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PHASE–SPACE ANALYSIS OF CHARGED AND OPTICAL BEAM TRANSPORT: WIGNER ROTATION ANGLE

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Abstract

The possibility of using the phase space formalism to establish a correspondence between the dynamical behaviour of squeezed states and optical or charged beams, propagating through linear systems, has received a great deal of attention during the last years. In this connection, it has been indicated how optical experiments may be conceived to measure the Wigner rotation angle. In this paper we address the topic within the context of the paraxial propagation of optical or charged beams and suggest a possible experiment for measuring the Wigner angle using an electron beam passing through quadrupoles and drift sections. The analogous optical system is also discussed.

1 Introduction

Lorentz group is the basic language of special relativity [1]. It has been recognized as a powerful tool in many other fields of modern physics as well. Many dynamical symmetry groups, underlying specific branches of physics, as quantum optics, classical and quantum mechanics, Hamiltonian optics, are locally isomorphic to the group $SO(2,1)$ of Lorentz transformations in two–space and one–time dimensions.

Actually, the (3 + 1)–dimensional Lorentz group $SO(3,1)$ is the full space–time symmetry group. However, we seldom discuss Lorentz transformations in the three–dimensional coordinate system, since computing for instance velocity additions and successive Lorentz boosts is quite complicated, 4×4 matrices being involved. Restricting thereby the problem from $SO(3,1)$ to $SO(2,1)$ may simplify significantly calculations.

Furthermore $SO(2,1)$ has both a physical and mathematical interest. It is indeed the little group, which leaves a space–like four–momentum invariant. Accordingly, studying $SO(2,1)$ amounts to studying free particles travelling faster than light (the so–called tachyons), which are intrinsically interesting from a theoretical point of view.

Above all, the group $SO(2,1)$ gave rise to a great amount of interest, since, as already remarked, it is locally isomorphic to other groups, as $Sp(2)$, $SU(1,1)$, $SL(2,C)$, and has therefore a very rich mathematical and physical content. The isomorphism with the above quoted groups offers many advantages from both analytical and theoretical point of view. In fact, it allows further simplification in the calculations involving Lorentz transformations, the groups $Sp(2)$, $SU(1,1)$, $SL(2,C)$ consisting of 2×2 matrices. In particular, the algebraic analogy with the symplectic group $Sp(2)$ offers a further advantage, as $Sp(2)$ consists of real matrices. Hence, it is possible to visualize Lorentz transformations in terms of transformations in a two–dimensional geometry, making a correspondence between Lorentz transformations in real space and linear canonical transformations in phase–space or, equivalently, between volume–preserving transformations and area–preserving transformations.

In addition, as firstly recognized by Han, Kim and Noz [2], the quoted isomorphism allows to conceive a kind of analog computer for testing the Lorentz group properties. In particular, as is well known, instead of rotations, pure Lorentz transformations do not form a subgroup. As a consequence, the product of two boosts along different directions is not a boost but a boost preceded or followed by a rotation. The angle of rotation is known as the Wigner angle and provides the kinematic basis for Thomas precession in atomic physics [1, 2].

Many suggestions have indeed proposed in order to perform optical experiments to observe the optical analog of the Wigner angle [3, 4].

In this connection, the paper is devoted at suggesting a possible experiment for measuring the Wigner angle within the context of electron beam transport [5]. The paraxial propagation of a charged particle along a magnetic channel is indeed governed by the symplectic symmetry. It can be therefore conceived an experiment involving electron beams for measuring the Wigner angle. The formal analogy between the propagation of charged beams and that of light beams through optical systems in the Gaussian approximation suggests to discuss the topic in full generality. However, the specific language used through the paper is that usually adopted in accelerator physics, whilst the symbology is just that of ray–optics.

The paper is organized as follows. Sec. 2 is devoted to a preliminary analysis of the problem, in order to introduce the formalism relevant to symplectic symmetry. In Sec. 3 the analogy between linear canonical transformations and optical systems is developed, thus leading to a specific design of the experiment for measuring the Wigner angle within the electron–beam optics, as illustrated in Sec. 4. Concluding remarks are given in Sec. 5.

