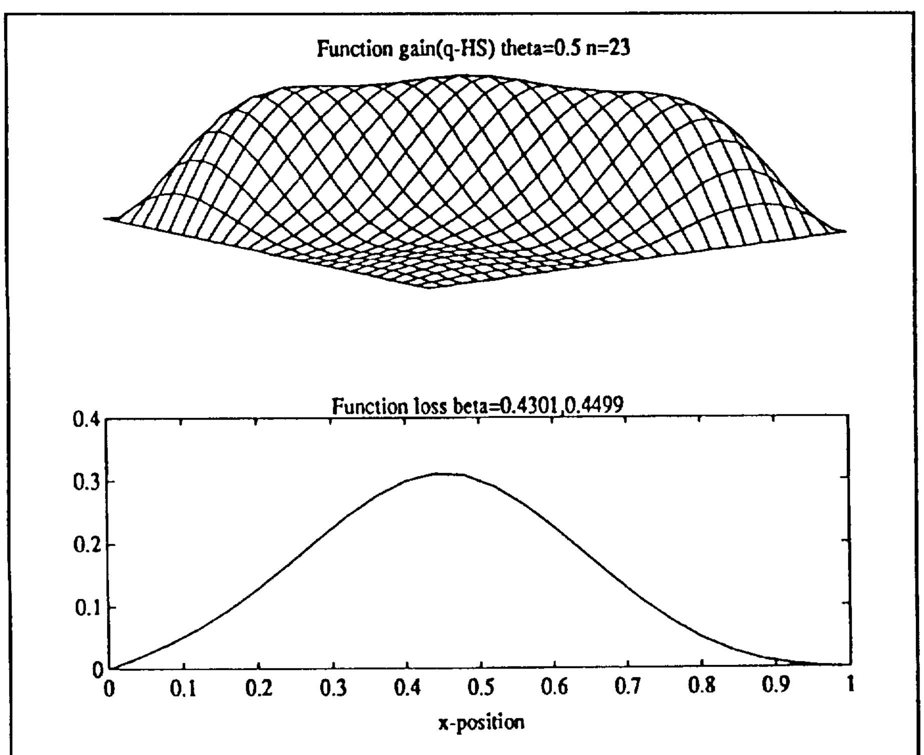

Figure (19) Heat Equation Riccati Kernel, N=32, θ=.5

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