N95- 13897

PROPERTIES OF TWO-MODE SQUEEZED NUMBER STATES

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Abstract

Photon statistics and phase properties of two-mode squeezed number states are studied. It is shown that photon number distribution and Pegg-Barnett phase distribution for such states have similar $(N+1)$-peak structure for nonzero value of the difference in the number of photons between modes. Exact analytical formulas for phase distributions based on different phase approaches are derived. The Pegg-Barnett phase distribution and the phase quasiprobability distribution associated with the Wigner function are close to each other, while the phase quasiprobability distribution associated with the Q function carries less phase information.

1 Introduction

Recent developments in quantum optics have led to new proposals to generate number states of the electromagnetic field using conditioned measurements techniques [1] or the properties of atom-field interactions in microwave cavities in the micromaser [2]. The precisely defined two-mode photon number state $|N+q, N\rangle$ can be used as an input field in a squeezing device, such as a parametric amplifier. The model involves a signal and an idler modes driven by a classical pump. The Hamiltonian for the two coupled modes is taken to be [3, 4] (we set $\hbar = 1$)

$$\hat{H} = \omega_a \hat{a}^\dagger \hat{a} + \omega_b \hat{b}^\dagger \hat{b} - i\{g\hat{a}\hat{b}\exp(i\omega t) - g^* \hat{b}^\dagger \hat{a}^\dagger \exp(-i\omega t)\},$$

where ω is the pump frequency and g is the effective intermode coupling constant. If we consider exact resonance $\omega = \omega_a + \omega_b$ then the Hamiltonian may be transformed into the interaction picture

$$\hat{H}_I = -i\{g\hat{a}\hat{b} - g^* \hat{b}^\dagger \hat{a}^\dagger\}.$$

In this picture the time-evolution operator is

$$\exp(-i\hat{H}_I t) = \exp\{-gt\hat{a}\hat{b} + g^* t\hat{b}^\dagger \hat{a}^\dagger\}.$$

and is immediately identifiable as a time-dependent two-mode squeezed operator:

$$\exp(-i\hat{H}_I t) = \hat{S}(gt),$$

with squeezing parameter $\xi = gt$. The output state at time t will be the two-mode squeezed number state

$$|\xi\rangle = \exp(-\xi\hat{a}\hat{b} + \xi^* \hat{b}^\dagger \hat{a}^\dagger)|N+q, N\rangle.$$

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The properties of this state are phase dependent and it should be interesting to study them.

The problem of the quantum description of the optical field phase has been the subject of considerable study for many years [5]. This is connected with the difficulty in constructing a Hermitian phase operator. Within the past few years the notion of phase variables in quantum systems has been greatly clarified. Pegg and Barnett [6]–[8] have shown how such an operator can be defined for quantized electromagnetic fields. This new formalism makes it possible to describe the quantum properties of optical phase in a direct way within quantum mechanics on the basis of the Hermitian phase operator and its eigenstates.

A quite different approach to the concepts of the phase variable has also been widely used in quantum optics [9]–[11] and which involves quantum quasiprobability distributions such as the Q function and the Wigner function rather than Hermitian operators and their eigenstates. These quasiprobability distributions depend upon the complex eigenvalue α of the non-Hermitian annihilation operator, which can be expressed in terms of a radial variable $|\alpha|$ and a "phase" θ both of which are real. If we integrate over the radius, the resulting distributions are periodic in the phase angle and, for the most of states they satisfy all properties required by a proper phase distribution. In recent papers, the Pegg–Barnett phase distribution have been compared with those distributions obtained from the Wigner and Q functions by integrating them over the radius for the multi-photon down-conversion [12], displaced number states and displaced thermal states [13], squeezed number states and squeezed thermal states [14]. In this paper we extend such comparison onto two-mode case.

The purpose of this paper is to study photon statistics and phase properties of the two-mode squeezed number states which can be considered as a natural generalization of a two-mode squeezed vacuum state.