2 $SO(2,1)$ and $Sp(2)$

As remarked in ref. [6], the *symplectic group*, originally introduced by Weyl in 1938, plays a central role in many branches of physics, as a consequence of that symplectic transformations preserve the skew symmetric products, which frequently appear in physics. In particular, symplectic geometry is the mathematical theory underlying Hamiltonian mechanics. It emerges especially in phase–space picture. The phase–space of a mechanical system is indeed recognized as a symplectic manifold and the time evolution of a conservative dynamical system is a one–parameter family of symplectic diffeomorphisms, or, linear canonical transformations.

Phase space formalism is becoming the unifying language for both classical and quantum mechanics.It is basic to the Hamiltonian formulation of classical mechanics. Within this context, indeed, the evolution of a dynamical system is described by a number n of independent coordinate variables and on the same number of canonically conjugate momenta. The cartesian space of these $2n$ coordinates is just the *phase–space*.

Correspondingly, phase space picture of quantum mechanics is becoming increasingly popular. Although, the concept of phase–space is not compatible with quantum mechanics, $\hat{q}$ and $\hat{p}$ being noncommuting operators, the Wigner phase–space representation allows to overcome this problem, since in this representation both the coordinate and momentum variables are c–numbers. Accordingly, it is possible to perform phase–space canonical transformations as in the case of classical mechanics, which correspond to unitary transformations in the Schrödinger picture of quantum mechanics.

Phase space concept appears therefore as the unifying context, where classical as well as quantum mechanics can be naturally framed, thus suggesting the possibility to transfer concepts and methods from quantum to classical mechanics and viceversa.

Furthermore, as discussed in ref. [2], phase–space picture provides the natural language for quantum optics as well, offering a geometrical view to coherent and squeezed states as circles and ellipses respectively. In this respect, taking advantages from the symmetry of the relevant Wigner phase space distribution function it is possible to calculate expectation values and transition probabilities for the above quoted states [2].

In the present paper, we are interested in the paraxial propagation of optical or charged–particle beams through *optical* systems [7, 8]. We are thereby led to consider the harmonic oscillator–type Hamiltonian

$$H = \frac{1}{2}\{p^2 + k(s)q^2\} \; , \tag{1}$$

with q and p being canonically conjugate variables. As noticed, the Hamiltonian (1) models the paraxial propagation of electron beams as well as light beams through optical systems. The p^2–term describes the free propagation for both electron beams and light rays, the variable p being understood as the particle–momentum and the ray reduced slope, respectively. On the other hand, the q^2–term accounts for the propagation through optical systems as quadrupoles or lens–like medium, the corresponding *strength* being measured by the coefficient $k(s)$. The coordinate s is measured along the symmetry axis of the system, also assumed as the direction of the beam propagation.

The basic tools of classical mechanics are the Poisson brackets and the canonical transformations. The latter can be derived from the former.

The properties of Poisson brackets, indeed, as the antisymmetry, the derivation property and the Jacobi's identity, assure that Poisson brackets make any commutative ring of functions defined on a domain $X \subset R^{2n}$ into a Lie algebra. It is thereby possible to associate with any Hamiltonian H the operator $\hat{\xi}_H \equiv \{H, \ldots\}$, which for the one–degree of freedom writes as

$$\hat{\xi}_H = \frac{\partial H}{\partial q}\frac{\partial}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial}{\partial q} \; . \tag{2}$$

Accordingly, the dynamics ruled by (1) is naturally framed within the group structure generated by the operators associated with the quadratic polynomials:

$$\begin{aligned}
&\frac{1}{2}p^2 \rightarrow -p\frac{\partial}{\partial q} \; , \\
&\frac{1}{2}q^2 \rightarrow q\frac{\partial}{\partial p} \; , \\
&pq \rightarrow p\frac{\partial}{\partial q} - q\frac{\partial}{pq} \; .
\end{aligned} \tag{3}$$

