PHOTON STATISTICS
OF A TWO-MODE SQUEEZED VACUUM

G. Schrade. V. M. Akulin [1], W. P. Schleich [2]
Abt. für Quantenphysik, Universität Ulm
D-89069 Ulm, Germany

V. I. Man'ko [3]
Dipartimento di Scienze Fisiche
Universita di Napoli Federico II
80125 Napoli, Italy

Abstract

We investigate the general case of the photon distribution of a two-mode squeezed vacuum and show that the distribution of photons among the two modes depends on four parameters: two squeezing parameters, the relative phase between the two oscillators and their spatial orientation. The distribution of the total number of photons depends only on the two squeezing parameters. We derive analytical expressions and present pictures for both distributions.

1 Introduction

Squeezing the quantum fluctuations of the radiation field has been demonstrated experimentally using various optical systems [1]. Most of them rely on two-mode squeezing [2, 3, 4]. Therefore the properties of a two-mode squeezed state have been studied extensively [5, 6]. However it appears, that the photon statistics of such a state has not been investigated in all details. Only some particular cases have been considered (see for example Refs. [7] and [8]). In the present paper we therefore extend these considerations to an arbitrary two-mode squeezed vacuum and address the following questions: (i) What is the most general case of the squeezed vacuum of a two-mode oscillator, and how many independent parameters are needed to describe this state? (ii) What is the photon statistics in this state? This is of interest in the context of the degenerate parametric amplifier [9] since the mathematical structure of the two-mode distribution function coincides with the transition probability function of a two-dimensional parametric oscillator [10].

---

[1] also Moscow Institute of Physics and Technology Dolgoprudny, Moscow, Russia

[2] also Max-Planck-Institut für Quantenoptik, D-85748 Garching, Germany

[3] permanent address: Lebedev Institute of the Academy of Sciences, Moscow, Russia

2 general case of the squeezed vacuum of a two-mode oscillator

We start our considerations with the question (i). The linear canonical transformation

$$(a^\dagger, a; b^\dagger, b) \longrightarrow (A^\dagger, A; B^\dagger, B) \tag{1}$$

of the creation operators $a^\dagger$, $b^\dagger$ and the annihilation operators a, b to their new counterparts $A^\dagger$, $B^\dagger$ and A, B corresponding to the two modes suggests a set of 10 parameters. Indeed, the generic transformation (1) of the four-dimensional vector $(a^\dagger, a, b^\dagger, b)$ into $(A^\dagger, A, B^\dagger, B)$ via a 4×4 matrix brings in 16 complex values, that is 32 parameters. Since a and $a^\dagger$, b and $b^\dagger$, A and $A^\dagger$, B and $B^\dagger$ are hermitian conjugate of each other, only half of these parameters are independent. Therefore the number of parameters reduces to 16. The condition to preserve the commutation relation $[A, B] = 0$, $[A, B^\dagger] = 0$, $[A, A^\dagger] = 1$, and $[B, B^\dagger] = 1$ provides additional constraints. Since the commutator of an operator with its hermitian conjugate is always real, the last two conditions provide only two constraints, whereas the first two conditions must hold for the real and the imaginary part separately. This decreases the number of parameters by six leaving us indeed with 10 parameters.

This finding is in accordance with the results of group theory. Reference [11] shows, that the group of rotations and squeezing of the four-dimensional phase space, that preserves the phase space volume of the two degrees of freedom, consists of 10 generators. Hence there are 10 parameters determining the elements of the real symplectic group Sp(4,R).

But what is the physical meaning of these parameters? They are associated with rotation and squeezing transformation of phase space of two oscillators. Generic rotations of four-dimensional phase space are described by 6 parameters but the symplectic rotations in four-dimensional space, that is the transformation preserving the commutation relations, are described by 4 parameters. They can be represented as a sequence of four rotations. The first rotation given by the transformation $e^{i\hat{M}\phi_1}$, where $\hat{M} = i(a^\dagger b - b^\dagger a)$ is the angular momentum operator, corresponds to the rotation in the coordinate space by the angle ϕ_1. The second and the third rotations are given by the operator $e^{i\psi a^\dagger a}$ and $e^{i\chi b^\dagger b}$, and correspond to rotations by the angles ψ and χ in the respective phase-spaces of the two oscillators. The last rotation again can be taken in the form $e^{i\hat{M}\phi_2}$. Thus the angles ϕ_1, ϕ_2, ψ and χ are chosen to be the parameters of the symplectic rotation. The general symplectic transformation also includes squeezing. We can represent the generic real symplectic transformation (1) as consisting of three consecutive transformations: symplectic rotation (4 parameters), followed by independent squeezing of the two modes given by the squeezing operators

$$\hat{S}_1 = e^{r_1(a^2 - a^{\dagger 2})/2}, \qquad \hat{S}_2 = e^{r_2(b^2 - b^{\dagger 2})/2}, \tag{2}$$

