Scattering Cross Section of Sound Waves by the Modal Element Method

Kenneth J. Baumeister
Lewis Research Center
Cleveland, Ohio

and

Kevin L. Kreider
The University of Akron
Akron, Ohio

National Aeronautics and
Space Administration

SCATTERING CROSS SECTION OF SOUND WAVES
BY THE MODAL ELEMENT METHOD

Kenneth J. Baumeister
National Aeronautics and Space Administration
Lewis Research Center
Cleveland, Ohio 44135

Kevin L. Kreider
The University of Akron
Department of Mathematical Sciences
Akron, Ohio 44325-4002

ABSTRACT

The modal element method has been employed to determine the scattered field from a plane acoustic wave impinging on a two dimensional body. In the modal element method, the scattering body is represented by finite elements, which are coupled to an eigenfunction expansion representing the acoustic pressure in the infinite computational domain surrounding the body. The present paper extends the previous work by developing the algorithm necessary to calculate the acoustic scattering cross section by the modal element method. The scattering cross section is the acoustical equivalent to the radar cross section (RCS) in electromagnetic theory. Since the scattering cross section is evaluated at infinite distance from the body, an asymptotic approximation is used in conjunction with the standard modal element method. For validation, the scattering cross section of the rigid circular cylinder is computed for the frequency range $0.1 \leq ka \leq 100$. Results show excellent agreement with the analytic solution.

INTRODUCTION

The modal element method, which couples finite element algorithms to eigenfunction expansions, has been employed in calculating the scattered field from an acoustical plane wave impinging on a two dimensional body. The primary reasons for employing the modal element method are (1) to accurately describe the radiation boundary condition at the computational boundary and (2) to reduce the size of the numerical grid. In fact, for hard scatterers, the modal element method can effectively reduce a two dimensional scattering problem to a one dimensional problem by employing a single line of elements circumscribing the scattering body.

The modal element method has been given various titles, such as the unimoment method and the transfinite element method. In electromagnetics, the method has been applied to scattering from dielectric cylinders (Chang & Mei (1976), Lee & Cendes (1987), Baumeister & Kreider (1992)) and propagation in ducts (Baumeister (1991)). In acoustics, the method has also been applied to scattering from cylinders (Baumeister & Kreider (1993)) and propagation in ducts (Astley & Eversman (1981)). In all of these applications, an eigenfunction expansion is used to represent the acoustic pressure field in the far field. An asymptotic approximation of this expansion presents a simple means of determining the scattering cross section.

This paper presents the numerical algorithm for evaluating the acoustic scattering cross section by the modal element method. The scattering cross section is the acoustical equivalent to the radar cross section (RCS) in electromagnetic theory. Since the scattering cross section is evaluated at infinite distance from the body, asymptotic approximations are used in conjunction with the standard modal element method. For validation, the method is applied to scattering from rigid circular cylinders, for which the analytic solution is known.

NOMENCLATURE

A_m^* modal amplitude of wave moving radially outwards

a dimensionless circular cylinder radius

$H_m^{(1)}$ Hankel function of the first kind

k wave number

m mode number

M_{coef} number of modal coefficients used in eigenfunction expansion

p	dimensionless perturbation acoustic pressure
r	dimensionless radial coordinate
ε	dimensionless property constant
Θ	angle between radius vector and positive x axis
μ	dimensionless property constant
σ^c	acoustic scattering cross section
ω	dimensionless frequency

