Higher-order squeezing of the quantum electromagnetic field and the generalized uncertainty relations in two-mode squeezed states

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It is found that the two-mode output quantum electromagnetic field in two-mode squeezed states exhibits higher-order squeezing to all even orders. And the generalized uncertainty relations are also presented for the first time.

The concept of higher-order squeezing of the single-mode quantum electromagnetic field was first introduced and applied to several processes by Hong and Mandel in 1985. Lately Li Xizeng and Shan Ying have calculated the higher-order squeezing in the process of degenerate four-wave mixing and presented the higher-order uncertainty relations of the fields in single-mode squeezed states. In this paper we generalize the above work to the higher-order squeezing in two-mode squeezed states. The generalized uncertainty relations are also presented for the first time.

1 Definition of higher-order squeezing in two-mode squeezed states

The definition of two-mode squeezed states was given by Caves and Schumaker:

\[ |\alpha_+, \alpha_-; \zeta \rangle = \hat{S}(\zeta)|\alpha_+, \alpha_- \rangle. \]  

(1)

Where \( \hat{S}(\zeta) \) is the two-mode squeezed operator

\[ \hat{S}(\zeta) = \exp\left[\frac{1}{2}(\zeta^* \hat{a}_+ \hat{a}_- - \zeta \hat{a}_+^* \hat{a}_-^*)\right], \]  

(2)

\[ |\alpha_+, \alpha_- \rangle \] is the two-mode coherent state, \( \hat{a}_\pm \) are two-mode annihilation operators, \( \alpha_\pm \) are eigenvalues of \( \hat{a}_\pm \) in \( |\alpha_+, \alpha_- \rangle \).

Define the two-mode squeezed annihilation operators by \( \hat{A}_\pm \),

\[ \hat{A}_\pm = \hat{S}(\zeta) \hat{a}_\pm \hat{S}^+(\zeta) = \mu \hat{a}_\pm + \nu \hat{a}_\pm^+. \]  

(3)

where

\[ \mu = \cosh r, \quad \nu = e^{i\theta} \sinh r, \]  

(4)

\( \zeta = re^{i\theta} \) is the squeeze parameter.
Then the two-mode squeezed states are the eigenstates of $\hat{A}_\pm$,

$$\hat{A}_\pm |\alpha_+, \alpha_-; \gamma> = \alpha_\pm |\alpha_+, \alpha_-; \gamma>,$$

and $\alpha_\pm$ are the eigenvalues of $\hat{A}_\pm$.

The real two-mode output field $\hat{E}$ can be decomposed into two quadrature components $\hat{E}_1$ and $\hat{E}_2$, which are canonical conjugates. The output field $\hat{E}$ exhibits higher-order squeezing to any higher-order (Nth order) in $\hat{E}_1$, if there exists such a phase angle $\phi$ that the higher-order moment $< (\Delta \hat{E}_1)^N >$ in a two-mode squeezed state is smaller than its value in a completely two-mode coherent state, viz.,

$$< (\Delta \hat{E}_1)^N >_{\text{two-mode}} < < (\Delta \hat{E}_1)^N >_{\text{coherent}}.$$

This is the definition of higher-order squeezing in two-mode squeezed states.

2 The quadrature components of the two-mode output field $\hat{E}$

The electric field operator for the two-mode output field has the form of

$$\hat{E}(x, t) = \hat{E}^{(+)}(x, t) + \hat{E}^{(-)}(x, t).$$

Where

$$\begin{align*}
\hat{E}^{(+)}(x, t) &= \sqrt{\frac{\omega_+}{2}} \hat{a}_+ e^{-i\Omega (t-x)} + \sqrt{\frac{\omega_-}{2}} \hat{a}_- e^{-i\Omega (t-x)}, \\
\hat{E}^{(-)}(x, t) &= \sqrt{\frac{\omega_+}{2}} \hat{a}_+ e^{i\Omega (t-x)} + \sqrt{\frac{\omega_-}{2}} \hat{a}_- e^{i\Omega (t-x)}.
\end{align*}$$

