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Joint and Angle-covariant Spin Measurements with a Quadrupole Magnetic Field

Hans Martens and Willem M. de Muynck
Theoretical Physics Group, Department of Physics, Eindhoven University of Technology, PO Box 513, 5600 MB Eindhoven, The Netherlands.

Abstract

We study a Stern-Gerlach type setup, with a quadrupole magnetic field, for neutral particles of arbitrary spin. The Hamiltonian is of a form proposed for joint measurements of incompatible observables. The measurement results are discussed, showing the limitation of such Hamiltonians. Some remarks are made on the relevance of covariance as a criterion for measurement schemes.

1 Introduction

The canonical form for the Hamiltonian of a measurement interaction is (operators are careted)

$$\hat{H}_I = \kappa \hat{a}'_p \hat{A}_o \; , \tag{1}$$

where the subscripts o and p denote object and pointer, respectively. The operator $\hat{A}_o$ is measured, whereas $\hat{a}'_p$ is conjugate to the read-out observable $\hat{a}_p$. One of the simplest measurement arrangements, often used as an example [1, 2, 3], is the Stern-Gerlach, where an inhomogeneous magnetic field effects an interaction amounting to [2, 4]

$$\hat{H}_I = -\kappa \hat{x} \hat{\sigma}_x \; , \tag{2}$$

Here the coupling constant κ is proportional to the strength of the inhomogeneity. The spin degrees of freedom represent the object, and the spatial degrees of freedom are the "measuring device". Thus we measure the spin in the x-direction, by reading out the x-momentum component. If we assume both $\hat{p}_x$ and $\hat{\sigma}_x$ to be conserved in the absence of interactions, the Hamiltonian (2) causes in the Heisenberg picture ($\hbar = 1$)

$$\hat{p}_x(t) = \hat{p}_x(0) + \kappa \hat{\sigma}_x t \; . \tag{3}$$

Thus, as soon as t is larger than the $\hat{p}_x$-width, the spatial part is separated into s packets, according to $\hat{\sigma}_x$ eigenvalue. Conversely, a read-out of $\hat{p}_x$ means an accurate measurement of $\hat{\sigma}_x$.

In order to be able to measure two incompatible observables jointly, Arthurs & Kelly [5] extended the basic scheme (1) to

$$\hat{H}_I = \kappa_A \hat{a}'_p \hat{A}_o + \kappa_B \hat{b}'_p \hat{B}_o \; . \tag{4}$$

In this interaction Hamiltonian, $\hat{a}'_p$ and $\hat{b}'_p$ are two compatible pointer system operators, with read-outs $\hat{a}_p$ and $\hat{b}_p$, respectively. The operators $\hat{A}_o$ and $\hat{B}_o$, not necessarily compatible, are to be measured. Arthurs & Kelly applied (4) to joint position-momentum measurements. They found that the probabilities of finding a result (a, b) are given by

$$\mathrm{prob}(a,b) = \frac{1}{\pi} Tr\Big(\hat{\rho}_o |\alpha\rangle\langle\alpha|\Big) \ , \ \alpha = a + ib \ , \tag{5}$$

where $|\alpha\rangle$ is a coherent or squeezed state, depending on the interaction balance κ_A/κ_B. Therefore the a marginal is related to the x probability distribution by a convolution

$$\mathrm{prob}(a) = \int_{-\infty}^{\infty} f(a-x)\ \langle x|\hat{\rho}_o|x\rangle\ \mathrm{d}x \tag{6}$$

with a Gaussian f, and analogously for b and p. This relation, providing the basis for the interpretation of b in terms of x, we have termed "non-ideality" elsewhere [6]. Sets of operators such as $\frac{1}{\pi}|\alpha\rangle\langle\alpha|$, which generate probability distributions but are not orthogonal, are called positive operator-valued measures (POVMs) [7].

