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# UNDERSTANDING SQUEEZING OF QUANTUM STATES WITH THE WIGNER FUNCTION


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**Abstract**

The Wigner function is argued to be the only natural phase space function evolving classically under quadratic Hamiltonians with time-dependent bilinear part. This is used to understand graphically how certain quadratic time-dependent Hamiltonians induce squeezing of quantum states. The Wigner representation is also used to generalize Ehrenfest's theorem to the quantum uncertainties. This makes it possible to deduce features of the quantum evolution, such as squeezing, from the classical evolution, whatever the Hamiltonian.


## 1 Introduction

The Wigner function [1] can be used to get a visual understanding of why certain time-dependent Hamiltonians induce squeezing of quantum states. We first address the question: Why the Wigner function, rather than, say, the Husimi function [2]? It will be showed that although other phase space functions may evolve classically under certain *specific* quadratic Hamiltonians, the Wigner function is the only one to do so under quadratic Hamiltonians having a *time-dependent* bilinear part. We consider, as an example, squeezing by a periodically modulated harmonic oscillator. We then discuss a generalization of Ehrenfest's theorem applying to the quantum uncertainties. This allows to deduce aspects of the quantum evolution, such as squeezing, from the classical evolution even in the case of arbitrary Hamiltonians.

Phase space variables (position and momentum) are denoted by $q$ and $p$. A caret is used to identify operators. Thus, $[\hat{q}, \hat{p}] = i$, where we take $\hbar = 1$. The time variable is denoted by $t$.

## 2 Quadratic Hamiltonians and phase space functions

Ehrenfest's theorem expresses the time derivatives of the expectations $\langle\hat{q}\rangle$ and $\langle\hat{p}\rangle$ in a way formally similar to Hamilton's equations. In the case of a Hamiltonian $\hat{H} = \frac{1}{2}\hat{p}^2/m + V(\hat{q})$, it reads

$$\frac{d}{dt}\langle\hat{q}\rangle = \langle\hat{p}\rangle/m, \qquad \frac{d}{dt}\langle\hat{p}\rangle = \langle F(\hat{q})\rangle \tag{1}$$

where $F(q) = -\partial V/\partial q$ is the force. If the potential $V(q)$ is quadratic, hence $F(q)$ linear (or, if the particle is in a state sufficiently localized on the scale of non-harmonic variation of V), then $\langle F(\hat{q})\rangle = F(\langle\hat{q}\rangle)$, and (1) become identical to Hamilton's equations, implying that $\langle\hat{q}\rangle$ and $\langle\hat{p}\rangle$ follow classical trajectories in phase space. This stays true in the case of a general quadratic Hamiltonian

$$\hat{H}(t) = \hat{H}_1(t) + \hat{H}_2(t) \tag{2}$$

where $\hat{H}_1(t)$ is linear, and $\hat{H}_2(t)$ is bilinear in $\hat{q}$ and $\hat{p}$ ($\alpha$, $\beta$, a, b, c scalar functions of time):

$$\hat{H}_1(t) = H_1(\hat{q},\hat{p},t) = \alpha(t)\hat{q} + \beta(t)\hat{p} \tag{3}$$

$$\hat{H}_2(t) = H_2(\hat{q},\hat{p},t) = a(t)\hat{q}^2 + b(t)(\hat{q}\hat{p}+\hat{p}\hat{q}) + c(t)\hat{p}^2 \tag{4}$$

Can one push this further, and associate with state vectors $|\psi\rangle$, or state operators $\hat{\rho}$, a phase space function $f(q,p,t)$ whose quantum evolution is classical for such Hamiltonians? That is, such that

$$\frac{\partial}{\partial t} f(q,p,t) = \{H, f\}_{PB} \equiv \left(\frac{\partial H}{\partial q}\frac{\partial}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial}{\partial q}\right) f(q,p,t) \qquad \text{or} \qquad f(q,p,t) = f(q_t,p_t,0) \tag{5}$$

where $(q_t,p_t)$ is the point which classically evolves into $(q,p)$ in the time interval $(0,t)$, under the Hamiltonian $H(q,p,t)$. It is natural to ask that $f(q,p)$ be linear in either $|\psi\rangle$ or $\hat{\rho}$, so that it has the form

$$f(q,p;\psi) = \langle\phi_{qp}|\psi\rangle \qquad \text{or} \qquad f(q,p;\hat{\rho}) = \mathrm{Tr}\{\hat{\Delta}_{qp}\hat{\rho}\} \tag{6}$$

where $|\phi_{qp}\rangle$ and $\hat{\Delta}_{qp}$ are kets and operators parametrized by phase space.