We now introduce two Hermitian quadrature components $\hat{E}_1$ and $\hat{E}_2$ of the electric field defined by

$$\begin{align*}
\hat{E}_1(x, t) &= \hat{E}^{(+)} e^{i[\Omega (t-x) - \phi]} + \hat{E}^{(-)} e^{-i[\Omega (t-x) - \phi]}, \\
\hat{E}_2(x, t) &= \hat{E}^{(+)} e^{i[\Omega (t-x) - (\phi + \xi)]} + \hat{E}^{(-)} e^{-i[\Omega (t-x) - (\phi + \xi)]}.
\end{align*}$$

Then, $\hat{E}(x, t)$ can be decomposed into two quadrature components $\hat{E}_1$ and $\hat{E}_2$, which are canonical conjugates

$$\hat{E}(x, t) = \hat{E}_1 \cos [\Omega (t-x) - \phi] + \hat{E}_2 \sin [\Omega (t-x) - \phi],$$

$$[\hat{E}_1, \hat{E}_2] = 2iC_0,$$

Where $\Omega$ is the carrier frequency

$$\Omega = \frac{\omega_+ + \omega_-}{2},$$

and $\phi$ is an arbitrary phase angle that may be chosen at will.
The units are chosen so that $\hbar = c = 1$.
Substituting Eqs. (7) and (8) into (9), we obtain
\[ \hat{E}_1(x,t) = g_+ a_+ + g_- a_- + g_+^* a_+^* + g_-^* a_-^*. \] (12)
where
\[ g_\pm = \sqrt{\frac{\omega \pm \epsilon}{2}} e^{-i(\omega \pm \epsilon t - x)}, \] (13)
and
\[ \epsilon = \omega_+ - \Omega = \Omega - \omega_- \] (14)
is the modulation frequency.
From Eq. (3), we get
\[ \hat{a}_\pm = \mu^* \hat{A}_\pm - \nu \hat{A}_\mp^*. \] (15)
Substituting (15) to (12), we obtain $\hat{E}_1$ in terms of $\hat{A}_\pm$
\[ \hat{E}_1(x,t) = (h_+ \hat{A}_+ + h_- \hat{A}_-) + (h_+^* \hat{A}_+^* + h_-^* \hat{A}_-^*). \] (16)
Where
\[ h_\pm = g_\pm \mu^* - g_\mp^* \nu^*. \] (17)
Define
\[ \hat{B} = h_+ \hat{A}_+ + h_- \hat{A}_-, \] (18)
Then
\[ \hat{E}_1 = \hat{B} + \hat{B}^+. \] (19)

3 Higher-order noise moment $\langle (\Delta \hat{E}_1)^N \rangle$ and Higher-order squeezing

By using the Campbell–Baker–Hausdorff formula, we get
\[ \langle (\Delta \hat{E}_1)^N \rangle = \langle (\hat{E}_1 - \hat{B})^N \rangle = \sum_{\gamma=0}^{N} \frac{N!}{\gamma!(N-\gamma)!} \langle (\hat{E}_1)^{N-\gamma} \rangle \langle (\hat{B})^\gamma \rangle \] (20)
where $N^{(r)} = N(N-1) \cdots (N-r+1)$, $C_0 = \frac{1}{2} [\hat{E}_1, \hat{E}_2] = [\hat{B}, \hat{B}^+]$, " ::= " denotes normal ordering with respect to $\hat{B}$ and $\hat{B}^+$.
Now we take the two-mode squeezed states, then
\[ \langle (\Delta \hat{E}_1)^N \rangle = \langle (\Delta \hat{B}^+)\gamma(\Delta \hat{B})^{N-\gamma} \rangle = 0, \] (21)
and

\[ C_0 = [\hat{B}, \hat{B}^+] = |a_1|^2 + |a_2|^2 = (|g_+|^2 + |g_-|^2)(|\mu|^2 + |\nu|^2) - \Omega \sqrt{1 - \frac{\epsilon^2}{\Omega^2}(\mu^* \nu e^{-2i\phi} + \nu^* \mu e^{2i\phi})}. \] (22)

