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Local Reduction of Certain Wave Operators to One-dimensional Form

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ABSTRACT

It is noted that certain common linear wave operators have the property that linear variation of the initial data gives rise to one-dimensional evolution in a plane defined by time and some direction in space. The analysis is given for operators arising in acoustics, electromagnetics, elastodynamics, and an abstract system.

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1 Introduction

In this paper we point out an interesting fact concerning three common instances of linear wave propagation (acoustic, electromagnetic, and elastic). In each case, we consider the initial-value problem in some region of space sufficiently small that the gradients in the data can be taken as constant. Then the evolution of that data in time corresponds precisely to that produced by one-dimensional waves travelling in some distinguished direction. In the three instances, this direction is defined ¹ by the pressure gradient, by $(\nabla \times \vec{B}) \times (\nabla \times \vec{D})$ (not the Poynting vector), and by any vector perpendicular to $\nabla \times \vec{v}$. We also consider an abstract system, to which all 2×2 systems in a certain class are equivalent.

2 Statement of Result

Consider a set of linear partial differential equations in n unknowns and d space dimensions,

$$\partial_t \mathbf{q} + \sum_{i=1}^{i=d} A_i \partial_{x_i} \mathbf{q} = 0. \quad (1)$$

A simple wave solution of (1) is a solution of the special form

$$\mathbf{q}(\vec{x}, t) = \mathbf{r}_\ell f(\vec{\ell} \cdot \vec{x} - \lambda t)$$

where $\vec{\ell}$ is a unit vector in the direction of wave propagation, λ is the speed of propagation, and f is an arbitrary scalar function. Such solutions exist if and only if \mathbf{r}_ℓ is a right eigenvector of

$$A_\ell = \sum_{i=1}^{i=d} \ell_i A_i.$$

Linearly varying initial data for (1) can be considered as a vector space \mathcal{D} in \mathbf{R}^{nd} , conveniently displayed as d column vectors

$$D = [\partial_{x_1} \mathbf{q}, \partial_{x_2} \mathbf{q}, \dots]$$

Data giving rise to a simple wave is of the special form

$$D_\ell = [\ell_1 \mathbf{r}_\ell, \ell_2 \mathbf{r}_\ell \dots].$$

Denote by \mathcal{S}_ℓ the subspace of \mathcal{D} containing such data, with $\vec{\ell}$ fixed and $\lambda \neq 0$, i.e., the set of data giving rise to one-dimensional wave motion in the direction $\vec{\ell}$.

The evolution operator A that produces $\partial_t \mathbf{q}$ from an element of \mathcal{D} is an $n \times nd$ matrix whose nullspace $\mathcal{N}(A)$ comprises the set of linearly varying steady solutions to (1). We claim that for some interesting evolution operators, such as those mentioned in the Introduction, an arbitrary element of \mathcal{D} can be represented as a sum of elements lying in \mathcal{N} and elements lying in some \mathcal{S}_ℓ ,

$$D = \sum_j \beta_j N_j + \sum_k \alpha_k S_{\ell,k} \quad (2)$$

¹The notation used in this paper is that bold type represents a vector such as a set of unknowns; vectors with geometric meaning in \mathbf{R}^3 have arrows surmounting regular type

Note that we do not claim that $\mathcal{N} + \mathcal{S}_\ell = \mathcal{D}$, but that (when the trick works) the direction $\vec{\ell}$ can be chosen so that $\mathcal{N} + \mathcal{S}_\ell$ includes any given element of \mathcal{D} . To evaluate the coefficients in (2) we operate on both sides with A , giving

$$\partial_t \mathbf{q} = \sum_k \alpha_k \lambda_k \mathbf{r}_{\ell,k} \quad (3)$$

which is merely a decomposition of the time derivative onto the eigenvectors for wave motion along $\vec{\ell}$. This differs from the one-dimensional decomposition (Riemann problem) in that the direction of $\vec{\ell}$ is unknown and enters into the equations nonlinearly. In the following sections we see how this works out for various special cases.

3 Examples

3.1 Examples with Unique Solutions

3.1.1 Acoustics

For the equations governing acoustic waves in a uniform medium we have

$$A_\ell = \begin{bmatrix} 0 & \ell_1 & \ell_2 & \ell_3 \\ \ell_1 & 0 & 0 & 0 \\ \ell_2 & 0 & 0 & 0 \\ \ell_3 & 0 & 0 & 0 \end{bmatrix}, \quad (4)$$

where the unknowns are $(p, \vec{v})^T$, the pressure and velocities, and units have been chosen to make the sound speed unity. The eigenvalues are $\lambda_{1,2,3,4} = -1, 0, 0, 1$ and the non-stationary eigenvectors are

$$\mathbf{r}_1 = \begin{bmatrix} 1 \\ -\vec{\ell} \end{bmatrix}, \quad \mathbf{r}_4 = \begin{bmatrix} 1 \\ \vec{\ell} \end{bmatrix}.$$

