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DISCRETIZED ENERGY MINIMIZATION IN A WAVE GUIDE WITH POINT SOURCES

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DISCRETIZED ENERGY MINIMIZATION IN A WAVE GUIDE WITH POINT SOURCES

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ABSTRACT

An anti-noise problem on a finite time interval is solved by minimization of a quadratic functional on the Hilbert space of square integrable controls. To this end, the one-dimensional wave equation with point sources and pointwise reflecting boundary conditions is decomposed into a system for the two propagating components of waves. Wellposedness of this system is proved for a class of data that includes piecewise linear initial conditions and piecewise constant forcing functions. It is shown that for such data the optimal piecewise constant control is the solution of a sparse linear system. Methods for its computational treatment are presented as well as examples of their applicability. The convergence of discrete approximations to the general optimization problem is demonstrated by finite element methods.

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1. INTRODUCTION

This article's optimization problem is motivated by a variant of active noise control in acoustics. In physical terms, we consider a wave guide of finite length with the following properties:

- (i) the evolution of sound in the guide is appropriately described by plane waves propagating along its axis without dissipation
- (ii) the sound in the guide originates from initial conditions and sources with small support located within the wave guide or at its front ends
- (iii) the system's total behaviour results from superposition of mutually independent sound waves that are generated by the sources and the initial conditions
- (iv) at the front ends of the guide incident waves are partially absorbed and partially reflected without alteration of shape.

The initial conditions at time $t = 0$ are given and the behaviour of some of the sources (offending but uncontrollable sources) is known. Our control problem is to determine the behaviour of the other sources (controllable sources) such as to reduce the noise by destructive interference in prespecified regions of the wave guide.

Most of the above assumptions are used in the literature on anti-noise problems in ducts (e.g. [Sw], [TN], [FW], see also [MI] p.467). Furthermore, in acoustics it is common practice to consider harmonic sound fields with time dependences of the form $\exp(-i\omega t)$. In contrast to this, the present paper works with non-harmonic sources and fields; for the time domain model considered here, a convenient choice for the states and sources are elements in Hilbert spaces of real-valued functions.

The basic time domain model in linear acoustics is the wave equation for the velocity potential ϕ

$$(1.1.1) \quad \phi_{tt}(t, x) - c^2 \phi_{xx}(t, x) = f(t, x) + u(t, x), \quad 0 < t < T, \quad 0 < x < L$$

$$(1.1.2) \quad \phi(0, x) = \phi_0(x), \quad \phi_t(0, x) = \psi_0(x), \quad 0 < x < L$$

with initial conditions $\phi_0, \psi_0 : (0, L) \rightarrow \mathbb{R}$, the constant $c > 0$ being the speed of sound. As is customary in anti-vibration models (e.g. [Sw],[NCEB1],[NCEB2],[P]) we consider point sources that are represented by δ distributions

$$(1.1.3) \quad f(t, x) = \sum_{i=2}^n f_i(t) \delta_{\xi_i}(x), \quad u(t, x) = \sum_{i=2}^m u_i(t) \delta_{\eta_i}(x), \quad 0 < t < T,$$

at distinct points $\xi_i, \eta_i \in (0, L)$ with time dependencies $f_i, u_i \in L^2(0, T)$.

According to (iv), the specific acoustic impedances of the boundary surfaces are real constants $\zeta_0, \zeta_1 \geq 0$. The interaction of a surface and the sound is such that the quotient p/v_{in} of the sound pressure p and the incident velocity v_{in} at the surface is equal to $\rho c \zeta_i$ ([MI] p.259f, ρ denotes the equilibrium density within the duct). In terms of the velocity potential, since $p(t, x) = \rho \phi_t(t, x)$, $v(t, x) = -\phi_x(t, x)$, this leads to the boundary conditions

$$(1.2) \quad \begin{aligned} \phi_t(t, 0) - c\zeta_0 \phi_x(t, 0) &= 0, \\ \phi_t(t, L) + c\zeta_1 \phi_x(t, L) &= 0, \end{aligned} \quad t > 0.$$

We include sources at the boundaries, assuming that sources and reflection combine by superposition. For example, the model's boundary at $x = L$ could be an artificial, non-reflecting one ($\zeta_1 = 1$) with independent waves entering from $x > L$. Let us consider the case of an offending source at $x = L$ and a controllable source at $x = 0$,

$$(1.1.4) \quad \begin{aligned} \phi_t(t, 0) - c\zeta_0\phi_x(t, 0) &= \frac{1 + \zeta_0}{2c}u_1(t), \\ \phi_t(t, L) + c\zeta_1\phi_x(t, L) &= \frac{1 + \zeta_1}{2c}f_1(t), \end{aligned} \quad t > 0$$

$f_1, u_1 \in L^2(0, T)$. The factors $(1 + \zeta_i)/2c$ have the effect that the waves generated at the boundaries and in the interior have equal amplitudes relative to $f_i(t), u_i(t)$.

$\rho(c^2\phi_x(t, x)^2 + \phi_t(t, x)^2)/2c^2$ being the acoustic energy density at time t ([MI], (6.2.15)), we define the performance index

$$(1.3) \quad J(u_1, \dots, u_m) = \int_0^T \left(\frac{\rho}{2c^2} \int_0^L q(t, x)(c^2\phi_x(t, x)^2 + \phi_t(t, x)^2) dx + \sum_{i=1}^m r_i(t)u_i(t)^2 \right) dt$$

with design parameters $r_i(t) \geq 0$, $i = 1, \dots, m$. The weighting function $q(t, x) \geq 0$ is non-zero on those subintervals of $(0, L)$ where the acoustic energy is to be reduced.

The optimal control problem in terms of (1.1) is: Given initial conditions ϕ_0, ψ_0 and functions $f_i \in L^2(0, T)$, $i = 1, \dots, n$, find functions $u_i \in L^2(0, T)$, $i = 1, \dots, m$ that minimize the functional J wherein ϕ is the solution to (1.1). Inserting the solution operators of the system into the cost functional J gives a quadratic functional in (u_1, \dots, u_m) whose unique minimizer abstractly is represented by use of the adjoint of the operator that maps the control functions to the waves they generate. The purpose of the present work is the construction of approximate optimization problems that are amenable to numerical computations.

To this end, equation (1.1) is substituted by a first order system according to the factorisation $\partial_{tt} - c^2\partial_{xx} = (\partial_t - c\partial_x)(\partial_t + c\partial_x)$. The components of that system are the two components of the velocity potential ϕ that are travelling in opposite directions. The delta inhomogeneities, generally speaking, involve non-classical but wellknown concepts of solution (method of transposition, e.g. [L]). Yet, for the boundary conditions used in this article, no directly quotable reference that would cover the wellposedness of the models in section 2 was found. Instead of following the general theory of [LM], for the one-dimensional systems at hand, the existence, regularity and uniqueness of weak solutions (in a sense analogous to [L]) can be proved directly by application of classical methods (d'Alembert's formula, Duhamel's principle, energy estimate) in the sense of distributions.

The regularity of the initial conditions and sources in section 2 is tailored precisely to the discretized minimization problem of section 3. By restriction to sources that are piecewise constant (in time) and to initial conditions that are first order splines (in space), the problem can be formulated in terms of matrices. This is due to the fact that – according to the simple process of propagation and reflection – the value of the cost functional is determined by the two components of ϕ at the points $x_j = jc\Delta t$ at discrete times $t_k = k\Delta t$. Thus, for this type of sources and initial conditions, the problem with finite time horizon ($T < \infty$) is reduced to the solution of a finite (possibly large) system of linear equations.

The coefficient matrix of this system is sparse. We present numerical methods that avoid the zero coefficients. To demonstrate their computational feasibility and to illustrate some characteristic details of the solutions, four examples are given.

Finally, twice differentiable initial conditions are interpolated by linear splines and square integrable sources are projected onto subspaces of piecewise constant functions. It is shown that the corresponding optimal controls converge to the solution of the original problem. The proof rests on finite element methods for elliptic problems.

As is generally the case for wave equations (cf.[DLP],[DW]), the performance of the controls is sensitive to time delays. This feature affects the open loop solution of the present paper as well as the feedback solution as in [BKSW]. It should be noted that both approaches lead to acausal controls in the sense that the behaviour of the offending sources at times later than t is used to determine the optimal control action at time t . This is necessary because of the global nature of the performance index and the unrestrictive choice of admissible control functions. In the examples we can observe the correlation of acausal actions with the finite speed of propagation.

For typical boundary materials that are of practical interest, assumption (iv) is not valid ([BG],[BPS]). However, modelling and approximating the smearing of incident pulse waves by the boundaries (see [MI] p.264f) would destroy the sparsity in the matrix of the discretized system and thereby drastically increase the computational requirements.

On the other hand, we believe that explicitly computable solutions to minimization problems for pointwise reflecting boundaries as in (iv) provide detailed insights into the mechanisms of the optimal control of waves in enclosures.

Notation. $L^2(a, b; X)$ is the Lebesgue space of square integrable functions on the interval (a, b) with values in X . For real-valued functions, $X = \mathbb{R}$, we simply write $L^2(a, b)$. $C([a, b]; X)$ denotes the space of X -valued functions that are continuous on the closed interval $[a, b]$. $H^1(a, b)$ is the Sobolev space of absolutely continuous functions $h : (a, b) \rightarrow \mathbb{R}$ with distributional derivative $h' \in L^2(a, b)$, normed by $\|h\|_1 = (\|h\|^2 + \|h'\|^2)^{1/2}$ ($\|\cdot\|$ being the L^2 -norm). As usual, $X^k = \{(x_1, \dots, x_k) | x_i \in X, i = 1, \dots, k\}$. In a Hilbert space X the inner product will be denoted by $\langle \cdot, \cdot \rangle_X$. We use the abbreviations $h(\xi^+)$ and $h(\xi^-)$ for the right and left hand limit of the function h at the point $\xi \in \mathbb{R}$. For an interval $I \subseteq \mathbb{R}$ we write χI for the characteristic function of I , i.e. $\chi I(t) = 1$ for $t \in I$, $\chi I(t) = 0$ for $t \in \mathbb{R} \setminus I$.

2. THE FIRST ORDER SYSTEM

Any unidirectional linear wave is the sum of two components propagating in positive and negative direction. The insertion of the sum of a function of $(t - x/c)$ and a function of $(t + x/c)$ into (1.2) shows, that the boundary conditions (1.2) couple the two components by pointwise reflection with reflection coefficients

$$-1 \leq R_i = \frac{\zeta_i - 1}{\zeta_i + 1} < 1, \quad i = 0, 1.$$

Therefore we consider the system

$$(2.1) \quad \begin{aligned} \phi_t^+(t, x) + c\phi_x^+(t, x) &= 0, \\ \phi_t^-(t, x) - c\phi_x^-(t, x) &= 0, \end{aligned} \quad 0 < x < L, \quad 0 < t < T;$$

$$(2.2) \quad \phi^+(t, 0) = R_0\phi^-(t, 0), \quad \phi^-(t, L) = R_1\phi^+(t, L), \quad 0 < t < T;$$

$$(2.3) \quad \phi^+(0, x) = \phi_0^+(x), \quad \phi^-(0, x) = \phi_0^-(x), \quad 0 < x < L.$$

This first order system is a generalization of the homogeneous second order equation

$$(2.4) \quad \phi_{tt}(t, x) - c^2\phi_{xx}(t, x) = 0, \quad 0 < x < L, \quad 0 < t < T$$

with initial and boundary conditions (1.1.2), (1.2) in the following sense: If ϕ^+, ϕ^- are functions satisfying (2.1) – (2.3) with initial conditions

$$\phi_0^+(x) = \frac{1}{2}\phi_0(x) - \frac{1}{2c} \int_0^x \psi_0 d\xi, \quad \phi_0^-(x) = \frac{1}{2}\phi_0(x) + \frac{1}{2c} \int_0^x \psi_0 d\xi,$$

then $\phi = \phi^+ + \phi^-$ is a solution of (2.4), (1.1.2), (1.2) in the sense of distributions [S2]; this can be seen by the application of the commuting product $(\partial_t + c\partial_x)(\partial_t - c\partial_x)$ to $\phi^+ + \phi^-$, and by verification of the initial and boundary conditions. Classical solutions that are twice continuously differentiable evolve only if the initial conditions satisfy certain regularity and compatibility requirements.