If we combine (4) with (2), we get a Hamiltonian like

$$\hat{H}_I = -\kappa_x \hat{x}\hat{\sigma}_x + \kappa_y \hat{y}\hat{\sigma}_y \ . \tag{7}$$

Such a Hamiltonian may be realized using a quadrupole magnetic field, which around the origin in the x, y-plane satisfies

$$\vec{A} \propto (0, 0, xy) \Rightarrow \vec{B} \propto (x, -y, 0) \ . \tag{8}$$

In the present paper we will investigate the properties of a measurement scheme based on the quadrupole Stern-Gerlach (7), where $\kappa = \kappa_x = \kappa_y$. Neutral particles of arbitrary spin s will be used. We shall focus especially on its relation to (6) and (5).

2 General description

If we consider the quadrupole Hamiltonian (7), we see that its rotational symmetry immediately implies that $\hat{J}_z = \hat{L}_z - \hat{\sigma}_z$ is conserved. It is therefore profitable to change into a polar momentum representation (p, φ). Denote the eigenvalues of $\hat{\sigma}_z$ by m_z. Now we can eliminate φ by writing the m_z component of the state $|\Psi\rangle$ as

$$\langle m_z, p, \varphi|\Psi\rangle = \exp[i(m_{\bar{J}} + m_z)\varphi]\ \phi_{m_z}(p) \ . \tag{9}$$

The Hamiltonian, seen as an operator on the $(2s+1)$-dimensional spin Hilbert space, then becomes

$$\hat{H} = p^2 - i\frac{\partial}{\partial p}\hat{\sigma}_x - \hat{\sigma}_y \frac{m_{\bar{J}} + \hat{\sigma}_z}{p} \ . \tag{10}$$

We have taken $\kappa = 2m = 1$, without loss of generality [8]. Note that $\hat{\sigma}_y\hat{\sigma}_z$ is not Hermitian, but then neither is $i\frac{\partial}{\partial p}$; the overall expression (10) is Hermitian.

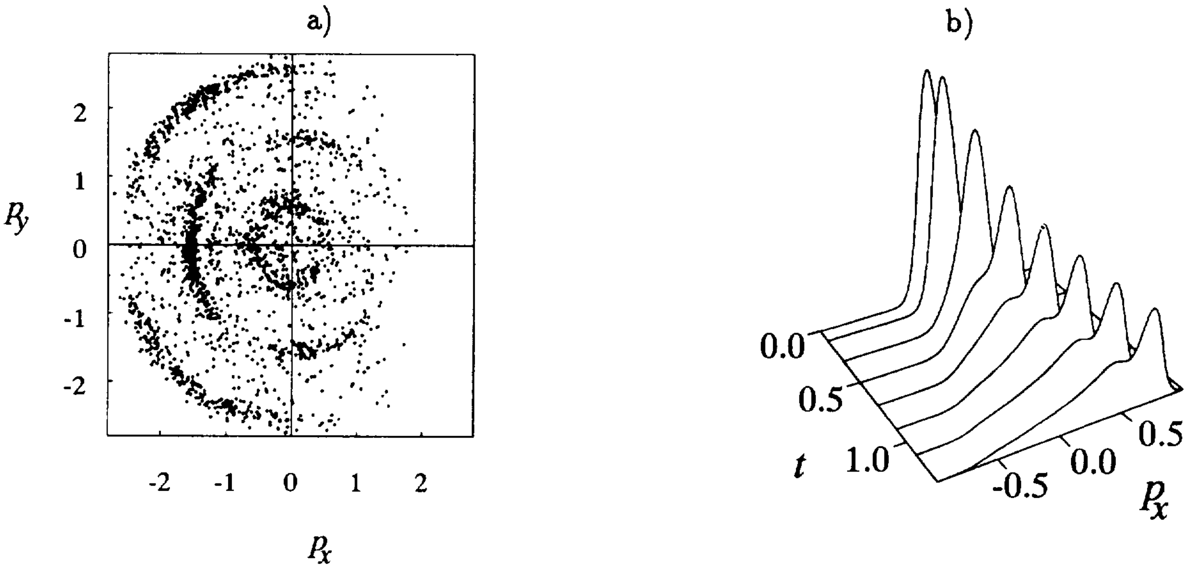

FIG. 1. Two views on the measurement results. (The Gaussian spatial part initially had variance $\Delta p^2 = 0.01$ and $L_z = 0$.)

a. Typical output distribution. The density of markers indicates the momentum probability density at $t = 1$. The input state was a $s = \frac{5}{2}$ state, with $m_x = -\frac{3}{2}$.

b. Matrix element $\langle \frac{1}{2}|\hat{M}(p_x)|\frac{1}{2}\rangle$ for $s = \frac{1}{2}$ as a function of time. The quality of the measurement can be characterized by the integral of the matrix element over $p_x > 0$. Note that this value does not approach 1.