The expansion (3) is

$$\begin{bmatrix} p_t \\ \vec{v}_t \end{bmatrix} = \alpha_1 \begin{bmatrix} -1 \\ \vec{\ell} \end{bmatrix} + \alpha_4 \begin{bmatrix} 1 \\ \vec{\ell} \end{bmatrix} \quad (5)$$

which can only be solved if

$$\vec{\ell} \parallel \vec{v}_t = -\text{grad } p,$$

when the wavestrengths are

$$\alpha_1 = \frac{1}{2}(\vec{\ell} \cdot \vec{v}_t - p_t), \quad \alpha_4 = \frac{1}{2}(\vec{\ell} \cdot \vec{v}_t + p_t).$$

3.1.2 Electromagnetics

Maxwells equations in free space, for the magnetic field \vec{B} , and the electric field \vec{D} , in units for which the speed of light is unity, give rise to

$$A_\ell = \begin{bmatrix} 0 & 0 & 0 & 0 & -\ell_3 & \ell_2 \\ 0 & 0 & 0 & \ell_3 & 0 & -\ell_1 \\ 0 & 0 & 0 & -\ell_2 & \ell_1 & 0 \\ 0 & \ell_3 & -\ell_2 & 0 & 0 & 0 \\ -\ell_3 & 0 & \ell_1 & 0 & 0 & 0 \\ \ell_2 & -\ell_1 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (6)$$

The eigenvalues $\lambda_{1,2,3,4,5,6}$ are $-1, -1, 0, 0, 1, 1$ and the non-stationary eigenvectors are

$$\mathbf{r}_1 = \begin{vmatrix} \vec{s} \\ \vec{t} \end{vmatrix}, \quad \mathbf{r}_2 = \begin{vmatrix} \vec{t} \\ \vec{s} \end{vmatrix}, \quad \mathbf{r}_5 = \begin{vmatrix} \vec{t} \\ -\vec{s} \end{vmatrix}, \quad \mathbf{r}_6 = \begin{vmatrix} \vec{s} \\ -\vec{t} \end{vmatrix},$$

where \vec{s}, \vec{t} are any pair of unit vectors such that $\vec{\ell}, \vec{s}, \vec{t}$ form a right-handed orthogonal system. The expansion (3) is

$$\begin{vmatrix} \vec{B}_t \\ \vec{D}_t \end{vmatrix} = \alpha_1 \begin{vmatrix} -\vec{s} \\ -\vec{t} \end{vmatrix} + \alpha_2 \begin{vmatrix} -\vec{t} \\ -\vec{s} \end{vmatrix} + \alpha_5 \begin{vmatrix} \vec{t} \\ -\vec{s} \end{vmatrix} + \alpha_6 \begin{vmatrix} \vec{s} \\ -\vec{t} \end{vmatrix}. \quad (7)$$

implying

$$\begin{aligned} \vec{B}_t &= (\alpha_6 - \alpha_1)\vec{s} + (\alpha_5 - \alpha_2)\vec{t} \\ \vec{D}_t &= -(\alpha_6 + \alpha_1)\vec{t} - (\alpha_5 + \alpha_2)\vec{s} \end{aligned}$$

For a solution to exist, both \vec{B}_t and \vec{D}_t must lie in the plane spanned by \vec{s} and \vec{t} , which is the plane normal to $\vec{\ell}$. Then

$$\vec{\ell} \parallel \vec{B}_t \times \vec{D}_t = \text{curl} \vec{B} \times \text{curl} \vec{D} \quad (8)$$

and we easily find

$$\begin{aligned} \alpha_1 &= -\vec{B}_t \cdot \vec{s} - \vec{D}_t \cdot \vec{t}, \\ \alpha_2 &= -\vec{B}_t \cdot \vec{t} - \vec{D}_t \cdot \vec{s}, \\ \alpha_5 &= \vec{B}_t \cdot \vec{t} - \vec{D}_t \cdot \vec{s}, \\ \alpha_6 &= \vec{B}_t \cdot \vec{s} - \vec{D}_t \cdot \vec{t}. \end{aligned} \quad (9)$$

3.2 Examples with Nonunique Solutions

3.2.1 'Cauchy-Riemann' Equations

Consider the system obtained by adding time-dependent terms to the Cauchy-Riemann equations;

$$\begin{aligned} u_t + u_x + v_y &= 0 \\ v_t - v_x + u_y &= 0 \end{aligned} \quad (10)$$

for which the matrix A_t is

$$A_t = \begin{bmatrix} \ell_1 & \ell_2 \\ \ell_2 & -\ell_1 \end{bmatrix}. \quad (11)$$

Gilquin, Laurens and Rosier [1] have shown that any strictly hyperbolic 2×2 system can be reduced to this case by a transformation of variables, and Noelle [2] finds that any linear initial data is the sum of three simple waves.