2.1. Wellposedness. In context of the present anti-noise problem it is adequate to work with continuous initial data, because typical initial conditions are either silence, or sound that has been generated by sources that are driven by $L^2(0, T)$ functions. Accordingly, we define the space of initial conditions

$$V = \{(\phi^+, \phi^-) \in H^1(0, L)^2 \mid \phi^+(0^+) = R_0\phi^-(0^+), \phi^-(L^-) = R_1\phi^+(L^-)\},$$

endowed with the $H^1(0, L)^2$ -norm.

Definition 2.1. Given $(\phi_0^+, \phi_0^-) \in V$, a pair (ϕ^+, ϕ^-) of functions $\phi^+, \phi^- : [0, T] \times (0, L) \rightarrow \mathbb{R}$ is called a solution of system (2.1) – (2.3) if

- (i) $\phi^+(0, x) = \phi_0^+(x)$, $\phi^-(0, x) = \phi_0^-(x)$, $0 < x < L$;
- (ii) the mapping $[0, T] \rightarrow V$, defined by $t \mapsto (\phi^+(t, \cdot), \phi^-(t, \cdot))$, is in $C([0, T]; V)$;
- (iii) for all $x \in (0, L)$, the functions $[0, T] \rightarrow \mathbb{R}$ defined by $t \mapsto \phi^+(t, x)$, $t \mapsto \phi^-(t, x)$ are absolutely continuous on $[0, T]$;
- (iv) for all $t \in (0, T)$, (2.1) holds for almost all $x \in (0, L)$.

Theorem 2.1. Given $(\phi_0^+, \phi_0^-) \in V$, system (2.1) – (2.3) has a unique solution. With $\ell(t) = \max\{\ell \in \mathbb{N} \mid \ell \leq ct/L\}$ the solution is given by

for $\ell(t) = \ell$ odd:

$$\begin{aligned} \phi^+(t, x) &= R_0(R_1 R_0)^{\frac{\ell-1}{2}} \phi_0^-(-(\ell-1)L - x + ct) + (R_0 R_1)^{\frac{\ell+1}{2}} \phi_0^+((\ell+1)L + x - ct), \\ \phi^-(t, x) &= R_1(R_0 R_1)^{\frac{\ell-1}{2}} \phi_0^+((\ell+1)L - x - ct) + (R_1 R_0)^{\frac{\ell+1}{2}} \phi_0^-(-(\ell+1)L + x + ct); \end{aligned}$$

for $\ell(t) = \ell$ even:

$$\begin{aligned}\phi^+(t, x) &= (R_0 R_1)^{\frac{\ell}{2}} \phi_0^+(\ell L + x - ct) + R_0 (R_1 R_0)^{\frac{\ell}{2}} \phi_0^-(-\ell L - x + ct), \\ \phi^-(t, x) &= (R_1 R_0)^{\frac{\ell}{2}} \phi_0^-(-\ell L + x + ct) + R_1 (R_0 R_1)^{\frac{\ell}{2}} \phi_0^+((\ell + 2)L - x - ct).\end{aligned}$$

These formulae are understood according to the convention $\phi_0^+(\tau) = \phi_0^-(\tau) = 0$ if $\tau \in \mathbb{R} \setminus (0, L)$, so that, for every $(t, x) \in [0, T] \times (0, L)$, at most one of the two terms in each formula is non-zero. At the points $x = -\ell L + ct$ and $x = (\ell + 1)L - ct$, where the values of ϕ^+ and ϕ^- are left open by this convention, these values are determined as the left or right hand limits, which coincide due to the boundary conditions in V .

Proof. As to uniqueness, the classical form of an energy estimate (eg.[S2] p.299) is not applicable here, because the initial conditions and solutions in Def. 2.1 are not continuously differentiable. However, for any solution (ϕ^+, ϕ^-) of (2.1) – (2.3) consider the function

$$E(t) = \int_0^L (\phi^+(t, x)^2 + \phi^-(t, x)^2) dx.$$

Because of (ii), (iii), E can be viewed as a distribution whose derivative is given by

$$E'(t) = \int_0^L (\phi^+(t, x)^2 + \phi^-(t, x)^2)_t dx$$

(see [Z], 2.8). Transforming the integrand using (iv) and integrating $(\phi^+(t, x)^2 + \phi^-(t, x)^2)_x$ from 0 to L we get, $\phi^+(t, \cdot)^2, \phi^-(t, \cdot)^2$ being absolutely continuous (eg. [HS], (18.16)),

$$\begin{aligned}E'(t) &= c[\phi^+(t, 0^+)^2 - \phi^+(t, L^-)^2 + \phi^-(t, L^-)^2 - \phi^-(t, 0^+)^2] \\ &= c[(R_0^2 - 1)\phi^-(t, 0^+)^2 + (R_1^2 - 1)\phi^-(t, L^-)^2] \leq 0.\end{aligned}$$

This shows that the derivative of E is a non-positive function; therefore ([S1], Chap.IV) $E(t)$ is a decreasing function. Thus, given $(\phi_0^+, \phi_0^-) \in V$, the difference of two solutions of (2.1)–(2.2) is a solution with zero initial conditions i.e. $E(0) = 0$. Consequently, for the difference of two solutions, $E(t) = 0, t \in [0, T]$. This implies the uniqueness of the solution.

The formulae for $\phi^+(t, x), \phi^-(t, x)$ describe the movement of the two components due to propagation and reflection for the $(\ell + 1)$ th cycle of complete reversion during $\ell L/c < t < (\ell + 1)L/c$ (cf. section 2.2). In light of the convention stated above, it is quite straightforward to verify that the given pair (ϕ^+, ϕ^-) satisfies (i) – (iv) of Definition 2.1. \square

With

$$H := L^2(0, T; L^2(0, L)^2)$$

we have the following immediate consequence of the formulae in Theorem 1.1.

Corollary 2.1. *The operator $S : V \rightarrow H$ that maps any initial condition in V to the spatial derivatives of the corresponding solution of (2.1) – (2.3), $S(\phi_0^+, \phi_0^-)(t) = (\phi_x^+(t, \cdot), \phi_x^-(t, \cdot))$, where ϕ^+, ϕ^- are given in Theorem 2.1, is a bounded linear operator.*

With regard to sources at the boundaries we consider the homogeneous first order system (2.1) with zero initial conditions

$$(2.5) \quad \phi^+(0, x) = \phi^-(0, x) = 0, \quad 0 < x < L$$

and inhomogeneous boundary conditions

$$(2.6) \quad \phi^+(t, 0^+) = R_0 \phi^-(t, 0^+), \quad \phi^-(t, L^-) = R_1 \phi^+(t, L^-) + F(t)$$

where F is related to the boundary source f for the wave equation (2.4)

$$(2.7) \quad \begin{aligned} \phi_t(t, 0) - c\zeta_0 \phi_x(t, 0) &= 0, \\ \phi_t(t, L) + c\zeta_1 \phi_x(t, L) &= \frac{1 + \zeta_1}{2c} f(t), \quad t > 0 \end{aligned}$$

by

$$(2.8) \quad F(t) = \frac{1}{2c} \int_0^t f \, dt.$$

For a source at the left boundary (2.6), (2.7) are to be replaced accordingly. The first order system with a point source at $\xi \in (0, L)$ is

$$(2.9) \quad \begin{aligned} \phi_t^+(t, x) + c\phi_x^+(t, x) &= cF(t)\delta_\xi, \\ \phi_t^-(t, x) - c\phi_x^-(t, x) &= cF(t)\delta_\xi, \end{aligned} \quad 0 < x < L, \quad 0 < t < T$$

where, as in (2.8), $2cF$ is the primitive of the point source f in the wave equation

$$(2.10) \quad \phi_{tt}(t, x) - c^2 \phi_{xx}(t, x) = f(t)\delta_\xi, \quad 0 < x < L, \quad 0 < t < T.$$

The application of $(\partial_t - c\partial_x)(\partial_t + c\partial_x)$ to $\phi^+ + \phi^-$, where ϕ^+, ϕ^- satisfy (2.9), (2.2), (2.5) or (2.1), (2.6), (2.5) shows that these systems are generalizations of (2.10), (1.2) or (2.4), (2.7) respectively, with zero initial conditions $\phi_0 \equiv \psi_0 \equiv 0$ in (1.1.2).

For $\xi \in (0, L)$ we denote by V_ξ the space

$$V_\xi = \{(\phi^+, \phi^-) \in H^1((0, L) \setminus \{\xi\})^2 \mid \phi^+(0^+) = R_0 \phi^-(0^+), \phi^-(L^-) = R_1 \phi^-(L^-)\}$$

endowed with the H^1 norm on $(0, L) \setminus \{\xi\}$.

Definition 2.2. Given $\xi \in (0, L)$ and $f \in L^2(0, T)$, a pair (ϕ^+, ϕ^-) of functions $\phi^+, \phi^- : [0, T] \times (0, L) \rightarrow \mathbb{R}$ with $\phi^+(0, x) = \phi^-(0, x) = 0$, $0 < x < L$, is called a solution

a) of system (2.1), (2.6), (2.5) if

- (i) the mapping $[0, T] \rightarrow H^1(0, L)^2$, defined by $t \mapsto (\phi^+(t, \cdot), \phi^-(t, \cdot))$, is in $C([0, T]; H^1(0, L)^2)$,
- (ii) for all $x \in (0, L)$, the functions $[0, T] \rightarrow \mathbb{R}$ defined by $t \mapsto \phi^+(t, x)$, $t \mapsto \phi^-(t, x)$ are absolutely continuous on $[0, T]$,
- (iii) for all $t \in (0, T)$, (2.6) holds,
- (iv) for all $t \in (0, T)$, (2.1) holds for almost all $x \in (0, L)$;

b) of system (2.9), (2.2), (2.5) if

- (i) the mapping $[0, T] \rightarrow V_\xi$, defined by $t \mapsto (\phi^+(t, \cdot), \phi^-(t, \cdot))$, is in $C([0, T]; V_\xi)$,
- (ii) for all $x \in (0, L) \setminus \{\xi\}$, the functions $[0, T] \rightarrow \mathbb{R}$ defined by $t \mapsto \phi^+(t, x)$, $t \mapsto \phi^-(t, x)$ are absolutely continuous on $[0, T]$,
- (iii) for all $t \in (0, T)$, $\phi^+(t, \xi^+) - \phi^+(t, \xi^-) = \phi^-(t, \xi^-) - \phi^-(t, \xi^+) = F(t)$,
- (iv) for all $t \in (0, T)$, (2.1) holds for almost all $x \in (0, L) \setminus \{\xi\}$.