Embedding the above operators into the following

$$\begin{aligned}\hat{K}_1 &= \frac{1}{2}\left(q\frac{\partial}{\partial p} - p\frac{\partial}{\partial q}\right) ,\\ \hat{K}_2 &= \frac{1}{2}\left(q\frac{\partial}{\partial p} + p\frac{\partial}{\partial q}\right) ,\\ \hat{K}_3 &= \frac{1}{2}\left(p\frac{\partial}{\partial p} - q\frac{\partial}{\partial q}\right) ,\end{aligned} \tag{4}$$

it is easy to recognize the symplectic structure of the corresponding group, as displayed by the commutation relations obeyed by the above introduced operators:

$$[\hat{K}_1, \hat{K}_2] = \hat{K}_3 \ , \quad [\hat{K}_1, \hat{K}_3] = -\hat{K}_2 \ , \quad [\hat{K}_2, \hat{K}_3] = -\hat{K}_1 \ . \tag{5}$$

As elements of $Sp(2)$, the operators $\hat{K}_1$, $\hat{K}_2$, $\hat{K}_3$ are amenable of the matrix representation

$$K_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \quad K_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad K_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{6}$$

Finally, let us say that the operators (4) are the generators of the canonical transformations in phase–space, which will be analysed with some details in the next section, within the context of the specific problem of charged beam motion in magnetic fields.

3 Linear canonical transformations and optical systems

Before analysing the specific role of the operators $\hat{K}_i$, $i = 1, 2, 3$ and discussing the optical analogs of the associated canonical transformations, let us make some preliminary considerations in order to visualize the phase–space canonical transformations within the specific context of electron–beam transport physics.

As is well known, an invariant quadratic form

$$\mathcal{I} = \underline{x}^T \underline{\underline{T}}(s) \underline{x} \tag{7}$$

can be associated to any dynamics described by quadratic Hamiltonians in canonical coordinates and momenta. In passing, it is worth stressing that within a quantum context the quadratic form $\mathcal{I}$ is reported as the Ermakov–Lewis invariant [9], the vector $\underline{x}$ containing obviously the position and momentum operators, whilst in classical mechanics it is known as the Courant–Snyder invariant [10], firstly introduced in the analysis of electron beam motion through magnetic channels. In the above expression, the two component vector $\underline{x} \equiv \begin{pmatrix} q \\ p \end{pmatrix}$ is acted by the real 2×2 matrix $\underline{\underline{T}}$, which furthermore is required to be symmetric and unimodular: $\det \underline{\underline{T}} = 1$. Just to share the language of accelerator physics, we refer to $\underline{\underline{T}}$ as the *Twiss* matrix[1] and write it in the form

$$\underline{\underline{T}} = \begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix} . \tag{8}$$

[1]The quadratic form (7) can also be regarded as the transcription in phase space of the quantum invariant, the vector $\underline{x}$ being formed by the expectation values of the position and momentum operators and the matrix $\underline{\underline{T}}$ being linked to the *quantum covariance* matrix.

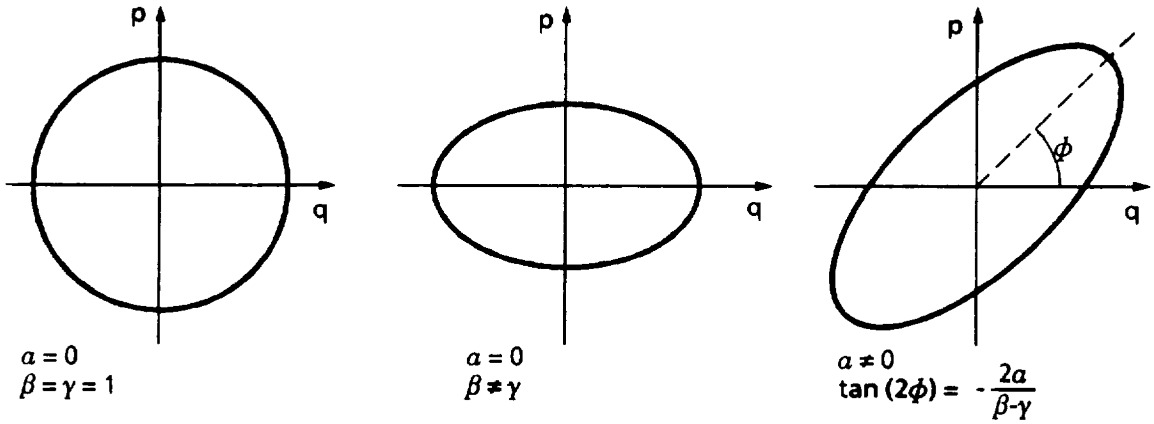

Figure 1: Phase–space ellipses for different values of the Twiss parameters α, β, γ.

