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A NEW APPROACH TO THE PHOTON LOCALIZATION PROBLEM

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Abstract

Since wavelets form a representation of the Poincaré group, it is possible to construct a localized superposition of light waves with different frequencies in a Lorentz-covariant manner. This localized wavelet satisfies a Lorentz-invariant uncertainty relation, and also the Lorentz-invariant Parseval's relation. A quantitative analysis is given for the difference between photons and localized waves. It is then shown that this localized entity corresponds to a relativistic photon with a sharply defined momentum in the non-localization limit. Waves are not particles. It is confirmed that the wave-particle duality is subject to the uncertainty principle.

1 Introduction

We propose a quantitative approach to the photon localization problem. Photons are relativistic particles requiring a covariant theoretical description. Classical optics based on the conventional Fourier superposition is not covariant under Lorentz transformations [1]. On the other hand, wavelets can be regarded as representations of the Lorentz group [2, 3, 4]. In Ref. [1], we discussed the difference between waves and wavelets without mentioning the word "wavelet." We have seen there that the lack of covariance of light waves is due to the lack of Lorentz invariance of the integral measure. We concluded there that an extra multiplicative factor is needed to make the Fourier optics covariant. We would like to point out that this procedure corresponds to the wavelet formalism of wave optics.

In spite of the covariance of wavelets, photons are not wavelets. Instead, we shall make a quantitative analysis of the difference between these two clearly defined physical concepts. The advantage of this quantitative approach is that we can see how close they are to each other. In this way, we can assert that photons are waves with a proper qualification.

Another convenient feature of the localized wavelet representation is that it is possible to introduce an cut-off procedure in a covariant manner, so as to preserve the information given in the distribution. By introducing the concept and word "window" [5, 6, 7, 8], it is possible to define the region in which the frequency distribution is non–zero. We can then compare the "windowed" wavelet to the photon operators in quantum field theory to pinpoint the difference between the photons and wavelets.

2 Localized Light Wavelets

For light waves, we start with the usual expression

$$F(z,t) = \frac{1}{\sqrt{2\pi}} \int g(k) e^{i(kz-\omega t)} dk \ . \tag{1}$$

Unlike the case of the Schrödinger wave, ω is equal to k, and there is no spread of wave packet. The velocity of propagation is always that of light. We might therefore be led to think that the problem for light waves is simpler than that for nonrelativistic Schrödinger waves. This is not the case. In Ref.[1], we have considered the following questions.

(1). We would like to have a wave function for light waves. However, it is not clear which component of the Maxwell wave should be identified with the quantal wave whose absolute square gives a probability distribution. Should this be the electric or magnetic field, or should it be the four–potential?

(2). The expression given in Eq.(1) is valid in a given Lorentz frame. What form does this equation take for an observer in a different frame?

(3). Even if we are able to construct localized light waves, does this solve the photon localization problem?

(4). The photon has spin 1 either parallel or antiparallel to its momentum. The photon also has gauge degrees of freedom. How are these related to the above-mentioned problems?

Even though light waves do not satisfy the Schrödinger equation, the very concept of the superposition principle was derived from the behavior of light waves. Furthermore, it was reconfirmed recently that light waves satisfy the superposition principle [9]. It is not difficult to carry out a spectral analysis on Eq.(1) and give a probability interpretation. The question then is whether this probability interpretation is covariant. We addressed this question in Ref. [1]. We concluded in effect that the localized light wave is covariant if we use the wavelet form. We regret however that we were not aware of the word "wavelet" at that time. In this section, we shall translate what we did in Ref. [1] into the language of wavelets.

The expression given in Eq.(1) is not covariant if $g(k)$ is a scalar function, because the measure dk is not invariant. If $g(k)$ is not a scalar function, what is its transformation property? We shall approach this problem using the light-cone coordinate system. We define the light-cone variables as

$$s = (z+t)/2, \qquad u = (z-t) \ . \tag{2}$$

The Fourier-conjugate momentum variables are

$$k_s = (k-\omega) \ , \qquad k_u = (k+\omega)/2 \ . \tag{3}$$

If we boost the light wave (or move against the wave with velocity parameter β), the new coordinate variables become

$$s' = \alpha_+ s \ , \quad u' = \alpha_- u \ , \quad k'_s = \alpha_- k_s \quad k'_u = \alpha_+ k_u \ , \tag{4}$$

where

$$\alpha_{\pm} = [(1 \pm \beta)/(1 \mp \beta)]^{1/2} \ . \tag{5}$$

If we construct a phase space consisting of s and k_s or u and k_u, the effect of the Lorentz boost will simply be the elongation and contraction of the coordinate axes. If the coordinate s is elongated by α_+, then k_s is contracted by α_- with $\alpha_+\alpha_- = 1$.