Theorem 2.2. Let $f \in L^2(0, T)$, $\xi \in (0, L)$.

a) The system (2.1), (2.6), (2.5) has the unique solution

$$\phi^+(t, x) = \sum_{k=0}^{\infty} R_0 (R_1 R_0)^k F \left(\frac{-(2k+1)L - x}{c} + t \right) \chi_{[L-ct, L]}(-2kL - x),$$

$$\phi^-(t, x) = \sum_{k=0}^{\infty} (R_1 R_0)^k F \left(\frac{-(2k+1)L + x}{c} + t \right) \chi_{[L-ct, L]}(-2kL + x).$$

b) The unique solution of (2.9), (2.2), (2.5) is given by

$$\begin{aligned} \phi^+(t, x) &= \sum_{k=0}^{\infty} (R_0 R_1)^k F \left(\frac{\xi - 2kL - x}{c} + t \right) \chi_{[\xi, \xi + ct]}(2kL + x) \\ &\quad + \sum_{k=0}^{\infty} R_0 (R_1 R_0)^k F \left(\frac{-\xi - 2kL - x}{c} + t \right) \chi_{[\xi - ct, \xi]}(-2kL - x), \\ \phi^-(t, x) &= \sum_{k=0}^{\infty} (R_1 R_0)^k F \left(\frac{-\xi - 2kL + x}{c} + t \right) \chi_{[\xi - ct, \xi]}(-2kL + x) \\ &\quad + \sum_{k=1}^{\infty} R_1 (R_0 R_1)^{k-1} F \left(\frac{\xi - 2kL + x}{c} + t \right) \chi_{[\xi, \xi + ct]}(2kL - x). \end{aligned}$$

Proof. The formulae are constructed by folding the solution of the problem on all of \mathbb{R} (no boundaries) into the domain $(0, L)$ with reflection coefficients R_0, R_1 . Note that for any $t > 0$ all the sums are finite, because for large enough $k \in \mathbb{N}$ all characteristic functions are evaluated outside their support. Thus, the solution-properties of the given pairs (ϕ^+, ϕ^-)

can be verified by considering the sums term by term. F being absolutely continuous, the regularity with respect to x and t follows from the fact that F is zero wherever the characteristic functions have time dependent steps. That the norm of $(\phi^+(t, \cdot), \phi^-(t, \cdot))$ varies continuously with t is due to the steady propagation with finite speed c (cf. section 2.2). As to uniqueness, the difference of two solutions of one of the inhomogeneous problems (extended continuously at ξ in case b)) is a solution of the homogeneous problem (2.1), (2.2), (2.5) and thus equals zero by Theorem 2.1. \square

For $\xi \in (0, L)$ let S_ξ denote the operator that maps any inhomogeneity $f \in L^2(0, T)$ to the restriction to $(0, L) \setminus \{\xi\}$ of the spatial derivatives of the solution of (2.9), (2.2), (2.5), i.e.

$$S_\xi f(t) = (\phi_x^+(t, \cdot), \phi_x^-(t, \cdot)) \in L^2((0, L) \setminus \{\xi\})^2,$$

where ϕ^+, ϕ^- are given in Theorem 2.2 b). The point ξ is spared out to exclude δ -terms in the derivatives of $\phi^+(t, \cdot), \phi^-(t, \cdot)$, whereas the velocity potential $\phi^+(t, \cdot) + \phi^-(t, \cdot)$ is continuous at ξ . Throughout we can choose $S_\xi f(t)(\xi) = (0, 0)$ as a possible extension to $(0, L)$. For notational convenience we let $\xi_1 = L, \eta_1 = 0$ and denote by S_0 and S_L the operators that map boundary sources $f \in L^2(0, T)$ to the spatial derivatives $(\phi_x^+(t, \cdot), \phi_x^-(t, \cdot))$ of the solution of (2.1), (2.6), (2.5), where (for the right hand boundary) ϕ^+, ϕ^- are given in Theorem 2.2 a). An immediate consequence of Theorem 2.2 is

Corollary 2.2. *For any $\xi \in [0, L]$, $S_\xi : L^2(0, T) \rightarrow H$ is a bounded linear operator.*

2.2. Propagation and reflection. The components of the solutions given in section 2.1 evolve by shifts with speed c in positive and negative direction combined with inversion and reduction by R_0, R_1 at the boundaries. The source f in (2.10) generates a wave that is symmetric about $\xi \in (0, L)$. More specifically, let the derivatives of the two components of the velocity potential $(\psi^+(t, \cdot), \psi^-(t, \cdot)) := S_\xi f(t)$ be given at some time $t \geq 0$ and let $0 < \tau \leq \tau_\xi = \min(\xi, L - \xi)/c$. Then, at time $t + \tau$

$$(2.11) \quad \begin{aligned} \psi^+(t + \tau, x) &= -\frac{1}{2c^2} f\left(\frac{\xi - x}{c} + t + \tau\right) \chi[\xi, \xi + c\tau](x) \\ &\quad + \begin{cases} \psi^+(t, x - c\tau), & c\tau < x < L \\ -R_0 \psi^-(t, c\tau - x), & 0 < x < c\tau \end{cases} \\ \psi^-(t + \tau, x) &= \frac{1}{2c^2} f\left(\frac{x - \xi}{c} + t + \tau\right) \chi[\xi - c\tau, \xi](x) \\ &\quad + \begin{cases} \psi^-(t, x + c\tau), & 0 < x < L - c\tau \\ -R_1 \psi^+(t, 2L - x - c\tau), & L - c\tau < x < L. \end{cases} \end{aligned}$$

This is to be understood almost everywhere in $(0, L) \setminus \xi$. For a boundary source at $\xi = 0$ ($\xi = L$) the first term in $\psi^-(t + \tau, x)$ ($\psi^+(t + \tau, x)$) is to be deleted with $\tau_0 = \tau_L = L/c$. For the pair $(\psi^+(t, \cdot), \psi^-(t, \cdot)) := S(\phi_0^+, \phi_0^-)(t)$, (2.11) applies for $0 < \tau \leq L/c$ with $f \equiv 0$. All this can be checked by differentiating and regrouping the formulae of section 2.1 and strict adherence to the convention at Theorem 2.1.

2.3. The minimization problem. For superposition of solutions to initial conditions ϕ_0^+, ϕ_0^- and multiple offending sources f_1, \dots, f_n at the points $\xi_1, \dots, \xi_n \in (0, L]$ define $g \in H$

by

$$g(t) = S(\phi_0^+, \phi_0^-)(t) + \sum_{i=1}^n S_{\xi_i} f_i(t).$$

For the controllable sources located at $\eta_1, \dots, \eta_m \in [0, L)$ we write

$$u = (u_1, \dots, u_m) \in U := L^2(0, T)^m$$

and denote by B the bounded linear operator $U \rightarrow H$

$$Bu(t) = \sum_{i=1}^m S_{\eta_i} u_i(t).$$

In this notation the spatial derivative of the solution of the first order system that corresponds to (1.1) is $g + Bu$. Rewriting the functional (1.3), note that for solutions (ϕ^+, ϕ^-) of (2.1)

$$c^2(\phi^+ + \phi^-)_x(t, x)^2 + (\phi^+ + \phi^-)_t(t, x)^2 = 2c^2[\phi_x^+(t, x)^2 + \phi_x^-(t, x)^2],$$

for all $t \in (0, T)$ and almost all $x \in (0, L)$. Therefore, we are considering the following minimization problem: Given $(\phi_0^+, \phi_0^-) \in V$ and $f_1, \dots, f_n \in L^2(0, T)$, determine $\hat{u} \in U$ such that

$$(2.12) \quad J(\hat{u}) = \min\{J(u) : u \in U\},$$

where

$$(2.13) \quad J(u) = \langle g + Bu, Q(g + Bu) \rangle_H + \langle u, Ru \rangle_U.$$

Here Q and R are bounded selfadjoint operators on H and U resp., defined by

$$\begin{aligned} Q(\phi^+, \phi^-)(t, x) &= \rho(q^+(t, x)\phi^+(t, x), q^-(t, x)\phi^-(t, x)) \\ R(u_1, \dots, u_m)(t) &= (r_1(t)u_1(t), \dots, r_m(t)u_m(t)). \end{aligned}$$

The two components of the waves may be weighted separately, but the functionals in (1.3) and (2.13) coincide if $q^+ \equiv q^- \equiv q$. We make the assumption that the weights $q^+, q^- \in L^\infty(0, T; L^\infty(0, L))$ and $r_i \in L^\infty(0, T)$ are such that

$$(2.14) \quad R + B^*QB > 0,$$

where $B^* : H \rightarrow U$ is the adjoint operator of B . Then the unique minimizer of the quadratic functional J is given by (eg. [B], (5.2.4))

$$\hat{u} = -(R + B^*QB)^{-1} B^*Qg,$$

3. DISCRETIZATION

In order to obtain a discrete version of the minimization problem, we consider a grid of characteristics of (2.1) in the (t, x) -plane with knots at the points (t_k, x_j) , where $x_j = j\Delta x$, $j = 0, \dots, N$ and $t_k = k\Delta t$, $k = 0, \dots, K$. We assume $L = N\Delta x$, $T = K\Delta t$ and $\Delta x = c\Delta t$. Furthermore all sources should be located at meshpoints, i.e. $\{\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_m\} \subset \{x_0, \dots, x_N\}$.

3.1. Piecewise constant sources and spline initial conditions. Define the finite-dimensional spaces

$$\begin{aligned} U_K &= \{f : [0, T] \rightarrow \mathbb{R} \mid f \text{ is constant on } [t_{k-1}, t_k), k = 1, \dots, K\}, \\ V_N &= \{(\phi^+, \phi^-) \in V \mid \phi^+, \phi^- \text{ are first order splines with respect to } \{x_j\}_{j=0}^N\}, \\ H_N &= \{(\psi^+, \psi^-) \in H \mid \psi^+, \psi^- \text{ are constant on } [x_{j-1}, x_j), j = 1, \dots, N\}. \end{aligned}$$

A first order spline with respect to $\{x_j\}_{j=0}^N$ is a continuous function $[0, L] \rightarrow \mathbb{R}$ that is a polynomial of degree one on $[x_{j-1}, x_j]$, $j = 1, \dots, N$. Being subspaces of $L^2(0, T)$ and H , U_K and H_N are endowed with the corresponding L^2 -products. For the k th basis element in U_K we choose $\chi[t_{k-1}, t_k)$, $k = 1, \dots, K$ and we use the pairs $(\chi[x_{j-1}, x_j), 0)$ and $(0, \chi[x_{j-1}, x_j])$ for the j th and $(j + N)$ th basis elements in H_N , $j = 1, \dots, N$.

Lemma 3.1. *If $(\phi_0^+, \phi_0^-) \in V_N$ and $f_i \in U_K$, $i = 1, \dots, n$ then $g(t_k) \in H_N$, $k = 0, \dots, K$. If $u_i \in U_K$, $i = 1, \dots, m$, then $Bu(t_k) \in H_N$, $k = 0, \dots, K$.*

Proof. Because of $\Delta x = c\Delta t$, the statements of the lemma follow from (2.11). \square

For initial conditions $(\phi_0^+, \phi_0^-) \in V_N$ and sources $f_i \in U_K$ the coordinate vector \underline{g} of the $(K + 1)$ -tuple $(g(t_0), \dots, g(t_K)) \in H_N^{K+1}$ has $2N(K + 1)$ real entries. The coordinate vector \underline{u} of $u = (u_1, \dots, u_m) \in U_K^m$ is in \mathbb{R}^{mK} . The operator that maps any $u \in U_K^m$ to $(Bu(t_0), \dots, Bu(t_K))$ has a $2N(K + 1) \times mK$ matrix representation \underline{B} . With this notation, the time-sampled spatial derivative of the solution to the first order system that corresponds to (1.1) is represented by the vector $\underline{g} + \underline{B}\underline{u}$.

3.2. Discrete minimization. We replace the minimization problem (2.12) by the following one: given $(\phi_0^+, \phi_0^-) \in V_N$ and $f_1, \dots, f_n \in U_K$, determine $\hat{u} \in U_K^m$ such that

$$(3.1) \quad J(\hat{u}) = \min\{J(u) : u \in U_K^m\},$$

where $J(u)$ again is defined by (2.13).