The entries α, β, γ usually named as *Twiss* parameters, play an important role in designing transport channels.

The quadratic form (7) can be depicted in the phase space as an ellipse, whose size and orientation are determined by the Twiss coefficients. The area of the ellipse, which is just the value of the invariant $\mathcal{I}$, is usually denoted in accelerator physics as $\mathcal{I} = \pi\varepsilon$, ε being referred to as the *beam emittance*. It plays a crucial role in characterizing the quality and the dynamics of the e–beam. In a single particle analysis, α, β and γ define the particle trajectory, as in an ensemble analysis they define the second order momentum of the phase space distribution function, thus providing information on its extent and maximum localization. Explicitly,

$$\begin{aligned} \varepsilon\gamma &= \sigma_{pp}^2 \equiv \langle p^2 \rangle - \langle p \rangle^2 , \\ \varepsilon\beta &= \sigma_{qq}^2 \equiv \langle q^2 \rangle - \langle q \rangle^2 , \\ \varepsilon\alpha &= -\sigma_{qp}^2 \equiv -(\langle qp \rangle - \langle q \rangle \langle p \rangle) , \end{aligned} \tag{9}$$

the averages being understood on the distribution function. Accordingly, the emittance can be given the further meaning:

$$\varepsilon^2 = \sigma_{qq}^2 \sigma_{pp}^2 - \sigma_{qp}^2 . \tag{10}$$

According to the above considerations,the dynamics of charged beams passing through transport channels naturally leads to a visualization of the problem in terms of circles and ellipses in the phase–space, which on the other hand have been recognized as useful pictures for the coherent and squeezed states of quantum optics as well (Fig. 1).

Acting on the vector $\underline{x}$ amounts to acting on the phase–space ellipse and correspondingly on the Twiss parameters. To be more precise, let us say that a linear canonical transformation U, represented by the matrix $\underline{\underline{U}} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, change the Twiss parameters as [8]

$$\begin{pmatrix} \alpha' \\ \beta' \\ \gamma' \end{pmatrix} = \begin{pmatrix} AD + BC & -AC & -BD \\ -2AB & A^2 & B^2 \\ -2DC & C^2 & D^2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} . \tag{11}$$

After these introductory remarks, let us consider the specific effect of the transformations generated by the operators $\hat{K}_i$, $i = 1, 2, 3$ and recognize the corresponding *optical* systems.

Using the matrix representation (6), we immediately get

$$e^{\eta \hat{K}_3} \equiv S(\eta, 0) = \begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix} , \tag{12}$$