Superscript

*	complex conjugate

METHOD OF ANALYSIS

The present study is concerned with computing the scattering cross section of a two dimensional axisymmetrical rigid body due to an impinging plane wave traveling in the +x direction. In order to determine the acoustic scattering cross section, the scattered field must be computed. To accomplish this, the spatial domain is divided into two subdomains, the homogeneous domain and the finite element domain, as shown in Fig. 1. The grid system in Fig. 1 is designed to allow various structures to be imbedded into the grid, as shown by inserts A and B. For the special case of rigid bodies, the grid system can be reduced to a single ring of elements, which effectively reduces the two dimensional problem to a one dimensional problem.
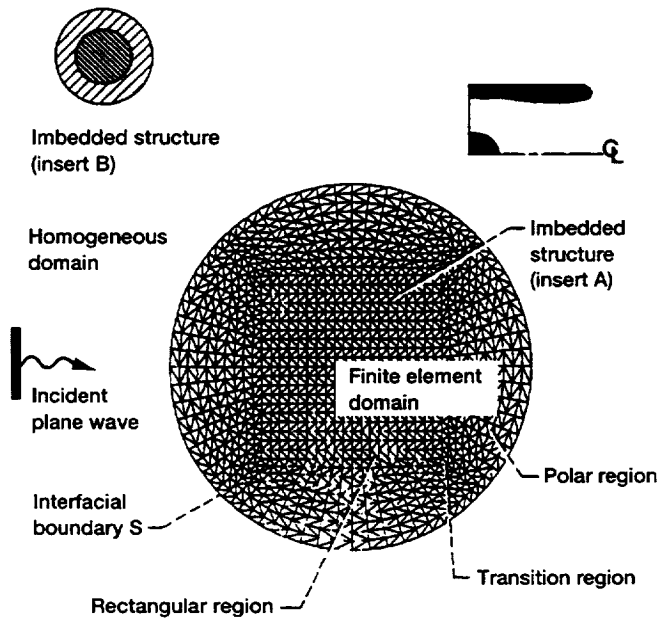

Figure 1.—Finite element grid system.

In the finite element domain, an approximate solution for the total (incident + scattered) pressure at the element nodes is calculated by the Galerkin method. Linear triangular elements are used and the subdomain interface is approximated by piecewise linear segments. In the homogeneous domain, which extends to infinity, the pressure is represented by an eigenfunction expansion. The modal element method couples the two solution forms by imposing continuity on the pressure and velocity at the interface between the two subdomains. This coupling results in a single matrix equation in which the eigenfunction coefficients and the pressure at the finite element nodes are calculated simultaneously, yielding a global representation of the acoustic pressure field.

GOVERNING EQUATION AND BOUNDARY CONDITIONS

The scalar form of the acoustic wave equation can be written as (Baumeister & Kreider (1993))

$$\frac{\partial}{\partial x}\left(\frac{1}{\varepsilon}\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{\varepsilon}\frac{\partial p}{\partial y}\right) + \omega^2 \mu p = 0 \qquad (1)$$

The harmonic time dependence $e^{-i\omega t}$ has been factored out. ε and μ are material property constants and ω is the dimensionless frequency. The wave number is

$$k = \sqrt{\omega^2 \mu \varepsilon} \qquad (2)$$

In the homogeneous region surrounding the body, $\varepsilon=1$ and $\mu=1$, while inside the body, ε and μ may assume other appropriate values depending on the material. Baumeister and Dahl (1989) present explicit expressions for ε and μ as a function of medium density, porosity, viscous loss coefficient and a heat transfer parameter.

At the interface between the finite element region and the analytic region, continuity is imposed on the pressure and velocity. The radiation boundary condition at infinity is automatically satisfied by the eigenfunction expansion introduced in the next section.

ANALYTIC SOLUTION

In the homogeneous domain, an exact expansion for the pressure can be derived (Morse and Ingard, 1968, p. 401) from eq. (1) by separation of variables:

$$p = p^i + p^s = e^{ikx} + \sum_{m=0}^{M_{coef}-1} A_m^+ H_m^{(1)}(kr)\cos(m\theta). \qquad (3)$$

p^i is the incident plane wave; p^s is the scattered wave. This expansion is valid for symmetric bodies. The modal coefficients A_m^+ are unknowns to be determined. Formulas to estimate the number M_{coef} of modes needed for convergence can be found in Baumeister & Kreider (1993).

2

FINITE ELEMENT SOLUTION

The solution to eq. (1) will be obtained by linear finite element theory. In Fig. 1, the finite element region is divided into triangular elements defined by corner nodal points. It is assumed that all material properties are constant in each element. In the conventional weighted residual approach, the unknown pressure field is described in terms of all the unknown nodal values of the pressure.