With the numbers

$$\begin{aligned}
p_{k,j}^+ &= \rho \int_{t_{k-1}}^{t_k} \int_{x_{j-1}}^{x_{j-1}+c(t-t_{k-1})} q^+(t,x) dx dt, & q_{k,j}^+ &= \rho \int_{t_{k-1}}^{t_k} \int_{x_{j-1}+c(t-t_{k-1})}^{x_j} q^+(t,x) dx dt, \\
p_{k,j}^- &= \rho \int_{t_{k-1}}^{t_k} \int_{x_{j-1}}^{x_j-c(t-t_{k-1})} q^-(t,x) dx dt, & q_{k,j}^- &= \rho \int_{t_{k-1}}^{t_k} \int_{x_j-c(t-t_{k-1})}^{x_j} q^-(t,x) dx dt, \\
r_{i,k} &= \int_{t_{k-1}}^{t_k} r_i(t) dt.
\end{aligned}$$

we define the $2N \times 2N$ matrices

$$\begin{aligned}
Q_0 &= \text{diag}(q_{1,1}^+, \dots, q_{1,N}^+, p_{1,1}^-, \dots, p_{1,N}^-), \\
Q_k &= \text{diag}(p_{k,1}^+ + q_{k+1,1}^+, \dots, p_{k,N}^+ + q_{k+1,N}^+, p_{k+1,1}^- + q_{k,1}^-, \dots, p_{k+1,N}^- + q_{k,N}^-), \\
& \hspace{20em} k = 1, \dots, K-1, \\
Q_K &= \text{diag}(p_{K,1}^+, \dots, p_{K,N}^+, q_{K,1}^-, \dots, q_{K,N}^-).
\end{aligned}$$

and the square matrices

$$\begin{aligned}
\underline{Q} &= \text{diag}(Q_0, \dots, Q_K), \\
\underline{R} &= \text{diag}(r_{1,1}, \dots, r_{1,K}, \dots, r_{m,1}, \dots, r_{m,K}).
\end{aligned}$$

of size $2N(K+1)$ resp. mK .

A minimization problem written in terms of these matrices is: determine $\hat{\underline{u}} \in \mathbb{R}^{mK}$ such that

$$(3.2) \quad \underline{J}(\hat{\underline{u}}) = \min\{\underline{J}(\underline{u}) : \underline{u} \in \mathbb{R}^{mK}\},$$

where

$$\underline{J}(\underline{u}) = (\underline{g} + \underline{B}\underline{u})^T \underline{Q}(\underline{g} + \underline{B}\underline{u}) + \underline{u}^T \underline{R}\underline{u}.$$

We assume $\underline{R} + \underline{B}^T \underline{Q} \underline{B} > 0$.

Theorem 3.1. *The coordinate vector $\hat{\underline{u}}$ of the solution \hat{u} of (3.1) is the solution of (3.2).*

Proof. Let $u \in U_K^m$ and $\underline{u} = \text{col}(u_{1,1}, \dots, u_{1,K}, \dots, u_{m,1}, \dots, u_{m,K})$ be its coordinate vector. Then

$$\langle u, Ru \rangle_U = \sum_{i=1}^m \sum_{k=1}^K \int_{t_{k-1}}^{t_k} r_i(t) u_{i,k}^2 dt = \underline{u}^T \underline{R} \underline{u}.$$

Given spline initial conditions and piecewise constant sources, let $h(t) = (h^+(t, \cdot), h^-(t, \cdot)) = (g + Bu)(t)$ and denote the coordinate vector of $(h(t_0), \dots, h(t_K)) \in H_N^{K+1}$ by $\underline{h} = \text{col}(h_{0,1}^+,$

$\dots, h_{0,N}^+, h_{0,1}^-, \dots, h_{0,N}^-, \dots, h_{K,1}^+, \dots, h_{K,N}^+, h_{K,1}^-, \dots, h_{K,N}^-$). The movement of the two components of $h(t)$ over (x_{j-1}, x_j) during $t_{k-1} < t < t_k$ is indicated in Fig. 3.1.

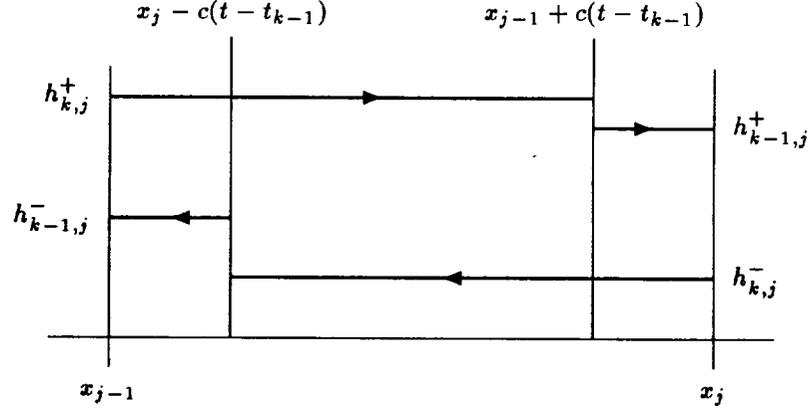


Fig. 3.1: Piecewise constant components at time $t \in (t_{k-1}, t_k)$.

We see that

$$\begin{aligned} \int_{t_{k-1}}^{t_k} \int_{x_{j-1}}^{x_j} (q^+(t, x)h^+(t, x)^2 + q^-(t, x)h^-(t, x)^2) dx dt \\ = p_{k,j}^+ h_{k,j}^{+2} + q_{k,j}^+ h_{k-1,j}^{+2} + p_{k,j}^- h_{k-1,j}^{-2} + q_{k,j}^- h_{k,j}^{-2} \end{aligned}$$

for all $k = 1, \dots, K$ and $j = 1, \dots, N$. Using this, we get

$$\begin{aligned} \langle h, Qh \rangle_H &= \sum_{k=1}^K \sum_{j=1}^N \rho \int_{t_{k-1}}^{t_k} \int_{x_{j-1}}^{x_j} (q^+(t, x)h^+(t, x)^2 + q^-(t, x)h^-(t, x)^2) dx dt \\ &= \sum_{j=1}^N q_{1,j}^+ h_{0,j}^{+2} + p_{1,j}^- h_{0,j}^{-2} + \sum_{j=1}^N p_{K,j}^+ h_{K,j}^{+2} + q_{K,j}^- h_{K,j}^{-2} \\ &\quad + \sum_{k=1}^{K-1} \sum_{j=1}^N p_{k,j}^+ h_{k,j}^{+2} + q_{k,j}^- h_{k,j}^{-2} + \sum_{k=2}^K \sum_{j=1}^N q_{k,j}^+ h_{k-1,j}^{+2} + p_{k,j}^- h_{k-1,j}^{-2} \\ &= \underline{h}^T \underline{Q} \underline{h}. \end{aligned}$$

Therefore, $J(u) = \underline{J}(\underline{u})$. \square

Thus, the vector representation \underline{u} of the solution to (3.1) solves the linear system

$$(3.3) \quad (\underline{R} + \underline{B}^T \underline{Q} \underline{B}) \underline{u} = -\underline{B}^T \underline{Q} \underline{g}.$$

4. NUMERICS

Without reference to the particular structure of system (3.3), its solution is a standard task. However, since the discretizations in time and space are coupled, $\Delta t = \Delta x/c$, the size of system (3.3) is large if high spatial resolution is required and/or if T is large. Then the number of entries in the full matrix \underline{B} is beyond the memory capacity of standard hardware. To overcome this problem, we present efficient algorithms for the computation of the right hand side of (3.3) and the nonzero entries of the sparse matrix $\underline{B}^T \underline{Q} \underline{B}$.

The vector \underline{g} can be produced by discrete simulation of the propagation and reflection of the components ϕ_x^+, ϕ_x^- . $\underline{g}(1 : 2N)$ is given by the initial conditions. Instead of shifting the arrays that hold ϕ_x^+, ϕ_x^- at each time step, it is more efficient, in particular when only a few of the time steps have to be assigned (Example 4.3), to keep the arrays, except of replacing the entry for $\phi_x^+(0^+)$ by $-R_0$ times the previous entry for $\phi_x^-(0^+)$ and the entry for $\phi_x^-(L^-)$ by $-R_1$ times the previous entry for $\phi_x^+(L^-)$. To accomplish this, an integer variable for the current index of $\phi_x^+(0^+)$ is initialized to 1, decreased by 1 at each time step, or reset to N if it is 1. The same boundary index is applicable for both components, if the array for ϕ_x^- is arranged in reversed order. To keep track of the indices where to add the contributions of the offending sources f_i , we have to initialize and update integers that point to the source locations ξ_i , $i = 1, \dots, n$. The assignment of the entries of ϕ_x^+, ϕ_x^- to $\underline{g}(2kN + 1 : 2(k+1)N)$ gives the k th group of \underline{g} , $k = 1, \dots, K$. For the subsequent multiplication with \underline{B}^T it is convenient to store the ϕ_x^- groups in reversed order. \underline{Q} being a diagonal matrix, $\underline{Q} \underline{g}$ is obtained from \underline{g} by $2N(K+1)$ multiplications.

For the computation of $-\underline{B}^T \underline{Q} \underline{g}$, consider the structure of \underline{B}^T . Using $2c^2$ as unit for the source amplitudes $f_i(t)$, $u_i(t)$, we omit the factors $\frac{1}{2c^2}$ in (2.11). Then the k th basis elements of U_K produces, at time t_k , a ϕ_x^- pulse $\chi[x_{j-1}, x_j]$ to the left and a ϕ_x^+ pulse $-\chi[x_j, x_{j+1}]$ to the right of the source location (boundary sources produce only one such pulse). These pulses travel one space index per time step and are reflected at the boundaries. Thus, the k th row of a source block in \underline{B}^T starts with k $2N$ -groups of zeros and continues with $K - (k - 1)$ $2N$ -groups each containing (at most) two nonzero propagating entries. Initializing these entries to ∓ 1 according to the source location, and updating their indices and amplitudes according to propagation and reflection, the k th entry of the i th source in $-\underline{B}^T \underline{Q} \underline{g}$ is accumulated by adding the products of the pulse amplitudes with those entries of $-\underline{Q} \underline{g}$ that are determined by the updated indices. Repeating the entire procedure for $i = 1, \dots, m$, $-\underline{B}^T \underline{Q} \underline{g}$ is computed avoiding the zeros in \underline{B}^T (which is a $mK \times 2N(K+1)$ matrix with at most $mK \times 2K$ nonzeros).

Next, we describe an algorithm to compute $\underline{B}^T \underline{Q} \underline{B}$ for time independent weights $q^\pm(t, x) = \frac{1}{\rho \Delta t \Delta x} \sum_{j=1}^N q_j^\pm \chi[x_{j-1}, x_j]$. Then $\underline{Q} = \text{diag}(\frac{1}{2}Q_1, Q_1, \dots, Q_1, \frac{1}{2}Q_1)$ with $Q_1 = \text{diag}(q_1^+, \dots, q_N^+, q_1^-, \dots, q_N^-) \in \mathbb{R}^{2N}$. The algorithm is based on the fact that two waves emerging from two sources contribute to an entry in $\underline{B}^T \underline{B}$ only if the pulses that are generated by the basis elements of U_K overlap during their travel forth and back in the duct. Two pulses overlap starting with the generation of the later one, or never. Therefore $\underline{B}^T \underline{B}$ consists of $K \times K$ blocks that are banded, each nonzero diagonal containing the accumulation of the products of two overlapping travelling pulses.