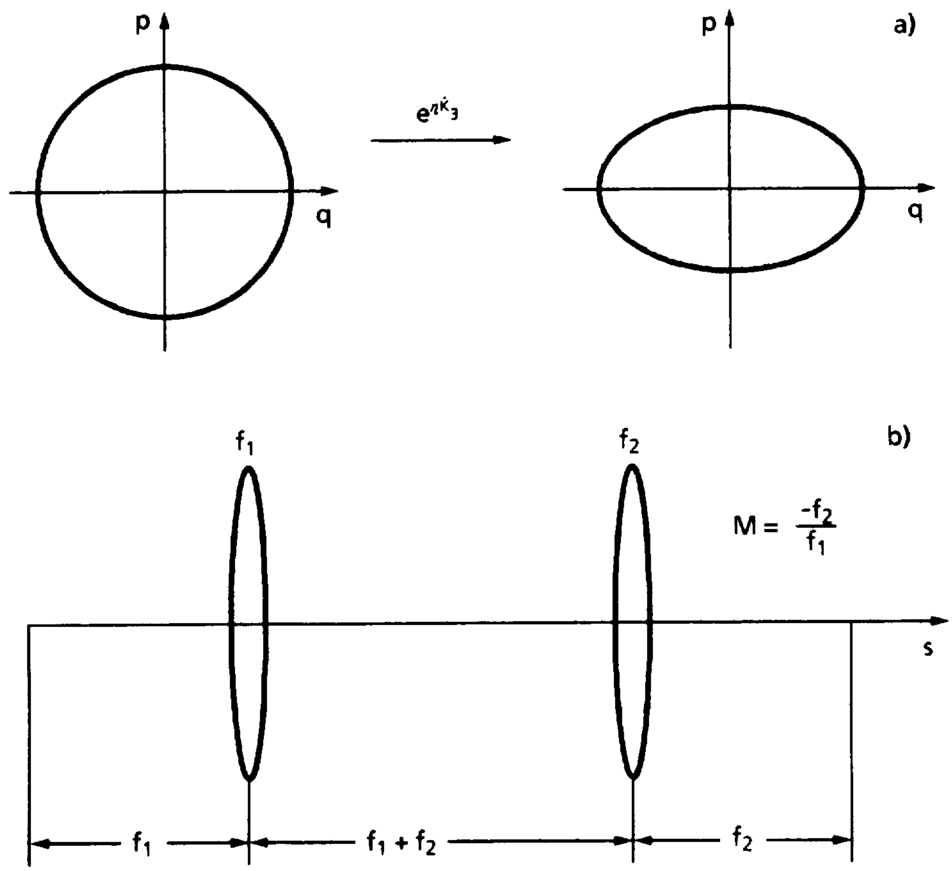

Figure 2: Phase–space canonical transformation (a) and optical system (b) corresponding to the operator $e^{\eta \hat{K}_3}$.

which represents an elongation in the q–direction and correspondingly a stretching in the p–direction, turning for instance a circle into an ellipse, as sketched in Fig. 2a.

It is needless to say that the transformation (12) preserves phase–space area, as a consequence of that $Sp(2)$ matrices leave invariant the cross products, naturally associated with areas.

In the optics of electron or light beams the transformation described by $S(\eta, 0)$ is realized by means of telescopic systems, consisting of two thin lenses appropriately combined, according to the scheme shown in Fig. 2b. In this regard, let us recall that the symbology for optics of electron beams and light beams is the same. Hence, thin lenses in ray optics correspond to quadrupoles in electron–beam optics.

The parameter M, which is just equal to minus the ratio of the foci of the two lenses, is reported in optics as the *magnification* of the system. It produces indeed an *image magnification* and a *ray–angle demagnification*.

As to the operator $\hat{K}_1$, it is easy to obtain the associated transformation as

$$e^{-\phi \hat{K}_1} \equiv R(\phi) = \begin{pmatrix} \cos\phi/2 & -\sin\phi/2 \\ \sin\phi/2 & \cos\phi/2 \end{pmatrix}, \tag{13}$$