In the case of light waves, k_s vanishes, and k_u becomes k or ω. In terms of the light-cone variables, the expression of Eq.(1) becomes

$$F(u) = \left(\frac{1}{2\pi}\right)^{1/2} \int g(k) e^{iku} dk \ . \tag{6}$$

We are interested in a unitary transformation of the above expression into another Lorentz frame. In order that the norm

$$\int |g(k)|^2 dk \tag{7}$$

be Lorentz-invariant, $F(u)$ and $g(k)$ should be transformed like

$$F(u) \rightarrow \sqrt{\alpha_+} F(\alpha_+ u) \ , \qquad g(k) \rightarrow \sqrt{\alpha_-} g(\alpha_- k) \ . \tag{8}$$

Then Parseval's relation:

$$\int |F(u)|^2 du = \int |g(k)|^2 dk \tag{9}$$

will remain Lorentz-invariant.

It is not difficult to understand why u in Eq.(2) and $k = k_u$ in Eq.(3) are multiplied by α_+ and α_- respectively. However, we still have to give a physical reason for the existence of the multipliers $(\alpha_{\pm})^{1/2}$ in front of $F(u)$ and $g(k)$. In Ref. [10], Kim and Wigner pointed out that the multipliers in Eq.(8) come from the requirement that the Wigner phase–space distribution function be covariant under Lorentz transformations [11, 7].

Let us illustrate this point using a Gaussian form. We can consider the g(k) function of the form

$$g(k) = \left(\frac{1}{\pi b}\right)^{1/4} \exp\left\{-(k-p)^2/2b\right\} \ , \tag{10}$$

which leads to the $F(u)$ of the form

$$F(u) = \left(\frac{b}{\pi}\right)^{1/4} \exp\left(bu^2/2\right) \exp\left(ipu\right) \ , \tag{11}$$

where b is a constant and specifies the width of the distribution, and p is the average momentum:

$$p = \frac{\int k|g(k)|^2 dk}{\int |g(k)|^2 dk} \ . \tag{12}$$

Under the Lorentz boost according to Eq.(8), $g(k)$ becomes

$$\left(\frac{1}{\pi b}\right)^{1/4} \sqrt{\alpha_-} \exp\left\{-\alpha_-(k - \alpha_+ p)^2/2b\right\} \ . \tag{13}$$

Thus, according to the transformation law given in Eq.(8), the transformed F(u) becomes

$$F(u) = \left(\frac{b}{\pi}\right)^{1/4} \sqrt{\alpha_+} \exp\left(b(\alpha_+ u)^2/2\right) \exp\left(i(\alpha_+ p)u\right) . \tag{14}$$

We note here that the average momentum p is now increased to $\alpha_+ p$. The average momentum therefore is a covariant quantity, and α_- can therefore be written as

$$\alpha_- = \sigma/p , \tag{15}$$

where σ is the average momentum in the Lorentz frame in which $\alpha_\pm = 1$, and $\beta = 0$.

As a consequence, in order to maintain the covariance, we can replace $F(u)$ and $g(k)$ by $F'(u)$ and $g'(k)$ respectively, where

$$F'(u) = \sqrt{\frac{p}{\sigma}} F(u) , \qquad g'(k) = \sqrt{\frac{\sigma}{p}} g(k) . \tag{16}$$

These functions will satisfy Parseval's equation:

$$\int |F'(u)|^2 du = \int |g'(k)|^2 dk \tag{17}$$

in every Lorentz frame without the burden of carrying the multipliers $\sqrt{\alpha_+}$ and $\sqrt{\alpha_-}$. We can simplify the above cumbersome procedure by introducing the form

$$G(u) = \frac{1}{\sqrt{2\pi p}} \int g(k) e^{iku} dk . \tag{18}$$

where the procedure for the Lorentz boost is to replace p by $\alpha_+ p$, and k in $g(k)$ by $\alpha_- k$. This is precisely the wavelet form for the localized light wave, and this definition is consistent with the form given in earlier papers on wavelets [2, 3].