The eigenvalues of A_t are $\lambda_1 = -1, \lambda_2 = 1$, with eigenvectors

$$\mathbf{r}_1 = \begin{vmatrix} \ell_1 - 1 \\ \ell_2 \end{vmatrix}, \quad \mathbf{r}_2 = \begin{vmatrix} \ell_1 + 1 \\ \ell_2 \end{vmatrix}.$$

These can be normalised and rewritten in terms of the wave direction θ as

$$\mathbf{r}_1 = \begin{vmatrix} -\sin \theta/2 \\ \cos \theta/2 \end{vmatrix}, \quad \mathbf{r}_2 = \begin{vmatrix} \cos \theta/2 \\ \sin \theta/2 \end{vmatrix}.$$

and the expansion (3) becomes

$$\begin{vmatrix} u_t \\ v_t \end{vmatrix} = \alpha_1 \begin{vmatrix} \sin \theta/2 \\ -\cos \theta/2 \end{vmatrix} + \alpha_2 \begin{vmatrix} \cos \theta/2 \\ \sin \theta/2 \end{vmatrix}. \quad (12)$$

This can be solved for any choice of θ :-

$$\alpha_1 = u_t \sin \theta/2 - v_t \cos \theta/2, \quad \alpha_2 = u_t \cos \theta/2 + v_t \sin \theta/2. \quad (13)$$

3.2.2 Elastodynamics

The equations governing elastic waves in a uniform isotropic medium are of the form (1) with $\mathbf{q} = (\vec{v}, p, \vec{\omega})^T$ where \vec{v} is the velocity, p is the trace of the strain tensor, and $\vec{\omega}$ forms the antisymmetric (rotational) part of the strain tensor. The strains are related to the displacements \vec{u} by

$$p = \text{div } \vec{u},$$

and

$$\vec{\omega} = \text{curl } \vec{u}.$$

The matrix A_t can be written as

$$\begin{bmatrix} 0 & 0 & 0 & b\ell_1 & 0 & \ell_3 & -\ell_2 \\ 0 & 0 & 0 & b\ell_2 & -\ell_3 & 0 & \ell_1 \\ 0 & 0 & 0 & b\ell_3 & \ell_2 & -\ell_1 & 0 \\ \ell_1 & \ell_2 & \ell_3 & 0 & 0 & 0 & 0 \\ 0 & -\ell_3 & \ell_2 & 0 & 0 & 0 & 0 \\ \ell_3 & 0 & -\ell_1 & 0 & 0 & 0 & 0 \\ -\ell_2 & \ell_1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (14)$$

Here units have been chosen so that transverse waves (S-waves) travel with unit speed, and longitudinal waves (P-waves) with speed \sqrt{b} , where the parameter b is

$$b = 1 + \frac{2M}{L}$$

and L, M are the Lamé coefficients.

The matrix (14) has eigenvalues $\lambda_{1,2,3,4,5,6,7} = -\sqrt{b}, -1, -1, 0, 1, 1, \sqrt{b}$, with non-stationary eigenvectors

$$\mathbf{r}_1 = \begin{vmatrix} \sqrt{b}\vec{\ell} \\ -1 \\ \vec{0} \end{vmatrix}, \quad \mathbf{r}_2 = \begin{vmatrix} \vec{s} \\ 0 \\ -\vec{t} \end{vmatrix}, \quad \mathbf{r}_3 = \begin{vmatrix} \vec{t} \\ 0 \\ -\vec{s} \end{vmatrix}, \quad (15)$$

$$\mathbf{r}_5 = \begin{vmatrix} \vec{t} \\ 0 \\ \vec{s} \end{vmatrix}, \quad \mathbf{r}_6 = \begin{vmatrix} \vec{s} \\ 0 \\ \vec{t} \end{vmatrix}, \quad \mathbf{r}_7 = \begin{vmatrix} \sqrt{b}\vec{\ell} \\ 1 \\ \vec{0} \end{vmatrix}. \quad (16)$$