2 Photon number statistics

Consider two modes of the electromagnetic field, which have annihilation operators $\hat{a}$ and $\hat{b}$. A two-mode squeezed number state (TMSNS) is defined by acting with the squeeze operator $\hat{S}(r,\varphi)$ on the two-mode number state $|N+q,N\rangle$, that is

$$|N+q,N\rangle_{(r,\varphi)} = \hat{S}(r,\varphi)|N+q,N\rangle, \quad q \geq 0, \tag{1}$$

where q is the difference in the number of photons between two modes and

$$\hat{S}(r,\varphi) \equiv \exp[r(\hat{a}\hat{b}e^{-2i\varphi} - \hat{b}^{\dagger}\hat{a}^{\dagger}e^{2i\varphi})]. \tag{2}$$

In problems in which photons are either created in pairs or destroyed in pairs the value of q remains constant. Note that in many applications where pair creation occurs starting from vacuum, the parameter q will be zero. The number state decomposition of TMSNS can be written as

$$\begin{aligned} |N+q,N\rangle_{(r,\varphi)} &= \sum_n |n+q,n\rangle\langle n+q,n|N+q,N\rangle_{(r,\varphi)} = \\ &= \sum_n b_n e^{i\varphi_n}|n+q,n\rangle, \end{aligned} \tag{3}$$

where

$$b_n = \frac{(\tanh r)^{N+n}}{(\cosh r)^{1+q}} \left(N!(N+q)!n!(n+q)!\right)^{1/2} \times \sum_{k=0}^{\min(n,N)} \frac{(-1)^{n-k}(\sinh r)^{-2k}}{k!(n-k)!(N-k)!(q+k)!} \tag{4}$$

and

$$\varphi_n = (n-N)\varphi \tag{5}$$

with φ being a phase of squeezing. The above amplitude is obtained by using the factored form of the two-mode squeeze operator [15]

$$\hat{S}(r,\varphi) = (\cosh r)^{-1} \exp[-\hat{a}^\dagger \hat{b}^\dagger e^{2i\varphi} \tanh r] \times \exp[-(\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b}) \ln(\cosh r)] \exp[\hat{a}\hat{b} e^{-2i\varphi} \tanh r]. \tag{6}$$

The mean number of photons in the TMSNS is

$$\langle \hat{a}^\dagger a + \hat{b}^\dagger b \rangle = (2N + q + 1)\cosh 2r - 1. \tag{7}$$

The joint probability to find n_a photons in mode a and n_b photons in mode b is given by

$$P(n_a, n_b) = |\langle n_a, n_b | N+q, N \rangle_{(r,\varphi)}|^2. \tag{8}$$

Using (3) and (4), we get

$$P(n_a, n_b) = P(n+q, n)\delta_{n_a, n+q}\delta_{n_b, n}, \tag{9}$$

where

$$P(n+q, n) \equiv P_q(n) = |b_n|^2. \tag{10}$$

As we can see in Fig. 1, photon number distribution $P_q(n)$ has an oscillatory behaviour. Such a behaviour is a consequence of interference in four-dimensional phase space [16]. We would like to emphasize a presence of $(N+1)$ peaks in the photon number distribution. The similar behaviour of the photon number distribution was observed for the displaced number states [17]. It should be stressed that such a peak structure for TMSNS can be revealed only for those values of the parameter q greater than a certain number. This number depends on the value of N and for large N we ought to choose large values for such a number. Otherwise, some adjacent peaks in the photon number distribution might overlap and thus $(N+1)$-peak structure cannot be certainly discerned.