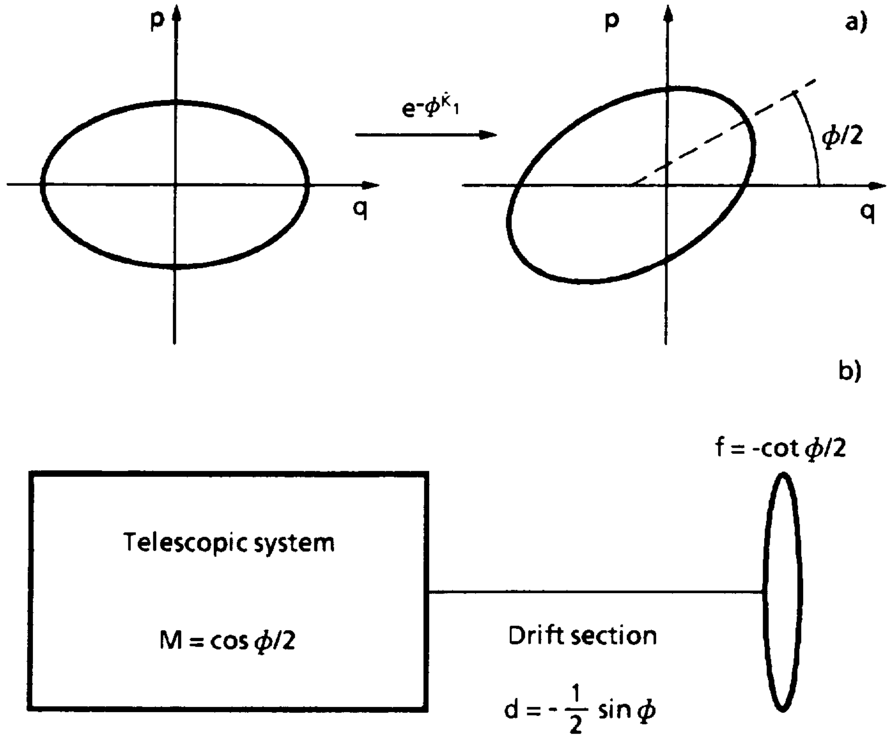

Figure 3: Phase–space canonical transformation (a) and optical system (b) corresponding to the operator $e^{-\phi \hat{K}_1}$.

immediately recognized to describe the rotation around the origin in the phase space by the angle $\phi/2$, as further confirmed by the transformation law (11) for the Twiss parameters according to which the associated ellipse rotates in the counterclockwise direction by $\phi/2$.

Since any group elements can be appropriately factorized, the canonical transformation corresponding to the matrix $R(\phi)$ can be realized by means of an appropriate sequence of the basic optical elements, that is thin lenses, drift sections and telescopic systems, which however are just a combination of the first two.

Accordingly, the optical system corresponding to a rotation in phase–space can be realized by means of the following sequence (Fig. 3):

1. telescopic system with magnification $M = \cos\phi/2$
2. drift section of length $d = -\frac{1}{2}\sin\phi$
3. thin lens of focus $f = -\cot\phi/2$

In this connection, it is worth stressing that the above scheme is only one of the possible ones, which can be obtained changing the ordering in the operator factorization, thus allowing to satisfy specific requests on the parameters of the optical components.

Finally, the operator $e^{\eta \hat{K}_2}$, which can be represented by the matrix:

$$e^{\eta \hat{K}_2} \equiv S(\eta, 90^\circ) = \begin{pmatrix} \cosh\eta/2 & \sinh\eta/2 \\ \sinh\eta/2 & \cosh\eta/2 \end{pmatrix}, \tag{14}$$

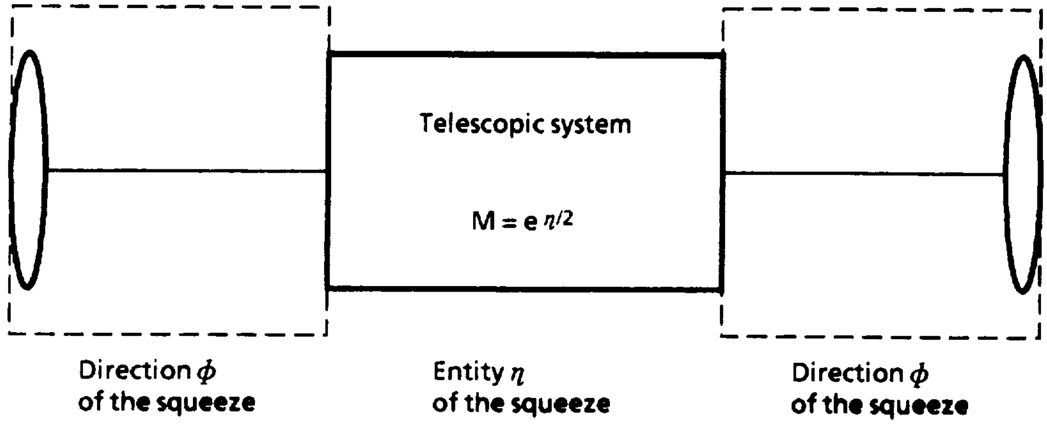

Figure 4: Optical analog of the squeeze $S(\eta, \phi)$.

is easily identified according to the transformation law (11) as producing a squeeze in the direction making an angle of 45° with the q–axis.