(2 squeezing parameters r_1 and r_2) and followed by another real symplectic rotation (4 parameters more). But how many of these parameters govern the two-dimensional squeezed vacuum state? The first four parametric rotation acting on the completely symmetric vacuum state leaves this state unchanged. Hence 2 squeezing parameters and 4 parameters associated with the second rotations define the two-dimensional squeezed vacuum state. In addition we can include an overall quantum phase factor $e^{i\rho}$ of this state. An explicit calculation of the general case of a two-mode squeezed vacuum wave function is given in Ref. [12].

3 Photon statistics for the total number of photons

We now address the question: how many of these parameters govern the photon statistics of such a two-dimensional squeezed vacuum state? In particular, how many of them determine the probability of (i) counting a total number n of photons in the two modes, and (ii) counting the number n_1 and n_2 of photons in the individual modes?

In this section we consider the case of (i) and ask for the total number n of photons in both modes, which corresponds to the surface of a four-dimensional sphere, $\frac{1}{2}(p_1^2 + x_1^2 + p_2^2 + x_2^2) = n$, centered at the origin of the four-dimensional phase space. The rotation of the phase space does not alter this sphere and hence only two squeezing parameters are essential. The probability of counting the total number n of photons for the case of two independently squeezed oscillators reads

$$W_n(s_1, s_2) = \sum_{n_1=0}^{n} W_{n_1}(s_1) W_{n-n_1}(s_2), \tag{3}$$

where $W_{n_j}(s_j)$ is the one-dimensional photon statistics [13]

$$W_{n_j}(s_j) = \begin{cases} 0 & \text{for } n_j \text{ odd} \\ \sqrt{1 - s_j}\, s_j^{n_j/2} 2^{-n_j} \binom{n_j}{n_j/2} & \text{for } n_j \text{ even} \end{cases} \tag{4}$$

of the squeezed vacuum wave function

$$\Psi_{sq}(x_j) = \frac{e^{r_j/2}}{\pi^{1/4}} \exp\left(-\frac{1}{2}\, e^{2r_j} x_j^2\right) \tag{5}$$

where

$$s_j = \tanh^2(r_j). \tag{6}$$

Equation (4) reduces Eq. (3) to

$$W_n(s_1, s_2) = \begin{cases} W_{2k}(s_1, s_2) = \mathcal{N} 4^{-k} \sum_{n_1=0}^{k} s_1^{n_1} s_2^{k-n_1} \binom{2n_1}{n_1} \binom{2(k-n_1)}{k-n_1} \\ W_{2k+1}(s_1, s_2) = 0 \end{cases} \tag{7}$$

where the normalization factor $\mathcal{N}$ reads

$$\mathcal{N} = \sqrt{1 - s_1} \sqrt{1 - s_2}. \tag{8}$$

The odd terms of W_n vanish because of Eq. (4) . The sum in Eq. (7) has been calculated in [12] and the probability of counting $n = 2k$ photons then reads

$$W_{2k}(s_1, s_2) = \mathcal{N} s_2^k\, {}_2F_1(-k, 1/2, 1; 1 - s_1/s_2) \tag{9}$$

where ${}_2F_1$ describes the hypergeometric function. For the special case of identical squeezing in the two modes, that is, $s_1 = s_2 = s$, Eq. (9) yields

$$W_{2k}(s) = \mathcal{N} s^k. \tag{10}$$

In Fig. 1 we show the photon statistics (9) for various magnitudes of the squeezing parameters s_1 and s_2. The solid and dashed curves correspond to weak and strong symmetric squeezing ($s_1 = s_2$), respectively. In accordance with Eq. (10) the photon distribution then displays an exponential dependence. Stronger squeezing results in a higher amount of quanta involved. The dotted curve shows the photon statistic for an asymmetric squeezing.
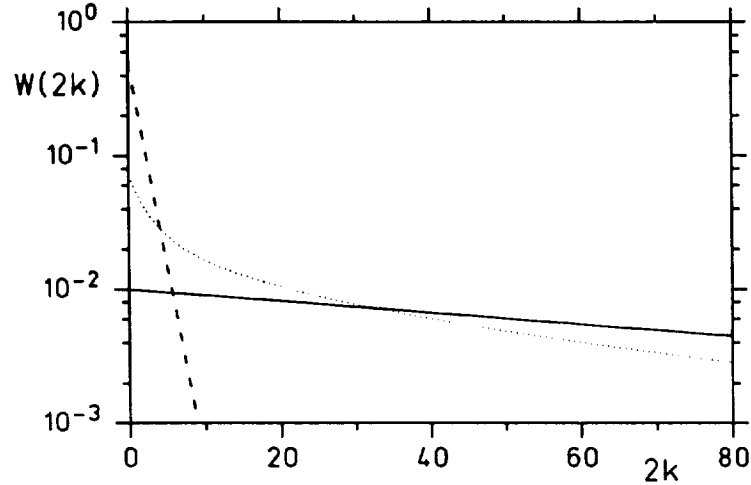