From (20), (4) and (13), we get

\[ < (\Delta \hat{E}_1)^N > = (N - 1)!!\Omega^{N/2}[\cosh(2r) - \sqrt{1 - \frac{\epsilon^2}{\Omega^2}\sinh(2r)\cos(\theta - 2\phi)}]^{N/2}. \] (23)

If \( \phi \) is chosen to satisfy \( \cos(\theta - 2\phi) = 1 \), then Eq(23) leads to the result

\[ < (\Delta \hat{E}_1)^N > = (N - 1)!!\Omega^{N/2}[\cosh(2r) - \sqrt{1 - \frac{\epsilon^2}{\Omega^2}\sinh(2r)}]^{N/2}. \] (24)

when \( \cosh r < \frac{\Omega}{\epsilon} \), the right hand side is smaller than \( (N - 1)!!\Omega^{N/2} \), which is the corresponding Nth order moment for two-mode coherent states.

It follows that the two-mode output field exhibits higher-order squeezing to all even orders.

4 The generalized uncertainty relations

[A]. Higher-order noise moment \( < (\Delta \hat{E}_2)^N > \)

\( \hat{E}_2 \) can be regarded as a special case of \( \hat{E}_1 \), in which if \( \phi \) is replaced by \( \phi + \pi/2 \), then from (23) it follows that

\[ < (\Delta \hat{E}_2)^N > = (N - 1)!!\Omega^{N/2}[\cosh(2r) + \sqrt{1 - \frac{\epsilon^2}{\Omega^2}\sinh(2r)\cos(\theta - 2\phi)}]^{N/2}. \] (25)

If \( \phi \) is chosen to satisfy \( \cos(\theta - 2\phi) = 1 \), then

\[ < (\Delta \hat{E}_2)^N > = (N - 1)!!\Omega^{N/2}[\cosh(2r) + \sqrt{1 - \frac{\epsilon^2}{\Omega^2}\sinh(2r)}]^{N/2}. \] (26)

When \( \cosh r < \frac{\Omega}{\epsilon} \), the right hand side is greater than \( (N - 1)!!\Omega^{N/2} \).

[B]. Generalized uncertainty relations

From (24) and (26), we obtain

\[ < (\Delta \hat{E}_1)^N > \cdot < (\Delta \hat{E}_2)^N > = [(N - 1)!!\Omega^N + \frac{\epsilon^2}{\Omega^2}\sinh^2(2r)]^N. \] (27)
Equation (27) shows that $(\Delta \hat{E}_1)^N$ and $(\Delta \hat{E}_2)^N$ in two-mode squeezed states cannot be made arbitrarily small simultaneously. We call Eq.(27) the generalized uncertainty relations in two-mode squeezed states, and the right hand side (constant) is dependent on $N, \epsilon, \Omega,$ and $r.$

Since

$$1 + \frac{\epsilon^2}{\Omega^2} \sinh^2(2r) > 1$$

so

$$(\Delta \hat{E}_1)^N > (\Delta \hat{E}_2)^N \geq (N - 1)!! \Omega^N. \quad (28)$$

If $r = 0,$ the two-mode squeezed states become two-mode coherent states, then

$$(\Delta \hat{E}_1)^N > (\Delta \hat{E}_2)^N \geq [(N - 1)!!] \Omega^N. \quad (29)$$

This is the generalized uncertainty relations in two-mode coherent states.

If $\epsilon = 0, N = 2,$ we obtain

$$(\Delta \hat{E}_1)^2 \cdot (\Delta \hat{E}_2)^2 = \Omega^2. \quad (30)$$

This is just the usual Heisenberg uncertainty relations in relevant references$^{1,3,4,6}.$

5 Application

As an application of the above result, we calculate the generation of higher-order squeezing by non-degenerate four-wave mixing (NDFWM). It can be shown that the field of the combined mode of the probe wave and the phase-conjugate wave exhibits higher-order squeezing to all even orders, and the generalized uncertainty relations still hold in NDFWM process.

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References

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