The finite element aspects of converting eq. (1) and (3) and boundary conditions into an appropriate set of global difference equations can be found in Baumeister & Kreider (1993). The resulting set of global difference equations is solved for the pressure at the nodes and the modal coefficients A_m^+. With the modal coefficients determined, a relatively simple algorithm has been developed to determine the scattered cross section and is described in the next section.

BISTATIC ACOUSTIC SCATTERING CROSS SECTION

The bistatic acoustic scattering cross section per unit length for a two dimensional body can be defined as

$$\sigma^c(\theta) = \lim_{r \to \infty} 2\pi r \frac{\left|p^s\right|^2}{\left|p^i\right|^2} = \lim_{r \to \infty} 2\pi r \frac{\left(p^s \bullet p^{s*}\right)}{\left(p^i \bullet p^{i*}\right)} \qquad (4)$$

The standard superscript c implies a two dimensional problem, where the cross section is often called the "echoing width" (Ruck, vol. 1, pp. 23). In three dimensions, the scattered field decays as $1/r^2$ while in two dimensions, considered herein, the scattered field decays as $1/r$. Thus, the $2\pi r$ is employed in eq. (4) to negate the 2D scattered field decay and to yield a finite measure of the reflected power at infinitey. The cross section defined in eq. (4) could be used in range equations specifically defined for 2D bodies.

For a unit incident plane wave, eq. (4) simplifies to

$$\sigma^c(\theta) = \lim_{r \to \infty} 2\pi r \left(p^s \bullet p^{s*}\right) \qquad (5)$$

Using conventional numerical grids, it is often necessary to extrapolate the nodal values of the field on the outer boundary to infinity. For example, Noack and Anderson (1992) include a boundary element approximation to determine the scattering cross section at infinity. Fortunately, no such additional work is required with the modal element method. Since the scattering cross section is evaluated at infinity, the asymptotic expansion for the Hankel functions (Abramowitz & Stegun, 9.2.3)

$$H_m^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}} e^{i\left(z - \frac{1}{4}\pi\right)} e^{-\frac{1}{2}m\pi} \qquad (6)$$

can be inserted into eq. (3), which is then substituted into eq. (4) to yield

$$\sigma^c(\theta) = \lim_{r \to \infty} 2\pi r \left(\sqrt{\frac{2}{\pi k r}} e^{i\left(kr - \frac{1}{4}\pi\right)} \sum_{m=0}^{M_{coef}-1} A_m^+ e^{-im\frac{\pi}{2}} \cos(m\theta) \right)$$

$$\left(\sqrt{\frac{2}{\pi k r}} e^{-i\left(kr - \frac{1}{4}\pi\right)} \sum_{m=0}^{M_{coef}-1} A_m^{+*} e^{im\frac{\pi}{2}} \cos(m\theta) \right)$$

$$= \frac{4}{k} \left(\sum_{m=0}^{M_{coef}-1} A_m^+ e^{-im\frac{\pi}{2}} \cos(m\theta) \right) \left(\sum_{m=0}^{M_{coef}-1} A_m^{+*} e^{im\frac{\pi}{2}} \cos(m\theta) \right) \qquad (7)$$

Once the A_m^+ have been calculated, the scattering cross section and the pressure field can be easily determined.

RESULTS AND COMPARISONS

Consider the following scattering problem: a unit plane wave, incident from the left, strikes a rigid circular cylinder of dimensionless radius a = 1 oriented with its axis normal to the propagation direction. The rigid body is simulated numerically through an impedance mismatch induced by setting $\varepsilon = 1 - 10^{19} i$ and $\mu = 1$ for each internal finite element and $\varepsilon = 1$ and $\mu = 1$ in the homogeneous domain surrounding the body. This feature allows flexibility in the numerical implementation of the method. Rigid bodies with absorber coatings may be studied with only slight modification to the computer code by adding several finite element rings with the appropriate values of ε and μ.

This problem is solved for frequencies ranging from ka = 0.1 to ka = 100 in order to validate the method. This problem was chosen because it has an exact solution, given by eq. (3) with modal coefficients (Bowman et al., eq. (2.38))

$$A_m^+ = \frac{-\left(2 - \delta_{m0}\right) i^m J_m'(ka)}{H_m^{(1)'}(ka)} \qquad (8)$$

and because it has been examined using other methods (Ling, 1986, Fig. 15).