Thus the expansion (3) for this case is

$$\begin{vmatrix} \vec{v}_t \\ p_t \\ \vec{\omega}_t \end{vmatrix} = \alpha_1 \begin{vmatrix} -b\vec{\ell} \\ \sqrt{b} \\ \vec{0} \end{vmatrix} + \alpha_2 \begin{vmatrix} -\vec{s} \\ 0 \\ \vec{t} \end{vmatrix} + \alpha_3 \begin{vmatrix} -\vec{t} \\ 0 \\ \vec{s} \end{vmatrix} + \alpha_5 \begin{vmatrix} \vec{t} \\ 0 \\ \vec{s} \end{vmatrix} + \alpha_6 \begin{vmatrix} \vec{s} \\ 0 \\ \vec{t} \end{vmatrix} + \alpha_7 \begin{vmatrix} b\vec{\ell} \\ \sqrt{b} \\ \vec{0} \end{vmatrix}. \quad (17)$$

From this we observe that $\vec{\omega}_t$ lies in the plane spanned by \vec{s}, \vec{t} , so that $\vec{\ell}$ must be chosen normal to $\vec{\omega}_t = \text{curl} \vec{v}$. For any such choice of $\vec{\ell}$,

$$\begin{aligned} \alpha_7 + \alpha_1 &= p_t / \sqrt{b}, \\ \alpha_7 - \alpha_1 &= \vec{\ell} \cdot \vec{v}_t / b, \\ \alpha_6 + \alpha_2 &= \vec{t} \cdot \vec{\omega}_t, \\ \alpha_6 - \alpha_2 &= \vec{s} \cdot \vec{v}_t, \\ \alpha_5 + \alpha_3 &= \vec{s} \cdot \vec{\omega}_t, \\ \alpha_5 - \alpha_3 &= \vec{t} \cdot \vec{v}_t. \end{aligned} \quad (18)$$

3.3 An Example with no Solution

The following is based on a suggestion of Jonathan Goodman (Courant Institute, NYU). Consider a system comprising two copies of the acoustic equations, with unknowns $(\vec{v}_1, p_1, \vec{v}_2, p_2)$. The resulting one-dimensional wave problem has a solution only for data with $\vec{v}_{1,t} \parallel \vec{v}_{2,t}$.

4 Applications

These results have potential application to the numerical analysis of wave motion, where the solution is often assumed to be linear within elements, and where the update strategy may depend on whether waves are entering or leaving some region. For a critique of methods where wave directions are determined relative to cell boundaries, see [3]. Dissatisfaction with such an approach leads to an interest in representing multidimensional waves in a way that is coordinate-free, but simple enough for computation. The lack of a general pattern may not be a drawback; exploiting behaviour that is very specific to a particular equation set may be the most effective approach.

Specific applications will turn on details of numerical technique as well as the analytical decompositions; for that reason they are not explored here, but reference may be made to computations of the compressible Euler equations by splitting finite-element gradients into steady and unsteady components [4], and to a boundary condition for Maxwell's equations that uses the above formulae to identify the direction in which waves exit the boundary [5].

In a non-numerical context, there is precedent for this analysis in the work of Frohn [6] who studied steady three-dimensional supersonic flow by the method of strained coordinates, where the objective is to find a coordinate perturbation such that the singularities of linear theory become realistic shocks and rarefactions. She selected a plane containing the streamline and the pressure gradient as the one in which a two-dimensional straining should be applied

5 Conclusions

Given a set of hyperbolic partial differential equations in D space dimensions ($D > 1$) and time, together with linearly varying initial data, one may seek directions in which the evolution of the problem takes place one-dimensionally. We have given examples in which the search succeeds

(and a rather artificial counterexample). Since a practical computational algorithm can only be based on a finite number of wave directions, this offers a possible basis for the creation of such algorithms.

6 References

1. H. Gilquin, J. Laurens, C. Rosier, Multi-dimensional Riemann problems for linear hyperbolic systems; Part I, in *Nonlinear Hyperbolic Problems: Theoretical, Applied and Computational Aspects*, eds A. Donato and F. Oliveri, Notes on Numerical Fluid Mechanics 43, Vieweg, 1993.
2. S. Noelle, Hyperbolic systems of conservation laws, the Weyl equation, and multidimensional upwinding, *Journal of Computational Physics*, to appear.
3. P.L. Roe, Beyond the Riemann problem I, in *Algorithmic Trends in Computational Fluid Dynamics*, eds M.Y. Hussaini, A. Kumar, M.D. Salas, Springer, 1993.
4. H. Pailliere, J-C Carette, H. Deconinck, Multidimensional upwind and SUPG methods for the solution of the compressible flow equations on unstructured grids, in *Lecture Series 1994-05, Computational Fluid Dynamics*, Von Karman Institute, Brussels, 1994.
5. B. Nguyen, P.L. Roe, Application of an upwind leapfrog method for electromagnetics, in *10th Annual Review of Progress in Applied Computational Electromagnetics*, vol 1, pp 446-458, 1994.
6. A. Frohn, An analytic characteristic theory for steady three-dimensional isentropic flow, *Journal of Fluid Mechanics*, vol 63, pp 81-96, 1974.

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