We integrated eq. (10) numerically. In view of the fact that the process is intended as a spin measurement, we took the initial state to be a product of spatial and spin parts (denoted by subscripts r and σ when necessary),

$$|\Psi\rangle = |\phi\rangle_r \otimes |\varsigma\rangle_\sigma \,. \tag{11}$$

The spatial part $|\phi\rangle_r$ was taken to be a axially symmetric Gaussian. The final state then turned out to be structured into a number of expanding rings, one for each $|m|$; only $m = 0$ (for integral spin particles) leads to a hump remaining around the origin. Fig. 1.a is a typical example, where $s = \frac{5}{2}$, and we have 3 rings. In the figure the spin part was initially directed in the x-direction, with $m_x = -\frac{3}{2}$, and thus we see that the distribution is peaked in the left part of the middle ring. More generally, states with spin initially directed in the xy-plane lead to correspondingly oriented distributions; only spins initially in the z-direction evolve into complete ring distributions. In the rings the spins are directed roughly outward, with appropriate magnitude.

The scheme is meant to be seen as a spin measurement. Accordingly, we trace out the spatial variables and generate the outcome probabilities from the spin density operator $\hat{\rho}_\sigma$ by means of a POVM $\hat{M}(p,\varphi)$ on spin Hilbert space,

$$\begin{aligned} \mathrm{prob}_{\hat{\rho}_\sigma}(p,\varphi) &= Tr[\hat{\rho}_\sigma\, \hat{M}(p,\varphi)] \,; \\ \hat{M}(p,\varphi) &= {}_r\langle\phi|\; \hat{U}^\dagger(t)\Big(|p,\varphi\rangle_r\langle p,\varphi| \otimes \hat{1}_\sigma\Big)\hat{U}(t)\; |\phi\rangle_r \,;\; \hat{U}(t) = \exp(-i\hat{H}t) \,. \end{aligned} \tag{12}$$

In the next two sections we will discuss some properties of the POVM $\hat{M}(p,\varphi)$.

3 Marginals

The output variables of the measurement are (p_x, p_y). Naively, one would suspect on the basis of (3) and (7) that a read-out of $\hat{p}_x$ can be seen as a measurement of $\hat{\sigma}_x$. Accordingly, we are interested in the p_x marginal of $\hat{M}$,

$$\hat{M}(p_x) = \int_{-\infty}^{\infty} \hat{M}(p_x, p_y)\, \mathrm{d}p_y \,. \tag{13}$$

Consider first the $s = \frac{1}{2}$ case. Here the spin operators anti-commute. Thus the interaction Hamiltonian (7) satisfies

$$\hat{S}\hat{H}_I\hat{S}^\dagger = \hat{H}_I \; ; \; \hat{S} = \hat{I}_y \otimes (2\hat{\sigma}_x) \,. \tag{14}$$

Hence, if the initial spatial state is $\hat{I}_y$-symmetric, the p_x marginal of the final (p_x, p_y) distribution must depend only on $\hat{\sigma}_x$. Accordingly we may write (13) as

$$\hat{M}(p_x) = \sum_{m=+\frac{1}{2},-\frac{1}{2}} f_m(p_x)\hat{E}_m(0) \,, \tag{15}$$

where the f_m are positive functions and $\hat{E}_m(0)$ denote the projectors onto the $\hat{\sigma}_x$ eigenstates. Thus we get an analog of the convolution (6), reproducing the basic result (3) with additional noise terms that do not depend on the spin degrees of freedom [4]. As is easily verified, an analogous relation holds between $\hat{p}_y$ and $\hat{\sigma}_y$. In fig. 1.b we plotted the probability that a particle with $m_x = +\frac{1}{2}$ will give the measurement result p_x. We see that the noise terms assure that the measurement quality is limited, in contrast to (3). This is necessary on account of the uncertainty principle [9, 10], as we are jointly measuring the two incompatible observables $\hat{\sigma}_x$ and $\hat{\sigma}_y$.