3 Windows

There are in physics many distributions, and we have a tendency to choose "smooth" or analytic functions to describe them. These functional forms usually extend from minus infinity to plus infinity. However, the distribution function of physical interest is usually concentrated within a finite interval. On the other hand, it is not uncommon in physics that mathematical difficulties in theory come from the region in which the distribution function is almost zero and is physically insignificant. Thus, we are tempted to ignore contributions from outside of the specified region. This is called the "cut-off" procedure.

One of the difficulties of this procedure is that a good cut-off approximation in one Lorentz frame may not remain good in different frames. The translational symmetry of wavelets allows us to define the cut-off procedure which will remain valid in all Lorentz frames.

We can allow the function to be nonzero within the interval

$$a \leq x \leq a + w , \tag{19}$$

while demanding that the function vanish everywhere else. The parameter w determines the size of the window. The window can be translated or expanded/contracted according to the operation of the affine group [8]. Indeed, it is possible to define the window in such a way that the boundary condition be covariant, and the information contained in the window be preserved [8].

4 Photon Localization Problem

The wavelet formalism allows the description of a localized wave function for light waves in a Lorentz covariant manner with Lorentz-invariant normalization. We shall now examine how close the wavelet can be to photons. First of all, it should be noted that there are no physical laws which dictate the functional form for $a(k)$ or $g(k)$. This depends on initial conditions. If we choose an analytic function, this is purely for mathematical convenience. If we choose functions which vanish near $k = 0$ and k greater than a certain value, this also satisfies our criterion for mathematical convenience. Indeed, the concept of window plays a decisive role in this form of localization.

In quantum electrodynamics, we start with the form

$$A(z,t) = \int \frac{1}{\sqrt{2\pi\omega}} a(k) e^{i(kz-\omega t)} dk \ . \tag{20}$$

This is a covariant expression in the sense that the norm

$$\int \frac{|a(k)|^2}{2\pi\omega} dk \tag{21}$$

is invariant under Lorentz transformations, because the integral measure $(1/\omega)dk$ is Lorentz-invariant.

We are quite familiar with the expression of Eq.(1) for wave optics, and with that of Eq.(20) for quantum electrodynamics. The wavelet form of Eq.(18) satisfies the same superposition principle as Eq.(1), and has the same covariance property as Eq.(20). It is quite similar to both Eq.(1) and Eq.(20), but they are not the same. The difference between $F(u)$ of Eq.(1) and the wavelet $G(u)$ is insignificant. Other than the factor $\sqrt{\sigma}$ where σ has the dimension of the energy, the wavelet $G(u)$ has the same property as $F(u)$ in every Lorentz frame [1]. However, the difference between $G(u)$ and $A(u)$ is still significant.

It is possible to give a particle interpretation to Eq.(20) after second quantization. However, $A(z,t)$ cannot be used for the localization of photons. On the other hand, it is possible to give a localized probability interpretation to $F(z,t)$ of Eq.(1), while it does not accept the particle interpretation of quantum field theory.

$A(u)$ of Eq.(20) and $G(u)$ of Eq.(18) are numerically equal if

$$a(k) = \sqrt{\frac{k}{p}} g(k) \ , \tag{22}$$

where the window is defined over a finite interval of k which does not include the point $k = 0$. It is thus possible to jump from the wavelet $G(u)$ to the photon field $A(u)$ using the above equation.

However, the above equality does not say that $a(k)$ is equal to $g(k)$. The photon intensity distribution is not directly translated into the photon-number distribution. This is the quantitative difference between wavelets and photons.

Of course, this difference becomes insignificant when the window becomes narrow. The narrower window in k means a wider distribution of the wave in the u coordinate system. We conclude therefore that photons become waves in non-localization limit. Particles are not waves. The wave-particle duality is subject to the uncertainty principle. The relation given in Eq.(22), together with the appropriate window, is a statement of this uncertainty relation.

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