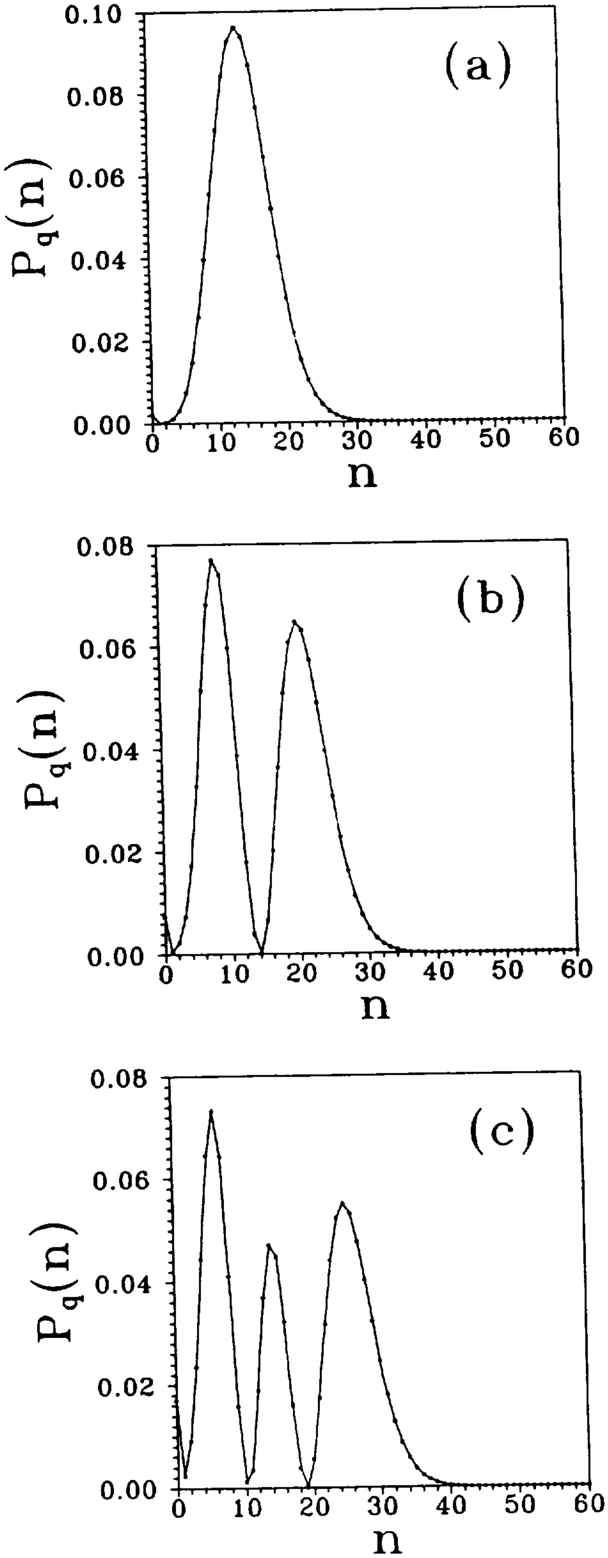

FIG. 1. Photon number distribution for the two-mode squeezed number state with $r = 0.5$, $q = 50$ and (a) $N = 0$, (b) $N = 1$, (c) $N = 2$.

3 Quasiprobability distributions

In this section we examine the representation of TMSNS by quasiprobability phase-space distributions. For convenience we choose the squeezing parameter to be real, $\xi = r$. The two-mode quasiprobability distributions are formed by a natural generalization of those for the single-mode fields [18]. The Glauber-Sudarshan $\mathcal{P}$ function, the Wigner function and the Q function are obtained by evaluating the Fourier transforms

$$V^{(s)}(\alpha,\beta) = \frac{1}{\pi^4}\int_{-\infty}^{\infty} d^2\eta d^2\xi \exp(\alpha\eta^* - \alpha^*\eta)\exp(\beta\xi^* - \beta^*\xi)C^{(s)}(\eta,\xi), \tag{11}$$

from the characteristic functions

$$C^{(s)}(\eta,\xi) = \exp\left(\frac{s}{2}(|\eta|^2 + |\xi|^2)\right) \mathrm{Tr}\left\{\hat{\rho}\exp(\eta\hat{a}^\dagger - \eta^*\hat{a})\exp(\xi\hat{b}^\dagger - \xi^*\hat{b})\right\} \tag{12}$$

where $s = 1$ if $V = \mathcal{P}$, $s = 0$ if $V = W$ and $s = -1$ if $V = Q$. We would like to notice that there no exists well-defined Glauber-Sudarshan $\mathcal{P}$ function for states under consideration owing to their nonclassical nature [18, 19].