Nevertheless, (15) means that an unambiguous relation between the $\hat{p_x}$ and $\hat{\sigma}_x$ distributions exists. It allows for the m_x estimation from p_x that we aimed at [11], albeit an imperfect one. In the spin-$\frac{1}{2}$ case, the spin observables are Fourier-pairs [12] and therefore close analogs of the position-momentum pair studied by Arthurs & Kelly [5]. Accordingly, the above conclusion matches theirs.

We might think that for $s > \frac{1}{2}$ this result may be generalized. In fig. 2.a the diagonal elements of the POVM (13) in $\hat{\sigma}_x$ representation are plotted vs. p_x. We see that indeed the various m_x values are roughly correlated to different p_x regions, as expected. Thus, it appears that from p_x an estimate of m_x can be made, just as for $s = \frac{1}{2}$. But there is a catch: neglecting the p^2-term (strong or impulsive interaction approximation), it follows after some calculation that

$$\begin{aligned} \hat{p}_x(t) = \hat{p}_x(0) \quad &+ \quad \hat{\sigma}_x(0)[t\cos^2\theta + \frac{\sin rt}{r}\sin^2\theta] \\ &+ \quad \hat{\sigma}_y(0)\frac{1}{2}[\frac{\sin rt}{r} - t]\sin 2\theta + \hat{\sigma}_z(0)\frac{\cos rt - 1}{r}\sin\theta \,; \end{aligned} \tag{16}$$

where we used the polar position representation $(x, y) = (r\cos\theta, r\sin\theta)$. Clearly $\hat{p}_x$ contains extra spin terms that do not commute with $\hat{\sigma}_x$. Indeed the non-diagonal elements of $\hat{M}(p_x)$ are not zero, as is evidenced by fig. 2.b. Consider again (16). The desired effect, $t\hat{\sigma}_x$, is contained in the second term , whereas the last two terms are the problematic ones. As the rings expand more and more, the $\frac{1}{r}$ terms will decay roughly as $1/t^2$. Still, even as $t \to \infty$, a significant term containing $\hat{\sigma}_y$ remains: $\hat{M}$ and $\hat{\sigma}_x$ are incompatible.

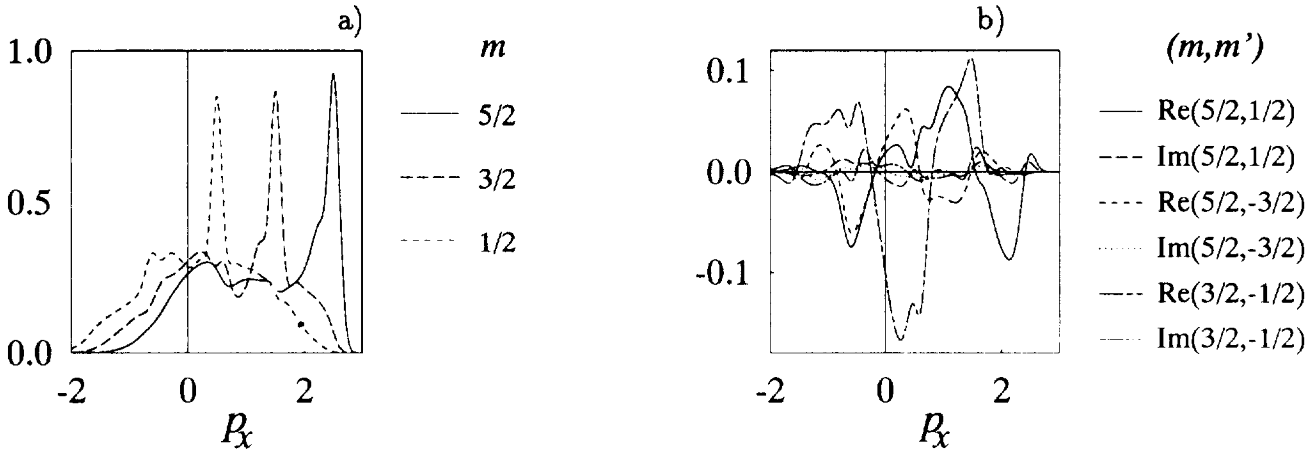

FIG. 2. The POVM $\hat{M}(p_x)$ in $\hat{\sigma}_x$ representation. (Data as in fig. 1.)
a. Diagonal matrix elements $\langle m|\hat{M}(p_x)|m\rangle = \langle -m|\hat{M}(-p_x)|-m\rangle$.
b. Independent non-zero non-diagonal matrix elements $\langle m|\hat{M}(-p_x)|m'\rangle$.