An entry in a diagonal can be computed from the one south east to it by adding the contribution that occurs because both sources are active one cycle earlier. For doing this, it is convenient to imagine the overlapping pulses travelling backward in time, adding contributions to what has happened later. At each time step, the product of the overlapping pulses has to be multiplied by q_j^+ or q_j^- , according to the current direction of propagation and the current location of the overlapping pulses (by half the value of q_j^+ or q_j^- for the last time step). For the four lower diagonals near (resp. on) the main diagonal of the block, the location of the entry in the most southern row is determined by the distance of the two sources of the block and their distances to the boundaries; the product of the overlapping pulses is 1, R_0 , R_1 , R_0R_1 respectively. These four diagonals are filled up one after the other: The elements to the north west of the most southern entry are computed until the overlapping pulses arrive at the boundary. From then on the diagonal entries are computed in groups of N elements corresponding to the propagation back and forth in the duct. At the beginning of each group the product of the pulses is multiplied by the square of the appropriate reflection coefficient. The diagonal is complete when the most western column of the block is reached.

Each time a nonzero entry to one of these four diagonals is determined, the entries within the block in the same row and $2jN$ columns to the west of it are computed by multiplication with $(R_0R_1)^j$. These represent the interaction of a source that was active at the same location but j full periods ($= j2N$ cycles) earlier than the one just considered – its pulse was reflected $2j$ times before overlapping with the later pulse.

Analogously, the diagonals above the main diagonal of the block are built up from south east to north west with the full periods north of the current entry.

These methods, coded in Fortran 90 (NAGWare compiler 1.2), create disk files that contain the right hand side and the nonzeros of the lower triangle of the coefficient matrix in (3.3). The latter one consists of lines of the form row index, column index, value of the entry. The files are loaded into Matlab 4.1 where the coefficient matrix is set up as a sparse symmetric matrix A and the right hand side is a full vector rhs . The solution \hat{u} is then obtained by the Matlab command $u = A \setminus rhs$ that invokes sparse matrix arithmetics [M]. The generation of the optimal waves is again done in Fortran 90, analogously to the generation of g . Finally, we use Matlab for the graphics.

In the following examples, t is given in seconds, x in meters, $c = 344$. The relation of the diagonal elements of \underline{Q} , denoted by $q^\pm(x)$, to the weights $q^\pm(t, x)$ is given above. We use time independent functions for $r_i(t)$, so that the entries of \underline{R} are $r_i = r_i(t)\Delta t$. The step functions $f_i(t)$, $u_i(t)$ are measured in units of $2c^2$. Then $J(u) = \underline{J}(u) = \underline{h}^T \underline{Q} \underline{h} + \underline{u}^T \underline{R} \underline{u}$.

Example 4.1. $L = 10, R_0 = 1, R_1 = 0.5, N = 40, K = 137, n = 1, \xi_1 = 4, m = 1, \eta_1 = 0, r_1 = 0, q^\pm(x) \equiv 1$, zero initial conditions. This introductory example is comparable to the problems in [CNE]. The offending source $f_1 = \chi[t_{19}, t_{20}]$ emits a positive and a negative pulse at time $t_{20} = 20\Delta x/c \sim 0.0145$. To reduce the rightgoing pulse of the offending wave, the optimal control source acausally emits a pulse wave at time t_4 : $\hat{u}_1(t) = -0.68$ for $t \in [t_3, t_4]$. To annihilate the pulse that comes from η_1 in negative direction, $\hat{u}_1(t) = -1$ for $t \in [t_{35}, t_{36}]$; to annihilate the reduced pulse that is reflected at the right boundary, $\hat{u}_1(t) = -0.16$ for $t \in [t_{83}, t_{84}]$. After that ϕ is constant, silence. The 3D graphics show the velocity potential at the grid points (t_k, x_j) connected by straight lines that are parallel to

the axes. In the plane of the t, x axes a contour plot indicates curves of constant $\phi(t, x)$. Compared to the uncontrolled process $J(0) = 116.22$, the optimal acoustic ($r_1 = 0$) energy is $J(\hat{u}) = 26.88$.

Example 4.2. The configuration is as in the previous example, except that the control source now is located in the interior, $\eta_1 = 2$. The optimal control thus generates leftgoing pulses so that the annihilation of ϕ_x^\pm is never completed. $J(\hat{u}) = 56.95$.

Example 4.3. $L = 10, R_0 = R_1 = 0.2, N = 25, K = 148, n = 1, \xi_1 = 10, m = 2, \eta_1 = 3.2, r_1 = 0.5, \eta_2 = 7.2, r_2 = 0.5, q^\pm(x) = \chi[0, 3.2)$. The offending source at the right boundary runs at 50 Hz, $f_1(t) = \sin 100\pi t_k$ for $t \in [t_{k-1}, t_k)$. As harmonic initial conditions we take ϕ_x^+, ϕ_x^- from the last time step of the (uncontrolled) generation of \underline{g} that starts from pre-initial conditions $\phi_x^+(-10, \cdot) \equiv \phi_x^-(-10, \cdot) \equiv 0$ with f_1 running for $-10 \leq t \leq 0$. The optimal control causes a transient ($t > 0$) toward another periodic state with a flat velocity potential on $(t_{18}, t_K) \times (0, 3.2)$. Note that, because of $r_2 \neq 0$, u_2 switches to 0 as soon as its signals cannot influence the waves in $(0, 3.2)$ any more. We get $J(0) = 659.46$, $J(\hat{u}) = 20.65$. The flatness of ϕ in $(0, 3.2)$ can be influenced by the magnitudes of r_1, r_2 relative to q^\pm . For our choice of r_1, r_2 we see some variation of ϕ throughout. For smaller design parameters r_1, r_2 the oscillations of ϕ become invisible on $(t_{18}, t_K) \times (0, 3.2)$. However, $r_2 = 0$ would yield a singular coefficient matrix since $q^\pm(x) = 0$ on $(3.2, 10)$.

Example 4.4. To test the applicability of our methods to somewhat more realistic acoustic events, we sample the first 2 seconds of a wellknown theme of classical music, $f_1(t) = \sum_{k=1}^3 \chi[\frac{k}{4}, \frac{2k+1}{8}) \sin 392\pi t + \chi[1, 2) \sin 312\pi t$, $t \in [0, 2]$, and feed it to a source at $\xi_1 = 2.5$ in a duct of length $L = 5$ with reflection coefficients $R_0 = 0, R_1 = 0.7$ and zero initial conditions. We seek to reduce the sound near the boundaries: $q^\pm(x) = \chi(0, 1) + \chi(4, 5)$. Let two control sources be located at $\eta_1 = 1, \eta_2 = 4$ weighted by $r_1 = r_2 = 0$. We would like to have spatial resolution of about 30 grid points for the shortest wave length of the offending wave, $344/196/30 = 0.059$, so we choose $\Delta x = 0.05, N = 100$. Then $T = 2 = K\Delta t$ leads to $K = 2c/\Delta x = 13760$.

The optimal control annihilates the components of the wave that cross η_1, η_2 in outgoing direction. This amounts to total reflection at the boundaries of (η_1, η_2) , which, roughly speaking, in general leads to increasing sound in its interior where the offending source is located. This is not the case for the symmetric configuration chosen here. We give graphics for the transients at $t = \frac{3}{8}$ and $t = \frac{7}{8}$. At the end of the first tone, the graphics show a transient toward a smoother velocity potential. At the end of the third tone, the controls generate waves that are absorbed by the offending source, so that $\phi = 0$ (although $r_1 = r_2 = 0$ and $q^\pm = 0$ in (η_1, η_2)) until the fourth tone of lower frequency begins at $t = 1$ (the contour curves reveal computational inaccuracies). Because of the totally reflecting behaviour of the control sources, there is no damping in $[\eta_1, \eta_2]$ and the process does not converge to a periodic state as $t \rightarrow 2$. In fact, the choice of R_0, R_1 does not affect the optimal control for this configuration; however, $R_0 = 0$ drastically reduces the number of nonzeros in (3.3). The full matrix \underline{B} would require over 600 gigabytes of storage (8 bytes per entry), whereas the sparse matrix A of density 0.0003 has 2.4 megabytes (including the memory for the indices). The sparse arithmetics to solve $Au = rhs$ require less than 10.3 megaflops.

5. APPROXIMATION AND CONVERGENCE

We again assume that L, T and the location of the sources are such that there exist integers $N, K \in \mathbb{N}$ with $\{\xi_2, \dots, \xi_n, \eta_2, \dots, \eta_m\} \subseteq \{jL/N, j = 1, \dots, N-1\}$ and $K = cTN/L$.

For $\ell \in \mathbb{N}$ and $\phi \in H^1(0, L)$ let $s_\ell \phi$ be the first order spline interpolation of ϕ with respect to the mesh $\{jL/\ell N\}_{j=0}^{\ell N}$, i.e.

$$s_\ell \phi(jL/\ell N) = \phi(jL/\ell N), \quad j = 0, \dots, \ell N.$$

There exists a $c_0 > 0$ such that for all $\ell \in \mathbb{N}$ and $\phi \in H^2(0, L)$

$$(5.1) \quad \|s_\ell \phi - \phi\|_{H^1(0, L)} \leq c_0 \|\phi''\|/\ell$$

(e.g. [Sch], Th. 2.5). Moreover, let π_ℓ denote the orthogonal projection of $L^2(0, T)$ onto $U_{\ell K}$, i.e.

$$\pi_\ell f(t) = \frac{\ell K}{T} \int_{(k-1)T/\ell K}^{kT/\ell K} f d\tau \quad \text{for } t \in [(k-1)T/\ell K, kT/\ell K), k = 1, \dots, \ell K.$$

For all $f \in L^2(0, T)$ (cf. [BB], p. 176)

$$(5.2) \quad \|\pi_\ell f - f\|_{L^2(0, T)} \rightarrow 0 \text{ as } \ell \rightarrow \infty.$$

Given $f_1, \dots, f_n \in L^2(0, T)$ and sufficiently smooth ϕ_0^+, ϕ_0^- , the solution \hat{u} of (2.12) is approximated by the functions \hat{u}_ℓ , that are the solutions of (3.1) wherein $(s_\ell \phi_0^+, s_\ell \phi_0^-)$ and $\pi_\ell f_1, \dots, \pi_\ell f_n$ are used for initial conditions and sources and the minimum is taken over $U_{\ell K}^m$. More precisely we have

Theorem 5.1. *If $(\phi_0^+, \phi_0^-) \in V \cap H^2(0, 2)^2$ or if $(\phi_0^+, \phi_0^-) \in V_{\ell_0 N}$ for some $\ell_0 \in \mathbb{N}$ then $\|\hat{u} - \hat{u}_\ell\|_U \rightarrow 0$ as $\ell \rightarrow \infty$.*

Proof. With $g_\ell = S(s_\ell \phi_0^+, s_\ell \phi_0^-) + \sum_{i=1}^n S_{\xi_i} \pi_\ell f_i$ the optimal control $\hat{u}_\ell \in U_{\ell K}^m$ is characterized by the variational equation

$$a(\hat{u}_\ell, u) = b_\ell(u) \text{ for all } u \in U_{\ell K}^m$$

where $a : U \times U \rightarrow \mathbb{R}$, $b_\ell : U \rightarrow \mathbb{R}$ are symmetric bilinear, resp. linear, continuous forms given by

$$\begin{aligned} a(u, v) &= \langle u, (R + B^*QB)u \rangle_U \\ b_\ell(u) &= -\langle B^*Qg_\ell, u \rangle_U. \end{aligned}$$