According to ref. [20], the Q function can alternatively be defined as

$$Q(\alpha,\beta) = \frac{1}{\pi^2}\langle\alpha,\beta|\hat{\rho}|\alpha,\beta\rangle. \tag{13}$$

From this definition we see that the Q function is always non-negative. Using the definition of the density matrix of TMSNS

$$\hat{\rho} = \hat{S}(r)|N+q,N\rangle\langle N+q,N|\hat{S}^\dagger(r) \tag{14}$$

and the factored form of the squeeze operator (6), we obtain

$$\begin{aligned} Q(\alpha,\beta) &= \frac{|\alpha|^{2q}}{\pi^2(\cosh r)^{4N+2q+2}}\exp[-(\alpha\beta + \alpha^*\beta^*)\tanh r]\exp[-(|\alpha|^2 + |\beta|^2)] \\ &\times \sum_{n=0}^{N}\sum_{k=0}^{N}\frac{(\frac{1}{2}\sinh r)^{n+k}N!(N+q)!(\alpha^*\beta^*)^{N-n}(\alpha\beta)^{N-k}}{n!k!(N-n)!(N-k)!(N+q-n)!(N+q-k)!}. \end{aligned} \tag{15}$$

As to the Wigner function, it can also be represented as [20]

$$W(\alpha,\beta) = \frac{4}{\pi^2}Tr\left\{\hat{\rho}\hat{D}_a(2\alpha)\hat{D}_b(2\beta)\exp[i\pi(\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b})]\right\}. \tag{16}$$

where $D_a(\gamma)$ and $D_b(\gamma)$ are the displacement operators for modes a and b respectively. It is straightforward to evaluate the Wigner function using eq. (14) and the operator transformations [15]

$$\hat{S}^\dagger(r)\hat{a}\hat{S}(r) = \hat{a}\cosh r - \hat{b}^\dagger\sinh r, \tag{17}$$

$$\hat{S}^\dagger(r)\hat{b}\hat{S}(r) = \hat{b}\cosh r - \hat{a}^\dagger\sinh r. \tag{18}$$

and their Hermitian conjugates. We find a quite simple analytical form for the Wigner function

$$\begin{aligned} W(\alpha,\beta) &= \frac{4}{\pi^2}(-1)^q \exp[-2\cosh 2r(|\alpha|^2+|\beta|^2) - 2\sinh 2r(\alpha\beta+\alpha^*\beta^*)] \\ &\quad \times L_N\left((2\sinh r|\alpha|)^2 + (2\cosh r|\beta|)^2 + 2\sinh 2r(\alpha\beta+\alpha^*\beta^*)\right) \\ &\quad \times L_{N+q}\left((2\cosh r|\alpha|)^2 + (2\sinh r|\beta|)^2 + 2\sinh 2r(\alpha\beta+\alpha^*\beta^*)\right). \end{aligned} \tag{19}$$

where $L_n(x)$ is the Laguerre polynomial of order n. From eqs. (15) and (19) one can see that the Q function and the Wigner function depend on the sum of the phases $\theta_a+\theta_b$ only. This fact clearly exhibits the correlated nature of the two-mode squeezed number states. In the next section we will employ the quasiprobability functions in consideration of phase properties of these states.

4 Phase distributions

Now we employ the two-mode Pegg-Barnett phase formalism [21], [22] to find the phase distribution function for such states. This formalism is based on the observation that the Hermitian phase operator can be defined in a finite-dimensional state space, spanned by the number states. The main idea of the Pegg-Barnett formalism is to evaluate all necessary expectation values on this finite-dimensional state space, and only after that the dimension of the space is allowed to tend to infinity. Having the number state decomposition (3) of TMSNS we can determine the continuous joint phase probability distribution for the continuous phase variables θ_a and θ_b, which is given by

$$P(\theta_a,\theta_b) = \frac{1}{(2\pi)^2}\left\{1 + 2\sum_{n>k}^{\infty} b_n b_k \cos[(n-k)(\theta_a+\theta_b)]\right\}, \tag{20}$$

where b_n are given by Eq. (4). The distribution (20) is normalized such that

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} P(\theta_a,\theta_b)d\theta_a d\theta_b = 1. \tag{21}$$