Fig. 1. Probability of counting $n = 2k$ photons in a two-mode squeezed vacuum as given by Eq. (9). For symmetric squeezing (dashed line for $s_1 = s_2 = 0.5$ and solid line for $s_1 = s_2 = 0.99$) the curve is a straight line and hence exponential whereas for asymmetric squeezing (dotted line for $s_1 = 0.5$ and $s_2 = 0.99$) the photon statistics is non-exponential. Here we have not specified the distribution of the $n = 2k$ photons among the two modes. Note that $W_{n=2k+1} = 0$ which we have omitted for simplicity.

4 Photon statistics in the individual modes

We now turn to the second case and calculate the photon statistics $W(n_1, n_2) = | < n_1, n_2 | \Psi_{sq} > |^2$ in the individual modes. For this purpose we start from the generic expression of a squeezed vacuum state $|\Psi_{sq} >$, which we produce in the following way:

$$\Psi_{sq}(x_1, x_2) \quad = e^{i\rho} \, e^{-i\phi \hat{M}} \, \hat{\Gamma}(\gamma) \, \hat{S}_1(r_1) \, \hat{S}_2(r_2) \, \Psi_{0,0}(x_1, x_2)$$

$$= \sqrt{\frac{2}{\pi}} \, (A_1 B_1 - C_1^2)^{1/4} \, e^{-A x_1^2 - B x_2^2 + 2C x_1 x_2 + i\rho} \tag{11}$$

with the vacuum wave function of the two-dimensional oscillator $\Psi_{0,0}(x_1, x_2) = \Psi_0(x_1) \Psi_0(x_2)$, the angular momentum operator $\hat{M} = i(a^\dagger b - b^\dagger a)$ and the operator $\hat{\Gamma}(\gamma) = \exp(i\gamma(b^\dagger b - a^\dagger a))$ which describes a mutual phase shift 2γ between the oscillators. The wave function of a two-dimensional squeezed vacuum state is a Gaussian described by the three complex numbers A, B, and C, which are functions of the four real parameters r_1, r_2, ϕ, and γ. Their explicit dependence is given in [12]. One also finds there the generic two-mode squeezed vacuum wave function depending on two more parameters that do not affect the photon distribution.

We contract Eq. (11) with the probability amplitude of the photon energy states $\psi_{n_1}(x_1)$ and $\psi_{n_2}(x_2)$ and arrive after calculating the resulting double integral [12] at

$$W(n_1, n_2) = \frac{8 \exp(-|\ln(n_1!/n_2!)|)}{|(2A+1)(2B+1) - 4C^2|} (A_1 B_1 - C_1^2)^{1/2} \times$$

$$\times \left| \frac{4AB + 2A - 2B - 1 - 4C^2}{4AB - 2A + 2B - 1 - 4C^2} \right|^{(n_1 - n_2)/2} \times$$

$$\times \left| \frac{4AB - 2A - 2B + 1 - 4C^2}{4AB + 2A + 2B + 1 - 4C^2} \right|^{(n_1 + n_2)/2} \times$$

$$\times \left| P_{|n_1 + n_2|/2}^{|n_1 - n_2|/2} \left(\frac{-4C}{\sqrt{4AB + 2A + 2B + 1 - 4C^2}\sqrt{4C^2 - 4AB + 2A + 2B - 1}} \right) \right|^2 .$$

Here P_l^k denotes the associated Legendre Polynomial. As in Eq. (7) the total number of photons $n = n_1 + n_2$ must be an even number hence n_1 and n_2 are either both even or both odd. Otherwise the photon distribution function vanishes.