This is so, because \hat{u}_ℓ minimizes the functional $J_\ell(u) = a(u, u) - 2b_\ell(u) + \langle g_\ell, Qg_\ell \rangle_H$ in $U_{\ell K}^m$ (cf. [C], Theorem 1.2; the constant term is included here for consistency with (2.13), but it

does not affect the minimization problem). Similarly, \hat{u} minimizes $J(u) = a(u, u) - 2b(u) + \langle g, Qg \rangle_H$ in U with

$$b(u) = -\langle B^*Qg, u \rangle_U$$

and thus it is characterized by

$$a(\hat{u}, u) = b(u) \text{ for all } u \in U.$$

By assumption (2.14) a is U -elliptic and we can apply the first Strang lemma ([C], Th. 26.1): there exist constants c_1, c_2 that are independent of ℓ , such that

$$\|\hat{u} - \hat{u}_\ell\|_U \leq c_1 \inf\{\|\hat{u} - u\|_U : u \in U_{\ell K}^m\} + c_2 \sup\{|b(u) - b_\ell(u)|/\|u\|_U : u \in U_{\ell K}^m\}$$

The first term on the right hand side tends to zero by (5.2). The second term is estimated using Schwarz' inequality and the boundedness of B^*Q

$$| \langle B^*Q(g - g_\ell), u \rangle_U | / \|u\|_U \leq \|B^*Q\| \|g - g_\ell\|_H$$

where

$$\|g - g_\ell\|_H \leq \|S\| \|(\phi_0^+, \phi_0^-) - (s_\ell \phi_0^+, s_\ell \phi_0^-)\|_V + \sum_{i=1}^n \|S_{\xi_i}\| \|f_i - \pi_\ell f_i\|_{L^2(0, T)}$$

so that (5.1), (5.2) imply the result. In case $(\phi_0^+, \phi_0^-) \in V_{\ell_0 N}$ we have $s_\ell \phi_0^+ = \phi_0^+$, $s_\ell \phi_0^- = \phi_0^-$ for $\ell \geq \ell_0$. \square

The second case in Theorem 5.1 is applicable when the problem is broken up into pieces over subintervals of $[0, T]$. But this is not equivalent to minimization over $[0, T]$ at once.

How do the discrete controls \hat{u}_ℓ perform when applied to the original data $(\phi_0^+, \phi_0^-) \in V \cap H^2(0, L)^2$ and sources $f_i \in L^2(0, T)$? The answer (in the sense of convergence) is contained in the theorem: $J(\hat{u}_\ell) \rightarrow J(\hat{u})$, since $J : U \rightarrow \mathbb{R}$ is continuous.

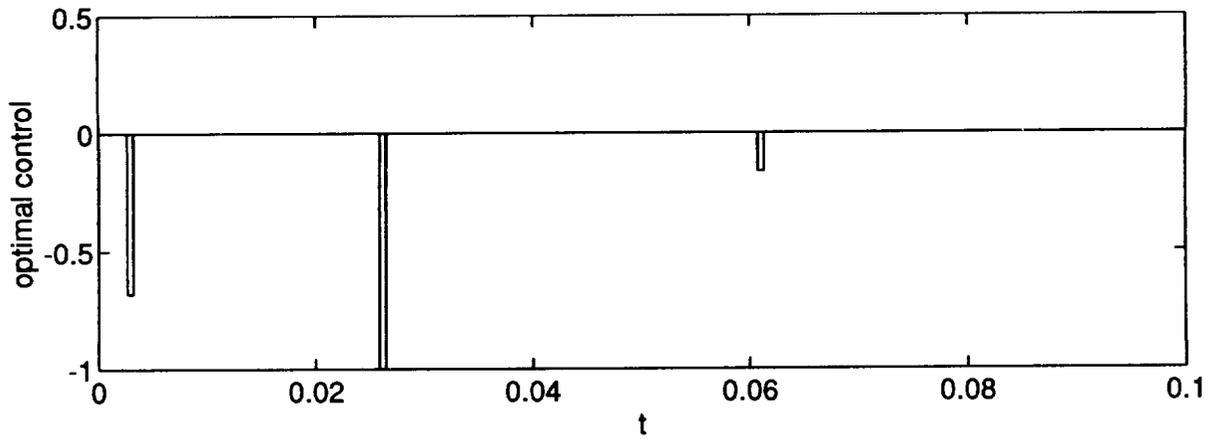
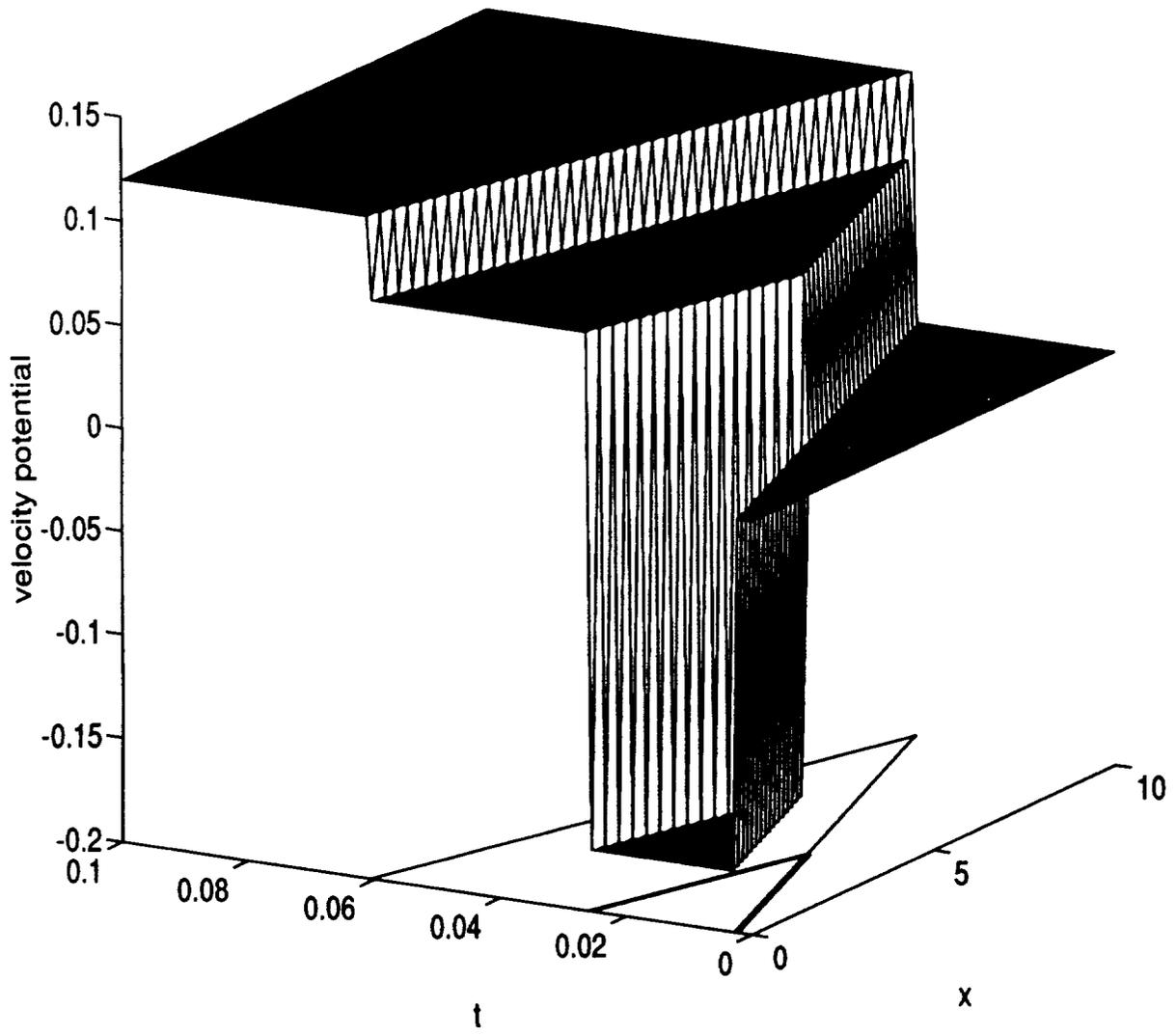
5. CONCLUSIONS

By decomposition of the one-dimensional wave equation with point sources and pointwise reflecting boundary conditions and by appropriate discretization in time and space, the minimization problem with piecewise constant sources was reduced to the solution of a sparse linear system. It was shown that the waves that are generated by discrete forcing functions and initial conditions are sufficiently regular so that the problem is wellposed. Four examples demonstrated the applicability of algorithms that efficiently set up the sparse linear system; such examples provide insights into the optimal control of waves in ducts. Finally, it was proved that the solutions of finite dimensional discretizations converge to the solution of the minimization problem with square integrable sources.

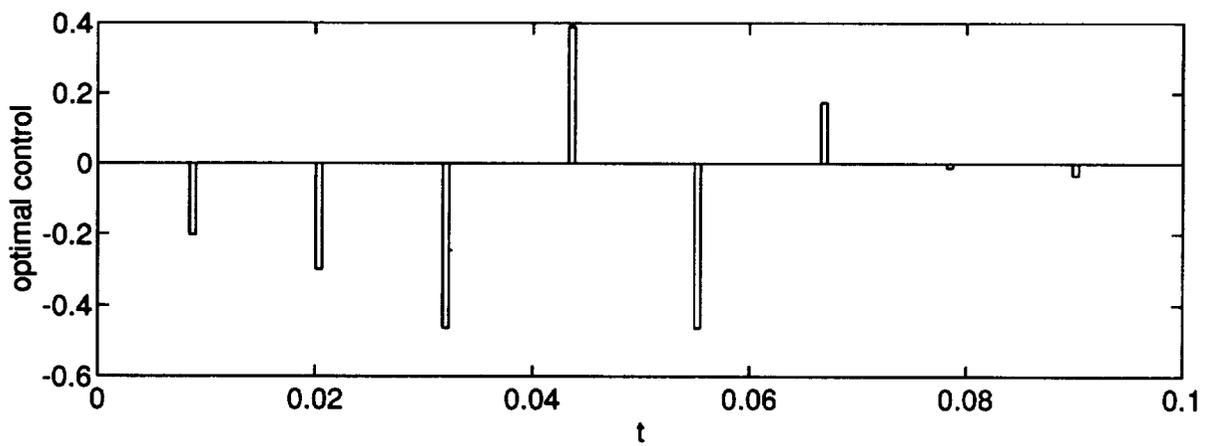
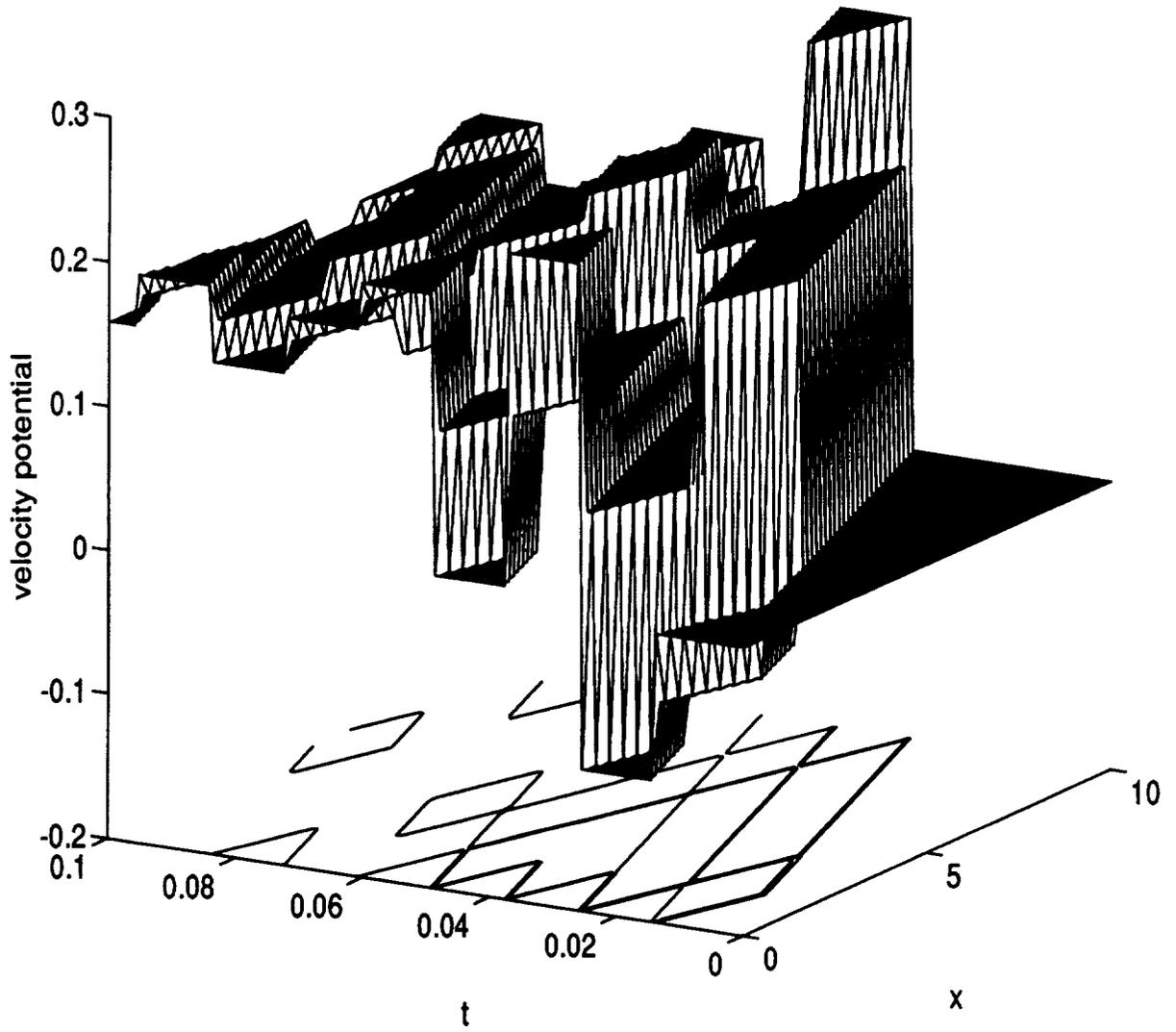
REFERENCES