The optical analog can be realized for instance by the same sequence as before, the relevant parameters being now $M = \cosh \eta/2$, $d = \sinh \eta/2$, $f = \tanh^{-1} \eta/2$.

Finally, let us discuss the squeeze $S(\eta, \phi)$ in the direction making an angle $\phi/2$ with respect to the q–axis. Since it can be obtained combining rotations and squeeze along the q–axis, as formally expressed by the composition

$$S(\eta, \phi) = R(\phi) S(\eta, 0) R(-\phi) , \tag{15}$$

it is easy to get the well known matrix representation [2]

$$S(\eta, \phi) = \begin{pmatrix} \cosh \eta/2 + \cos \phi \sinh \eta/2 & \sin \phi \sinh \eta/2 \\ \sin \phi \sinh \eta/2 & \cosh \eta/2 - \cos \phi \sinh \eta/2 \end{pmatrix} . \tag{16}$$

According to the above discussion, a possible optical configuration can be realized by a telescopic system, preceded and followed by the same sequence of thin lens–drift section, symmetrically disposed. It is worth stressing that the magnification of the telescopic system is determined by the entity η of the squeeze as the parameters of the optical system drift section–thin lens are determined by the squeeze direction ϕ (Fig. 4). The quantities η, ϕ are usually combined into the squeeze parameter $\zeta \equiv \eta e^{i\phi}$, so that η and ϕ can be regarded as the modulus and the phase of the squeeze parameter ζ.

In conclusion, we have stated a correspondence between the linear canonical transformations in phase–space, as squeezes and rotations, and *optical systems*, which can be conceptually conceived as realizing such transformations, acting effectively on electron or light beams.

4 Electron–beam transport channels and Wigner angle

The correspondence between linear canonical transformations and Lorentz transformations has been already recognized [2]. Boosts correspond indeed to squeezes in phase–space and rotations in real space to rotations in phase–space and rotations in real space to rotations in phase–space. Similarly, as the product of two boosts along different directions is not a boost, but a boost preceded or followed by a rotation, so the product of two squeeze along different directions does not result into a single squeeze, but into a squeeze and a rotation. It can be verified that

$$S(\lambda,\phi)S(\eta,0) = S(\xi,\theta)R(\omega)\ . \tag{17}$$

The parameters ξ, θ, specifying the entity and the direction of the resulting squeeze, and the angle ω, referred to as the Wigner angle, are determined by λ, ϕ, η according to the well–known formulae:

$$\begin{aligned}
\cosh\xi &= \cosh\eta\cosh\lambda + \cos\phi\sinh\lambda\sinh\eta\ , \\
\tan\theta &= \frac{\sin\phi[\sinh\lambda + \cos\phi\tanh\eta(\cosh\lambda - 1)]}{\cos\phi\sinh\lambda + \tanh\eta[1+\cos^2\phi(\cosh\lambda - 1)]}\ , \\
\tan\frac{\omega}{2} &= \frac{\sin\theta\sinh\frac{\eta}{2}\sinh\frac{\lambda}{2}}{\cosh\frac{\lambda}{2}\cosh\frac{\eta}{2} + \cos\phi\sinh\frac{\lambda}{2}\sinh\frac{\eta}{2}}\ .
\end{aligned} \tag{18}$$

Within the context of the optical analogy, developed in the previous section, the above relations can be recast in terms of the parameters of the optical systems corresponding to $S(\lambda,\phi)$ and $S(\eta,0)$, according to