In Fig. 2, the computed cross sections (square symbols) are compared to the corresponding exact solutions (solid line). The results agree very well with the exact solutions over a wide frequency range. Results for ka = 50 and ka = 100 show similar agreement but are omitted for brevity. In addition, the results agree well with the low frequency calculations (k = 3.2) of Ling (1986, Fig. 15).

Additional validation of the algorithm is obtained by examining high frequency (ka > 20) asymptotic approximations. In the backscattered direction ($\theta = 180$), the optical approximation is valid (Ruck, et al. (1970), eq. (4.1) to (38)) and the backscattering cross section becomes
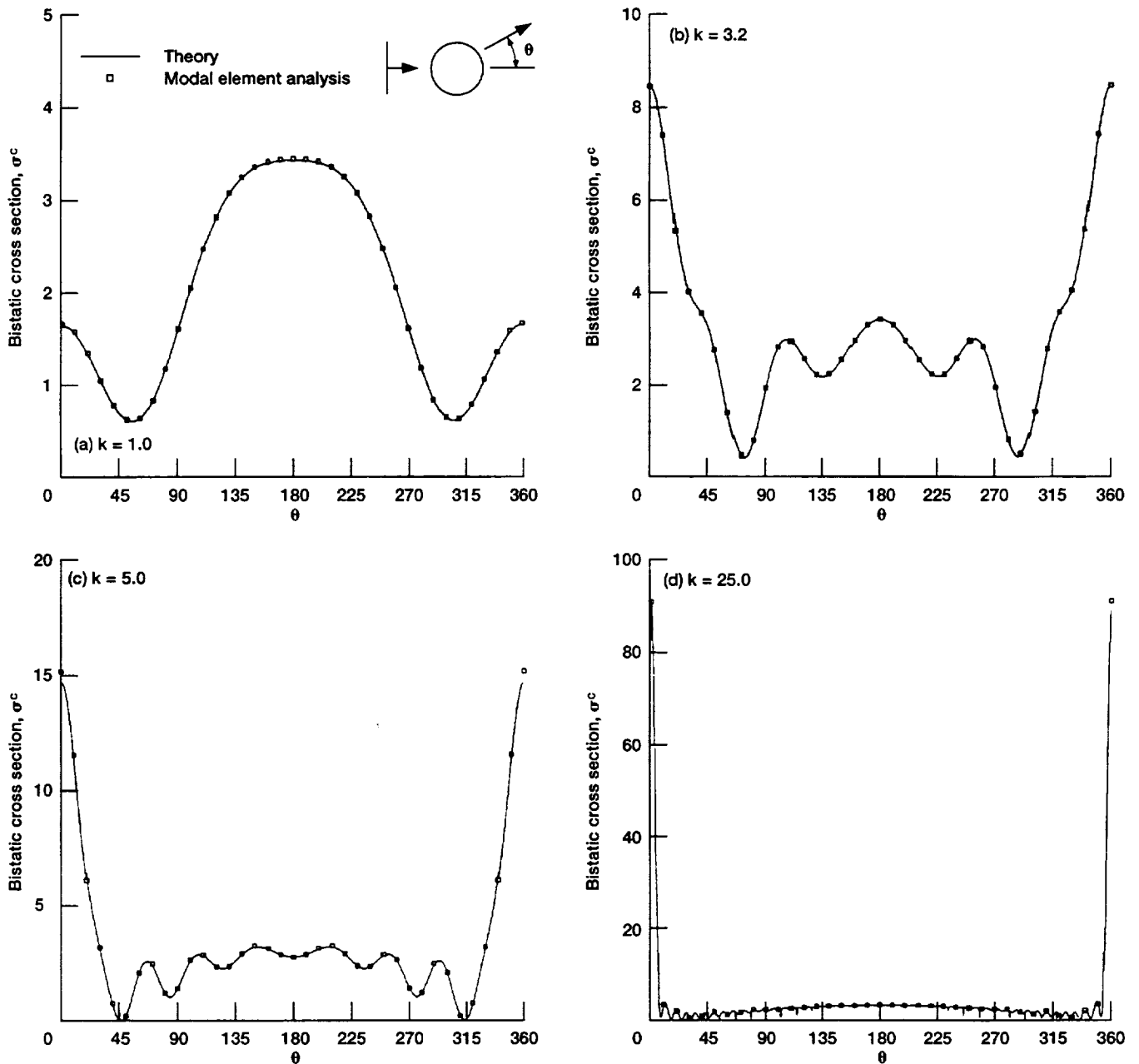
$$\sigma^c = \pi a = \pi \qquad (a = 1) \qquad (9)$$