One important phase property of TMSNS is seen directly from the form of formula (20). It is clear that the joint probability distribution depends on the sum of the two phases only

$$P(\theta_a,\theta_b) = P(\theta_+ = \theta_a + \theta_b). \tag{22}$$

This means the strong correlations of the two modes. Integrating $P(\theta_a,\theta_b)$ over one of the phases gives a marginal phase distribution $P(\theta_a)$ or $P(\theta_b)$ for the phases θ_a or θ_b, which are uniformly distributed

$$P(\theta_a) = \int_{-\pi}^{\pi} P(\theta_a,\theta_b)d\theta_b = \frac{1}{2\pi}. \tag{23}$$

$$P(\theta_b) = P(\theta_a) = \frac{1}{2\pi}. \tag{24}$$

Thus the phases θ_a or θ_b of the individual modes are uniformly distributed, and the only nonuniformly distributed phase quantity is the phase sum $\theta_+ = \theta_a + \theta_b$. In Fig. 2 we plot the Pegg–Barnett phase distribution for TMSNS in polar coordinates for different values of parameter q. For nonzero values of q the phase distribution shows $(N+1)$–lobe structure, and the greater q the more distinct lobes become. However, when $q = 0$ the phase distribution has only one lobe for all N. It is important to notice a remarkable resemblance in a behaviour of the phase distribution and the photon number distribution for TMSNS: they both display the $(N+1)$–peak structure. Another significant feature of the joint phase distribution is a property of the phase locking — the phase sum is locked to the argument of the squeezing parameter in the limit of large squeezing [21, 23].

Now consider phase quasiprobability distributions which can be obtained by integrating quasiprobability distribution functions (11) over the radial variables [9], [10]. As we have noticed above, $\mathcal{P}$ function is not well–defined function for the states under consideration and therefore it is impossible to determine corresponding phase quasiprobability distribution. As a result of integration of $Q(\alpha,\beta)$ and $W(\alpha,\beta)$ over $|\alpha|$ and $|\beta|$, we arrive to the following formula:

$$P^{(V)}(\theta_+) = \frac{1}{(2\pi)^2}\left\{1 + 2\sum_{n>k} b_n b_k \cos[(n-k)\theta_+] G^{(V)}(n,k) G^{(V)}(n+q,k+q)\right\}, \tag{25}$$

where the coefficients $G^{(V)}(n,k)$ distinguish between two distributions, and they are:

(i) for the Q function

$$G^{(Q)}(n,k) = \frac{\Gamma[(n+k)/2+1]}{\sqrt{n!k!}}, \tag{26}$$

(ii) for the Wigner function

$$\begin{aligned} G^{(W)}(n,k) &= \sum_{m=0}^{\lambda} (-1)^{\lambda-m} 2^{(|n-k|+2m)/2} \\ &\quad \times \sqrt{\binom{\lambda}{m}\binom{\nu}{\lambda-m}}\, G^{(Q)}(m,|n-k|+m), \end{aligned} \tag{27}$$

where $\lambda = \min(n,k), \nu = \max(n,k)$. All the coefficients $G^{(V)}(n,k)$ are symmetrical, $G^{(V)}(n,k) = G^{(V)}(k,n)$, and $G^{(V)}(n,n) = 1$. Note, that such expressions (20), (25) for the phase distributions are valid for all two-mode states with the number state decomposition like in (3). In Fig. 3, we show the plots of the three phase distributions in polar coordinates for TMSNS calculated according to formulas (20) and (25) with the coefficients (26) and (27) for different values of N and nonzero q. It is seen that the Pegg–Barnett phase distribution and $P^{(W)}(\theta_+)$ are similar and have the $N+1$ lobes, while $P^{(Q)}(\theta_+)$ is much broader and has only one lobe. In the case $q = 0$ all three distributions have the same form of one lobe. So, as in the case of displaced number states [13], there is an essential difference in the phase information carried by $P^{(Q)}(\theta_+)$ and $P^{(W)}(\theta_+)$. Because of the averaging procedure with the "probabilities" $G^{(Q)}(n,k)G^{(Q)}(n+q,k+q)$