- [B] A.V. Balakrishnan, *Applied Functional Analysis*, Springer, New York, 1981.
- [BB] H.T. Banks and J.A. Burns, *Hereditary control problems: Numerical methods based on averaging approximation*, SIAM J. Control Optimization **16** (1978), 169-208.
- [BKS] H.T. Banks, S.L. Keeling, R.J. Silcox and C. Wang, *Linear quadratic tracking problems in Hilbert space: Application to optimal active noise suppression*, Control of Distributed Parameter Systems, 1989: Selected Papers from the Fifth IFAC Symposium (M. Amouroux and A. El Jai, eds.), Pergamon Press, Oxford, 1990, pp. 5-10.
- [BPS] H.T. Banks, G. Propst and R.J. Silcox, *A comparison of time domain boundary conditions for acoustic waves in wave guides*, Quarterly of Applied Mathematics (to appear).
- [BG] J.S. Bolton and E. Gold, *The determination of acoustic reflection coefficients by using cepstral techniques, I: Experimental procedures and measurements of polyurethane foam*, Journal of Sound and Vibration **110** (1986), 179-202.
- [C] P.G. Ciarlet, *Basic Error Estimates for Elliptic Problems*, Handbook of Numerical Analysis, Vol. II, Finite Element Methods (Part 1) (P.G. Ciarlet and J.L. Lions, eds.), North-Holland, Amsterdam, 1991.
- [CNE] A.R.D. Curtis, P.A. Nelson, S.J. Elliott, *Active Control of one-dimensional enclosed sound fields*, Inter-Noise 85, Wirtschaftsverlag NW, Bremerhaven.
- [DLP] R. Datko, J. Lagnese and M.P. Polis, *An example on the effect of time delays in boundary feedback stabilization of wave equations*, SIAM J. Control Optimization **24** (1986), 152-156.
- [DW] W. Desch and R.L. Wheeler, *Destabilization due to delay in one dimensional feedback*, Control and Estimation of Distributed Parameter Systems (F. Kappel, K. Kunisch and W. Schappacher, eds.), Birkhäuser Verlag, Basel, 1989, pp. 61-83.
- [FW] J.E. Ffowcs Williams, *Anti-sound*, Proc. R. Soc. Lond. A **395** (1984), 63-88.
- [HS] E. Hewitt and K. Stromberg, *Real and Abstract Analysis*, Springer, Berlin, 1969.
- [L] J.L. Lions, *Pointwise control for distributed systems*, Control and Estimation of Distributed Parameter Systems (H.T. Banks, ed.), SIAM, Philadelphia, 1992, pp. 1-39.
- [LM] J.L. Lions and E. Magenes, *Nonhomogeneous Boundary Value Problems and Applications*, Vols. I and II, Springer, Berlin, 1972.
- [M] Matlab, *High-Performance Numeric Computation and Visualization Software*, The MathWorks, Inc.
- [MI] P.M. Morse and K.U. Ingard, *Theoretical Acoustics*, McGraw-Hill, New York, 1968.
- [NCEB1] P.A. Nelson, A.R.D. Curtis, S.J. Elliott and A.J. Bullmore, *The minimum power output of free field point sources and the active control of sound*, Journal of Sound and Vibration **116** (1987), 397-414.
- [NCEB2] P.A. Nelson, A.R.D. Curtis, S.J. Elliott and A.J. Bullmore, *The active minimization of harmonic enclosed sound fields, Part I: Theory*, Journal of Sound and Vibration **117** (1987), 1-13.
- [P] J.T. Post, *Active control of the forced and transient response of a finite beam*, NASA Contractor Report 181957, 1990.
- [Sch] M.H. Schultz, *Spline Analysis*, Prentice-Hall, Englewood Cliffs, 1973.
- [S1] L. Schwartz, *Théorie des Distributions*, Hermann, Paris, 1966.
- [S2] L. Schwartz, *Méthodes Mathématiques pour les Sciences Physiques*, Hermann, Paris, 1965.
- [Sw] M.A. Swinbanks, *The active control of sound propagation in long ducts*, Journal of Sound and Vibration **27** (1973), 411-436.
- [TN] M.C.J. Trinder and P.A. Nelson, *Active noise control in finite length ducts*, Journal of Sound and Vibration **89** (1983), 95-105.
- [Z] A.H. Zemanian, *Distribution Theory and Transform Analysis*, Dover, New York, 1987.

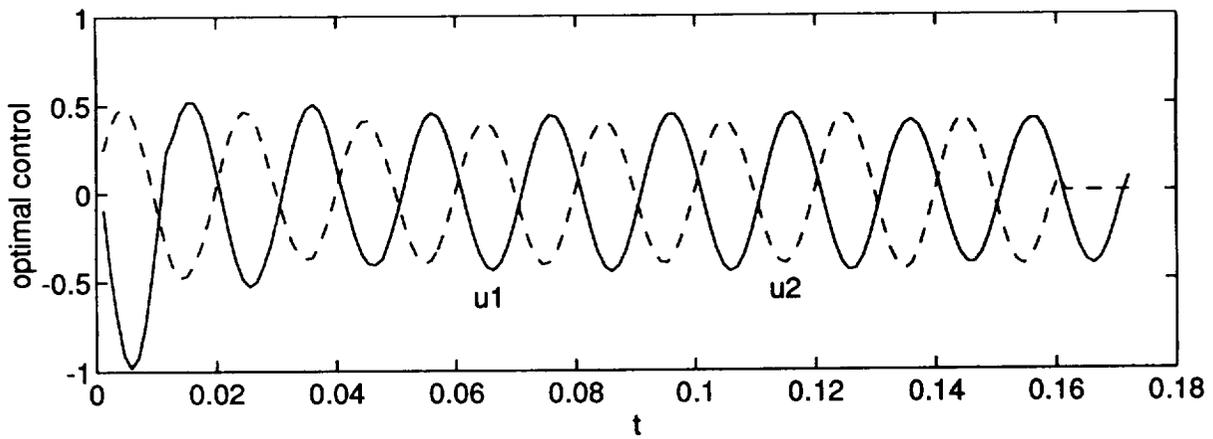
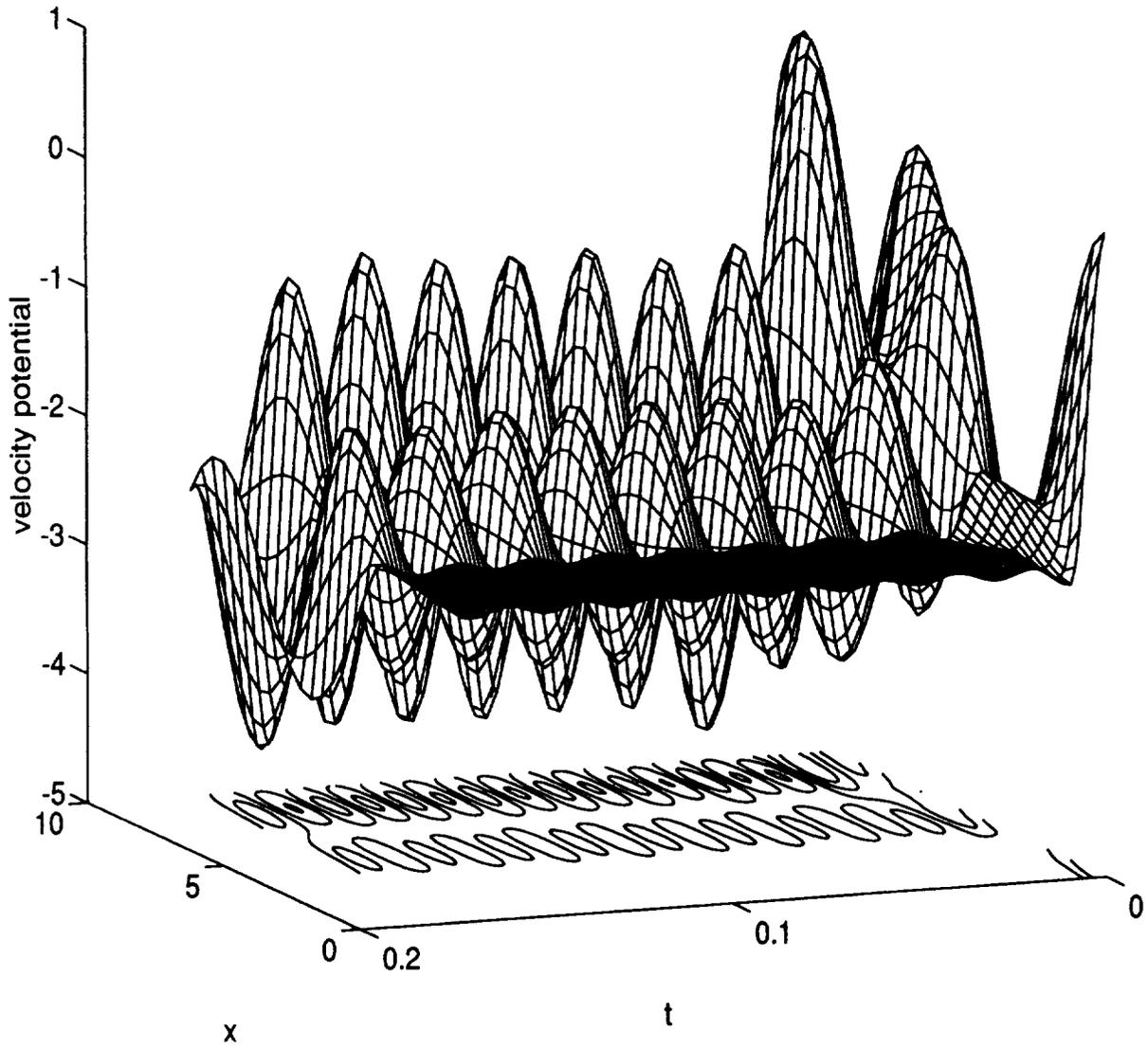
Example 4.1