3

Figure 2.—Bistatic scattering cross section.

In the forward scattered direction ($\theta = 0$), commonly referred to as the diffracted field, the optical approximation yields (Ruck, et al. (1970), eq. (4.1) to (41)) the forward cross section

$$\sigma^c = 4ka^2 = 4k \qquad (a = 1) \qquad (10)$$

Figure 3 shows that the asymptotic limits are attained as frequency increases in both the forward and backscattered directions. Table I contains the numerical values obtained for various frequencies in these directions. Even at $ka = 100$, the relative error in the forward direction is only about 3 percent.

Some error at high frequencies can be attributed to the approximation involved with the asymptotic expansion given by eq. (7). However, the major source of error is in the calculated values of the mode coefficients A_m^* in eq. (3). For example, in the $k = 100$ calculation, over 130 modal coefficients need to be determined. Consequently, some significant errors are involved in the calculation of the higher order modes.
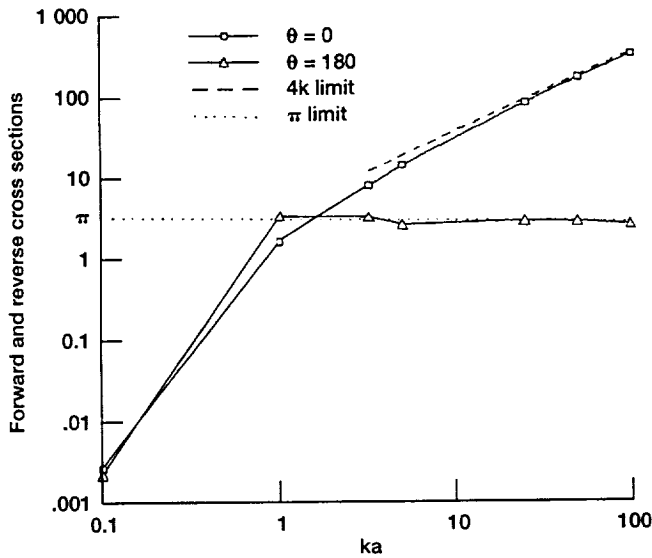
4

Figure 3.—Scattering cross section as a function of ka.

TABLE I.—COMPARISON OF EXACT AND
MODAL ELEMENT (ME) BISTATIC
CROSS SECTIONS

k	Forward ($\theta = 0$)		Backward ($\theta = 180$)	
	Exact	ME	Exact	ME
0.1	0.00261	0.00263	0.0022	0.00219
1.0	1.643	1.659	3.427	3.437
3.2	8.459	8.470	3.412	3.413
5.0	14.77	15.22	2.795	2.778
25	90.31	91.00	3.151	3.153
50	187.6	189.3	3.140	3.140
100	378.7	369.7	3.106	2.957

CONCLUDING REMARKS

In the modal element method, the scattering body is imbedded in a
finite element grid to calculate the acoustic pressure field. In the far
field, the pressure is represented analytically by a Hankel function
expansion. The asymptotic approximation of the Hankel functions are
used to calculate the scattering cross section. The method is applicable
to problems involving very high or low frequency scattering from
bodies. In the validation examples, the numerical results are in good
agreement with the corresponding exact solutions and asymptotic
analytical limits.

REFERENCES

Abramowitz, M. and Stegun, I.A., 1964, Handbook of Mathematical Functions With Formulas, Graphs, and Mathematical Tables, National Bureau of Standards Applied Mathematics.