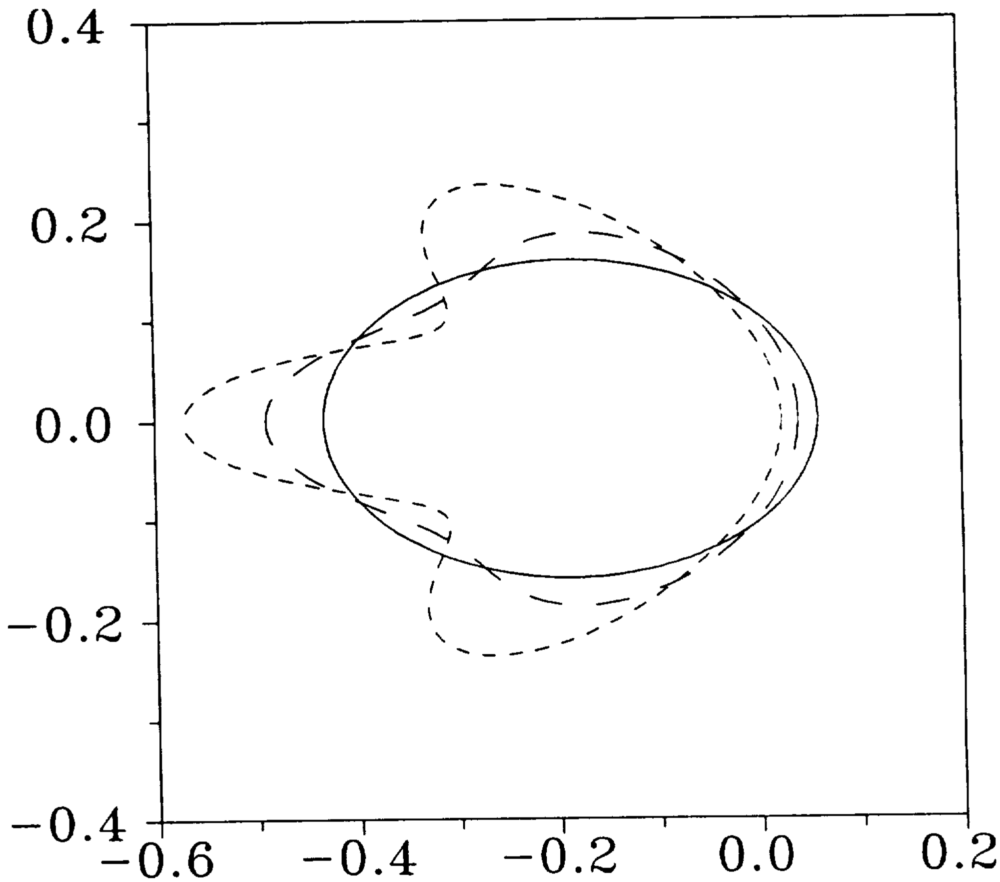

FIG. 2. Phase distribution $P^{(\mathrm{PB})}(\theta_+)$ for the two-mode squeezed number state with $r = 0.5$, $N = 2$ and $q = 0$ (solid line), $q = 3$ (long-dashed line) and $q = 6$ (short-dashed line).