But all the problematic terms in (16) are seen to contain a factor $\sin 2\theta$ or $\sin\theta$. Denote reflection in the xz-plane by $\hat{I}_y$. The sine terms in $\langle \hat{p}_x\rangle$ vanish if we choose the spatial part of the initial state $\hat{I}_y$-symmetric, like we do in our calculations. But if we consider $\langle \hat{p}_x^2\rangle$, terms with factors like $\hat{\sigma}_y^2 \sin^2 2\theta$ emerge. Such terms do not vanish, so that higher moments do not commute with $\hat{\sigma}_x$. Only if $s = \frac{1}{2}$, we have $\hat{\sigma}_y^2 \propto \hat{1}$ so that $\langle \hat{p}_x^2\rangle$ commutes with $\hat{\sigma}_x$ after all, in accordance with our earlier result (15).

But in what sense can we consider this scheme to be a measurement of $\hat{\sigma}_x$ if $s > \frac{1}{2}$? Clearly no analog of (15) holds; only the expectation value $\langle \hat{p}_x\rangle$ is free of incompatible spin terms, so that we can use the measurement only to estimate the $\hat{\sigma}_x$ expectation value. Higher moments, or even m_x probabilities, cannot be established. Thus it is a $\hat{\sigma}_x$ measurement only in the weak sense of expectation value estimation [11].

4 Covariance

The measurement is therefore not generally a useful joint measurement. But it may have another use. Remember, the Hamiltonian (7) is rotationally symmetric. If we now consider the outcomes in polar coordinates, it is easily derived that the POVM $\hat{M}(p,\varphi)$ is angle covariant:

$$\hat{M}(p,\varphi+\Delta\varphi) = \hat{R}(\Delta\varphi)\hat{M}(p,\varphi)\hat{R}^\dagger(\Delta\varphi)\ ;\ \hat{R}(\Delta\varphi) = \exp(i\Delta\varphi\hat{\sigma}_z)\ . \tag{17}$$

Covariance is a criterion that is often used to characterize classes of measurements e.g. time or photon phase measurements. Here we therefore speculate that $\hat{M}(p,\varphi)$ realizes some kind of spin-angle measurement. Define the projector onto the eigenstate of the operator $\cos\theta\hat{\sigma}_x + \sin\theta\hat{\sigma}_y$ with eigenvalue m as $\hat{E}_m(\theta)$. Then we choose as spin-angle observable [13]

$$\sum_m c_m \int_{-\pi}^{\pi} \hat{E}_m(\theta)\ \mathrm{d}\theta = \hat{1}\ , \tag{18}$$

where the summation runs over all relevant $|m|$ values, and $c_m = \frac{1}{2\pi}$ for $m = 0$ and $\frac{1}{\pi}$ otherwise. Accordingly, $c_m \hat{E}_m(\theta)$ defines a POVM that may be seen as a "spin-angle" observable [7]. We can now attempt to link (18) to (17). As in the previous section, this is possible for $s = \frac{1}{2}$. There a convolution-type relation between the realized angle measurement and the ideal (18) holds [4]. But, as in the previous section, this cannot be generalized to higher spins: $\hat{M}$ and 18 are incompatible, although the incompatibility is generally smaller than that of fig. 2.b. Thus, although the POVM is angle-covariant, it is not clear in what sense the "angle of spin" is measured, if at all. Analogously we may therefore conclude that photon-phase covariance and time covariance must also give many POVMs without unambiguous interpretation. Like spin-angle covariance, they are weak criterions.

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[13] The Fourier transform of $\hat{\sigma}_z$ is another candidate, but it fares even worse than (18) for the present (low) spin values: the incompatibility with $\hat{M}$ is much larger.