Astley, R.J. and Eversman, W., 1981, "Acoustic Transmission in Non-Uniform Ducts with Mean Flow, Part II: The Finite Element Method," Journal of Sound and Vibration, Vol. 74, pp. 103–121.

Baumeister, K.J., 1991, "Effect of Surface Deposits on Electromagnetic Propagation in Uniform Ducts," IEEE Trans. on Magnetics, Vol. 27, No. 5.

Baumeister, K.J. and Dahl, M.D., "Acoustic Wave Propagation in Heterogeneous Structures Including Experimental Validation," AIAA 89–1044.

Baumeister, K.J. and Kreider, K.L., 1992, "Modal Ring Method for the Scattering of Electromagnetic Waves," NASA TM–105966.

Baumeister, K.J. and Kreider, K.L., 1993, "Modal Element Method for Scattering of Sound by Absorbing Bodies," ASME Journal of Vibration and Acoustics, Vol. 115, No. 3, pp. 314–323.

Bowman, J.J., Senior, T.B.A., and Uslenghi, P.L.E., 1969, "Electromagnetic and Acoustic Scattering by Simple Shapes," North-Holland, Amsterdam.

Chang, S.K. and Mei, K.K., 1976, "Application of the Unimoment Method to Electromagnetic Scattering of Dielectric Cylinders," IEEE Trans. on AP, pp. 35–42.

Lee, Jin-Fa and Cendes, Z.J., 1987, "The Transfinite Element Method for Computing Electromagnetic Scattering From Arbitrary Lossy Cylinders," AP–S International Symposium, paper AP03–5, Blacksburg, Virginia.

Ling, R.T., 1986, "Numerical Solution For the Scattering of Sound Waves by a Circular Cylinder," AIAA 86–1880.

Morse, P.M. and Ingard, K.U., 1968, "Theoretical Acoustics," McGraw-Hill, New York.

Noack, R.W. and Anderson, D.A., 1992, "Time Domain Solutions of Maxwell's Equations Using a Finite-Volume Formulation," AIAA Paper 92–0451.

Ruck, G.T., Barrick, D.E., Stuart, W D. and Krichbaum, C.K., 1970, "Radar Cross Section Handbook," Volume 1 and 2, Plenum Press.

5

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. AGENCY USE ONLY *(Leave blank)*	2. REPORT DATE	3. REPORT TYPE AND DATES COVERED
	July 1994	Technical Memorandum

4. TITLE AND SUBTITLE

Scattering Cross Section of Sound Waves by the Modal Element Method

5. FUNDING NUMBERS

WU–505–62–52

6. AUTHOR(S)

Kenneth J. Baumeister and Kevin L. Kreider

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)

National Aeronautics and Space Administration
Lewis Research Center
Cleveland, Ohio 44135–3191

8. PERFORMING ORGANIZATION REPORT NUMBER

E–8989

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)

National Aeronautics and Space Administration
Washington, D.C. 20546–0001

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

NASA TM–106667

11. SUPPLEMENTARY NOTES

12a. DISTRIBUTION/AVAILABILITY STATEMENT

Unclassified - Unlimited
Subject Category 71

12b. DISTRIBUTION CODE

13. ABSTRACT *(Maximum 200 words)*

The modal element method has been employed to determine the scattered field from a plane acoustic wave impinging on a two dimensional body. In the model element method, the scattering body is represented by finite elements, which are coupled to an eigenfunction expansion representing the acoustic pressure in the infinite computational domain surrounding the body. The present paper extends the previous work by developing the algorithm necessary to calculate the acoustics scattering cross section by the modal element method. The scattering cross section is the acoustical equivalent to the radar cross section (RCS) in electromagnetic theory. Since the scattering cross section is evaluated at infinite distance from the body, an asymptotic approximation is used in conjunction with the standard modal element method. For validation, the scattering cross section of the rigid circular cylinder is computed for the frequency range $0.1 \leq ka \leq 100$. Results show excellent agreement with the analytic solution.

14. SUBJECT TERMS

Acoustic; Scattering; Cross section; Modal element

15. NUMBER OF PAGES
7

16. PRICE CODE
A02

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
Unclassified	Unclassified	Unclassified	