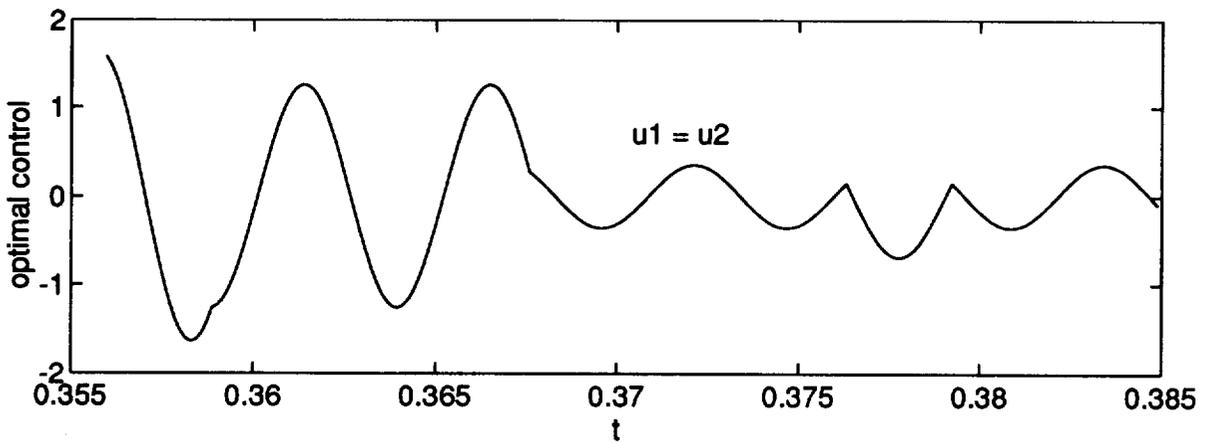
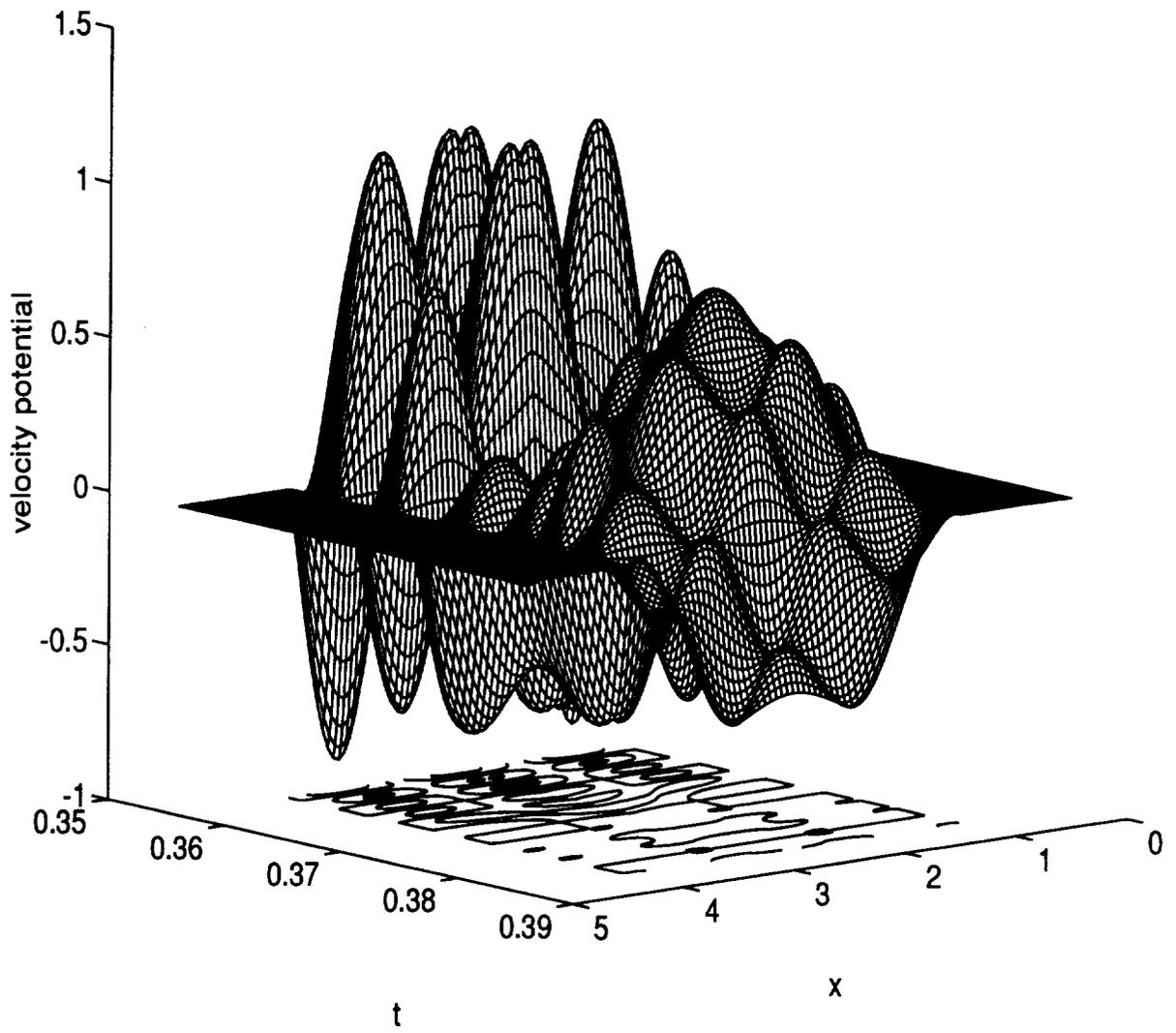
Example 4.2



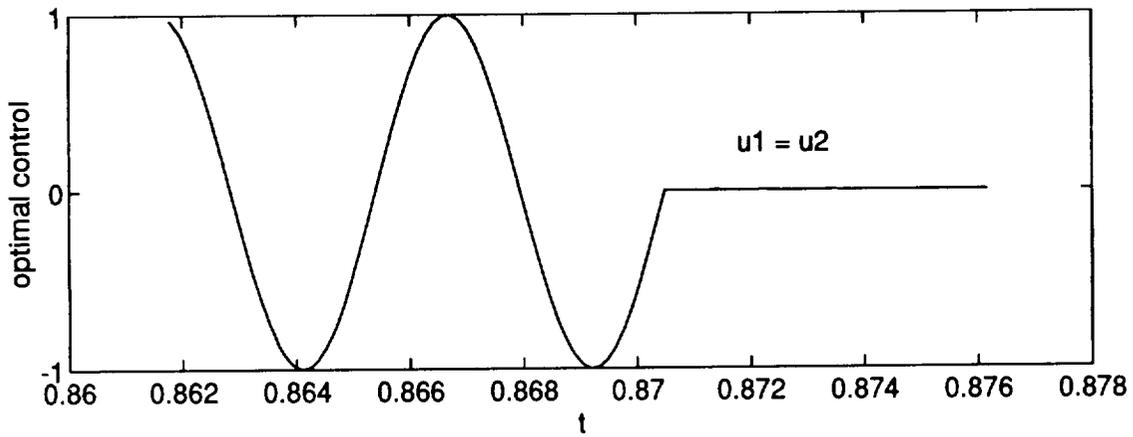
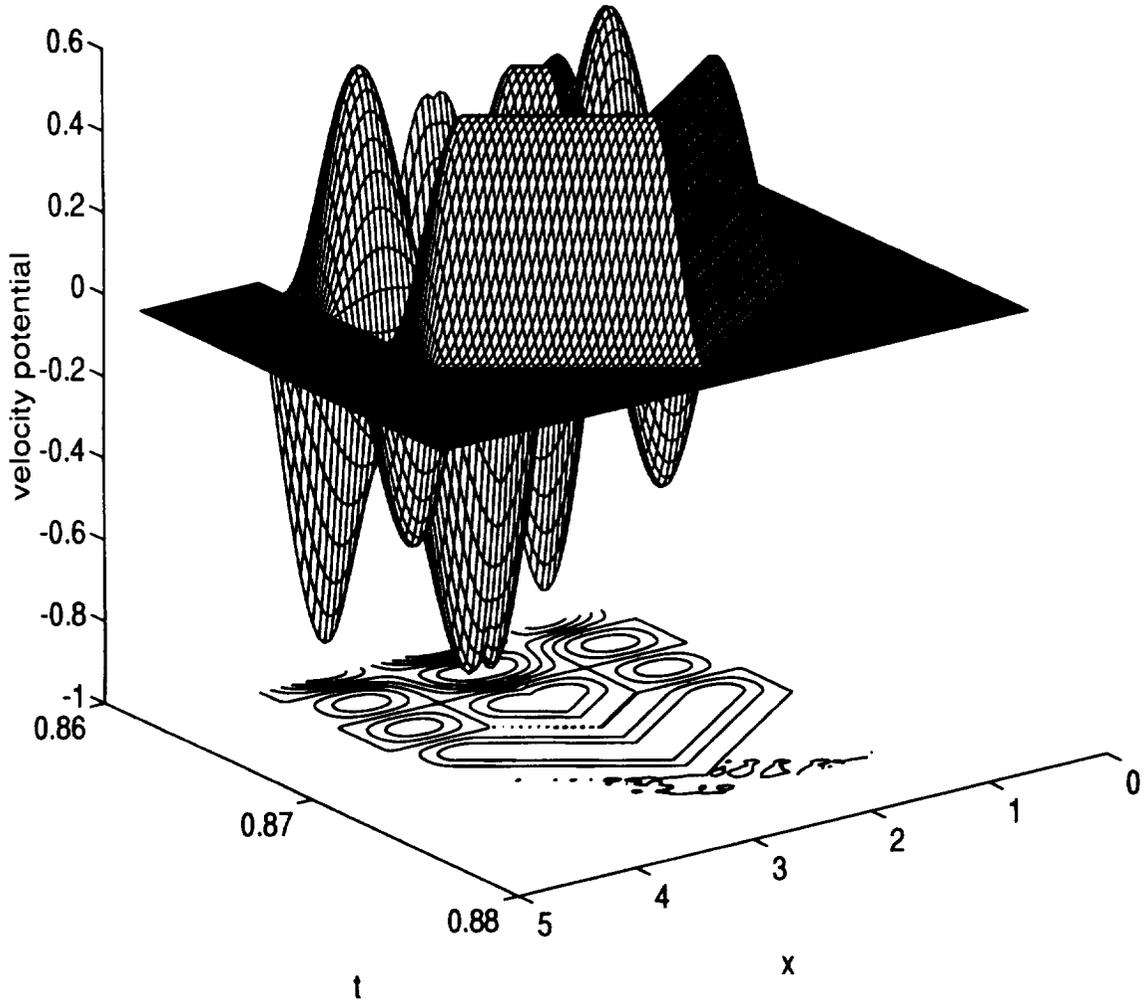
Example 4.3



Example 4.4



Example 4.4



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