$$\begin{aligned}
\cosh\xi &= \frac{1}{4M_\eta^2M_\lambda^2}\left\{(M_\eta^4+1)(M_\lambda^4+1) + \left(1-\frac{2d}{f}\right)(M_\eta^4-1)(M_\lambda^4-1)\right\}\ , \\
\tan 2\theta &= 2d\frac{(M_\lambda^4-1)(M_\eta^4+1) - \left(1-\frac{2d}{f}\right)(M_\eta^4-1)(M_\lambda^2-1)^2}{(M_\lambda^4-1)(M_\eta^4+1)\left(1-\frac{2d}{f}\right) + (M_\eta^4-1)\left[2M_\lambda^2 + \left(1-\frac{2d}{f}\right)(M_\lambda^2-1)^2\right]}\ , \\
\tan\frac{\omega}{2} &= 2d\frac{(M_\eta^2-1)(M_\lambda^2-1)}{(M_\eta^2+1)(M_\lambda^2+1) + \left(1-\frac{2d}{f}\right)(M_\eta^2-1)(M_\lambda^2-1)}\ ,
\end{aligned} \tag{19}$$

where M_η specifies the magnification of the optical system $S(\eta,0)$, whilst M_λ, d and f denote the parameters of the system $S(\lambda,\phi)$ according to the scheme of Fig. 4.

Accordingly, one can design an appropriate sequence of quadrupoles and drift sections to perform the transformation represented by the product of three squeeze:

$$S(-\xi,\theta)S(\lambda,\phi)S(\eta,0)\ , \tag{20}$$

with ξ, θ given according to (19).

As stated in (17), such a transformation does not leave unchanged the Twiss parameters of the beam, which indeed change as in a rotation of the angle $\omega/2$. Explicitly,

$$\begin{aligned}
\alpha_2 &= \alpha_1 - \frac{1}{2}(\beta_1 - \gamma_1)\sin\omega\ , \\
\beta_2 &= \alpha_1\sin\omega + \beta_1\cos^2\frac{\omega}{2} + \gamma_1\sin^2\frac{\omega}{2}\ , \\
\gamma_2 &= -\alpha_1\sin\omega + \beta_1\sin^2\frac{\omega}{2} + \gamma_1\cos^2\frac{\omega}{2}\ .
\end{aligned} \tag{21}$$

It is evident that if $\alpha_1 = 0$ and $\gamma_1 = \beta_1 = 1$, that is depicted in phase–space by a circle of unitary radius, the rotation does not have any effect, since $\alpha_2 = 0$ and $\gamma_2 = \beta_2 = 1$ as well. As a consequence, a beam having different variance in the q and p directions should be used; in other words, sharing the language of quantum optics, a *squeezed* beam should be used to produce a rotation of the beam ellipse.

In that case, indeed, the initial ellipse will rotate in the phase–space just by the angle $\omega/2$. Measuring then the Twiss parameters of the electron beam before and after being acted by the optical system, performing the transformation (20), it is possible to infer the Wigner angle, ω, for which the following link with the Twiss parameters can be deduced:

$$\tan\omega = 2\frac{\alpha_2(\beta_1 - \gamma_1) - \alpha_1(\beta_2 - \gamma_2)}{(\beta_2 - \gamma_2)(\beta_1 - \gamma_1) + 2\alpha_1\alpha_2} \,. \tag{22}$$

5 Concluding remarks

The consideration developed in the previous sections are basically grounded on the algebraic analogy between the $SO(2,1)$ Lorentz group and the symplectic group $Sp(2)$, which is basic to the Hamiltonian dynamics. Exploiting this analogy, it is possible to conceive and design an optical system for electron beams, which allows to get a measure of the Wigner angle by detecting the variations occurred in the electron–beam Twiss parameters as a result of the motion through the magnetic channel. In this connection, the two–slits method [11] may offer an appropriate tool to visualize the rotation of the beam ellipse and thus to measure the Wigner angle. However, the measure is strongly limited by space–charge effects and transverse coupling, induced by sextupolar contributions to the quadrupole magnetic field.

The discussion has been put forward in full generality comprehending also light beams, whose paraxial propagation through optical systems is governed by the symplectic symmetry as well. An experiment using light beams can be conceptually conceived, but its realization is rather difficult, since the measure of the corresponding Twiss parameters, which in the specifically optical context, can be understood as linked to the beam spot–size and divergence, is limited by diffraction effects.

In conclusion, let us stress the relevance of the above results, according to which an analog computer for the Lorentz group can be recognized within the purely classical context of electron beam transport or optical ray propagation.

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