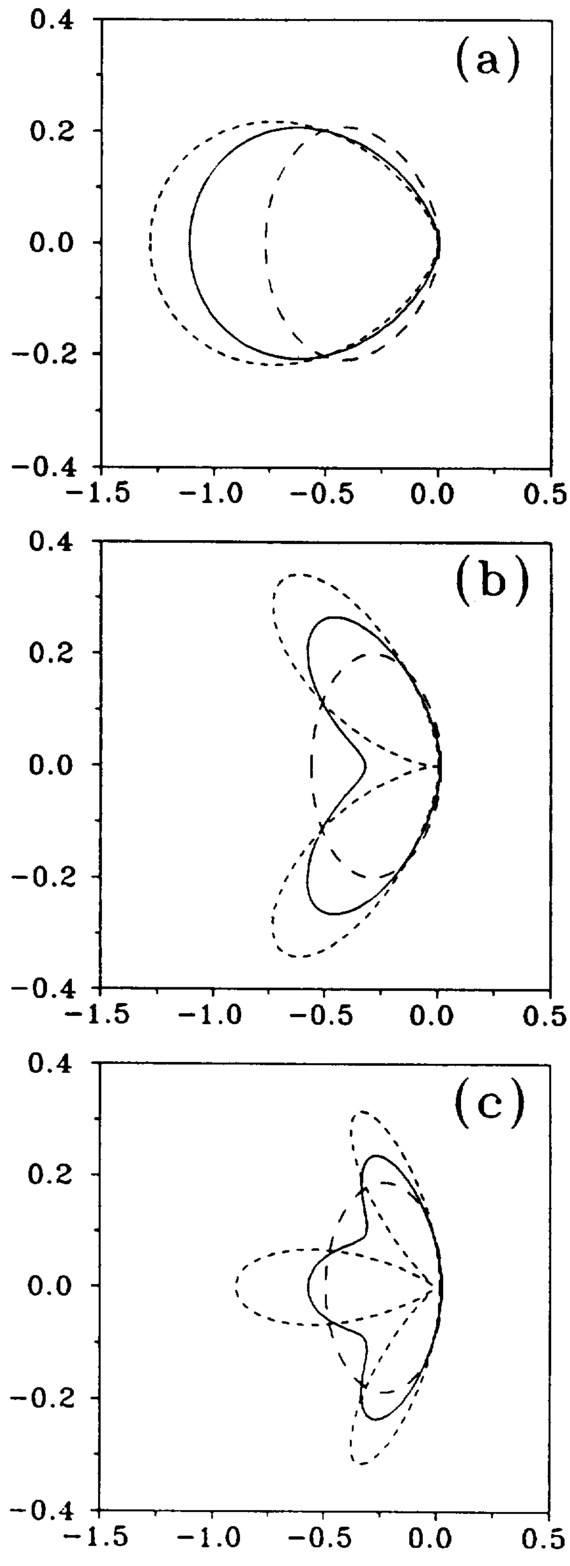

FIG. 3. Phase distribution $P^{(\mathrm{PB})}(\theta_+)$ (solid line) and phase quasiprobability distributions $P^{(W)}(\theta_+)$ (short-dashed line) and $P^{(Q)}(\theta_+)$ (long-dashed line) for the two-mode squeezed number state with $r = 0.5$, $q = 6$ and (a) $N = 0$, (b) $N = 1$, (c) $N = 2$.

some phase information is lost in $P^{(Q)}(\theta_+)$. The Pegg–Barnett phase distribution is very close to the distribution $P^{(W)}(\theta_+)$, although it is not identical to it. The phase peaks of $P^{(W)}(\theta_+)$ are slightly narrower than those of $P^{(PB)}(\theta_+)$. The greater the difference in number of photons q the closer these two distributions. Basically they carry the same phase information. This similarity is in agreement with the area of overlap in phase space arguments, which are that the Wigner function represents quantum states in the phase space [10]. However, the Wigner function can take on negative values and the positive definiteness of $P^{(W)}(\theta_+)$ is not automatically guaranteed, while there are no such problems with the Pegg–Barnett phase distribution.

5 Conclusions

We have discussed photon statistics and phase properties of the two-mode squeezed number states showing that the photon number distribution and the Pegg–Barnett phase distribution for such states exhibit the similar $N+1$-peak structure for nonzero values of the difference in the number of photons q between modes. We have compared the Pegg–Barnett phase distribution with the phase quasiprobability distributions $P^{(Q)}(\theta_+)$ and $P^{(W)}(\theta_+)$ obtained by integrating the Q function and the Wigner function over the radial coordinates. We have shown that the Pegg–Barnett phase distribution and the distribution $P^{(W)}(\theta_+)$ carry basically the same phase information, while the distribution $P^{(Q)}(\theta_+)$ loses an essential part of the phase information.

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