In Fig. 2 we display the probability to find n_1 photons in mode 1 and n_2 photons in mode 2. This is the generic case of the photon distribution of a two-mode squeezed vacuum. It depends on four parameters: two squeezing parameters, the orientation of the distribution function with respect to our laboratory system, and the correlation between the two modes. Beside the even-odd oscillations the maxima lie on curves which are symmetric with respect to the main diagonal $n_1 = n_2$. This behavior is similar to the distribution function of a displaced two-mode squeezed state discussed in Ref. [7, 8].
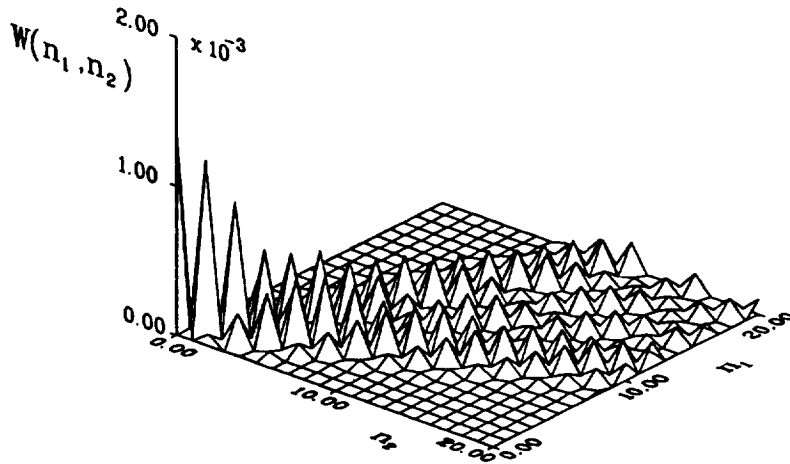

Fig. 2. Probability of counting n_1 and n_2 photons in the two-mode squeezed vacuum calculated via Eq. (12). We have chosen $r_1 = 3$, $r_2 = 5$, $\delta = \frac{\pi}{5}$, and $\gamma = \frac{2\pi}{9}$. The wavy structure of this distribution is confined to an angle in the photon number plane.

5 Conclusion

We conclude by summarizing our main results. The wave function of the two-mode squeezed vacuum depends on 6 parameters (besides the phase factor). However only 4 of them, – the squeezing parameters r_1 and r_2 of the two modes, the phase difference γ between the two oscillators, and the rotation of the reference system by the angle ϕ – manifest themselves in the generic case in the distribution of the photons among the two modes. This distribution function can be expressed explicitly in terms of Legendre polynomials. Only two parameters r_1 and r_2 govern the distribution of the total number of photons, which we express explicitly in terms of a hypergeometric function. In conclusion we want to make the remark that similar consideration for N-mode squeezed vacuum state shows that the photon distribution $W(n_1, n_2, ..., n_N)$ depends on N^2 parameters.

6 Acknowledgments

References

[1] For a review on squeezed states see, for example, special issues on squeezed states in J. Opt. Soc. Am. B **4**, No. 10 (1987); and Appl. Phys. B **55**, No. 3 (1992).

[2] C. M. Caves and B.L. Schumaker, Phys. Rev. A **31**, 3068 (1985).

[3] S. M. Barnett and P. L. Knight, J. Opt. Soc. Am. B **2**, 467 (1985).

[4] B. Yurke, S. McCall, and J. R. Klauder, Phys. Rev. A **33**, 4033 (1986).

[5] B. L. Schumaker, Phys. Rep. **135**, 317 (1986).

[6] A. K. Ekert and P. L. Knight, Am. J. Phys. **57**, 692 (1989).

[7] C. M. Caves, C. Zhu, G. J. Milburn, and W. Schleich, Phys. Rev. A **43**, 3854 (1991).

[8] M. Artoni, U. P. Ortiz, and J.L. Birman, Phys. Rev. A **43**, 3954 (1991).

[9] For the photon statistics of a parametric oscillator see for example R. Vyas and S. Singh, Phys. Rev. A **40**, 5147 (1989); and Opt. Lett. **14**, No. 20, 1110 (1989);

[10] I. A. Malkin, V.I. Man'ko, and D. A. Trifonov, J. Math. Phys. **14**, 576 (1973); V. V. Dodonov and V. I. Man'ko in *Invariants and the evolution of nonstationary quantum systems*, ed. by M. A. Markov, (Nova Science Publ., New York, 1989).

[11] G. J. Milburn, J. Phys. A **17**, 737 (1984).

[12] G. Schrade, V. M. Akulin, V. I. Man'ko, and W. P. Schleich, Phys. Rev. A **48**, 2398 (1993).

[13] W. Schleich and J. A. Wheeler, Nature **326**, 574 (1987).

SECTION 2

QUANTUM MEASUREMENT THEORY