Introduce the unitary time-evolution operator $\hat{U}(t,t')$, defined by

$$\frac{\partial}{\partial t}\hat{U}(t,t') = -i\hat{H}(t)\hat{U}(t,t'), \qquad \frac{\partial}{\partial t'}\hat{U}(t,t') = i\hat{U}(t,t')\hat{H}(t'), \qquad \hat{U}(t,t')\hat{U}(t',t) = \hat{U}(t,t) = \hat{1} \tag{7}$$

and similarly $\hat{U}_1$ and $\hat{U}_2$ corresponding to $\hat{H}_1$ and $\hat{H}_2$. Let us first assure classical evolution under linear time-independent Hamiltonians $\hat{H}_1 = \alpha\hat{q}+\beta\hat{p}$: Here, $\hat{U}_1(t,0) = e^{-it(\alpha\hat{q}+\beta\hat{p})}$, and one gets classical evolution, $f(q,p,t) = f(q-\beta t, p+\alpha t, 0)$, up to a phase, iff

$$|\phi_{qp}\rangle = \hat{D}_{qp}|\phi\rangle, \qquad \hat{\Delta}_{qp} = \hat{D}_{qp}\hat{\Delta}\hat{D}_{qp}^{-1} \tag{8}$$

where

$$\hat{D}_{qp} = e^{ip\hat{q}-iq\hat{p}} \tag{9}$$

are phase space displacement operators, and $|\phi\rangle$ and $\hat{\Delta}$ are some fiducial ket and operator. We now note the relations ($[\ ]_+$ denotes anticommutators, $\{\ \}_{PB}$ Poisson brackets):

$$[\hat{H}_1, \hat{D}_{qp}] = H_1(q,p)\hat{D}_{qp}, \quad \tfrac{1}{2}i[\hat{H}_1, \hat{D}_{qp}]_+ = \{H_1, \hat{D}_{qp}\}_{PB} \equiv (\frac{\partial H_1}{\partial q}\frac{\partial}{\partial p} - \frac{\partial H_1}{\partial p}\frac{\partial}{\partial q})\hat{D}_{qp} \tag{10a}$$

$$i[\hat{H}_2, \hat{D}_{qp}] = \{H_2, \hat{D}_{qp}\}_{PB} \tag{10b}$$

where $H_{1,2} \equiv H_{1,2}(q,p)$. Using these results, and refering to (5), one finds that

$$\hat{U}(0,t)\,\hat{D}_{qp} = \hat{D}_{q_tp_t}\,\hat{U}_2(0,t)\,e^{i\chi(q,p,t)} \tag{11}$$

where $\hat{U}_2$ is the time evolutor for $\hat{H}_2$, and $\chi(q,p,t)$ is a phase determined by the equation

$$\frac{\partial}{\partial t}\chi - \{H(t),\chi\}_{PB} = \tfrac{1}{2}H_1(t), \qquad \chi(q,p,0) \equiv 0 \tag{12}$$

One verifies (11) by verifying that both sides of $\hat{U}(0,t)\hat{D}_{qp}\hat{U}_2(t,0) = \hat{D}_{q_tp_t}e^{i\chi}$ satisfy the same differential equation. We then get, from (6) and (11):

$$f_\phi(q,p;\,\psi(t)) \equiv \langle\phi|\hat{D}_{qp}{}^\dagger\hat{U}(t,0)|\psi\rangle = e^{-i\chi(q,p,t)}\,f_{\phi_t}(q_t,p_t;\,\psi(0)), \qquad |\phi_t\rangle \equiv \hat{U}_2(0,t)|\phi\rangle \tag{13}$$

$$f_\Delta(q,p;\,\hat{\rho}(t)) \equiv \mathrm{Tr}\{\hat{D}_{qp}\hat{\Delta}\hat{D}_{qp}{}^{-1}\,\hat{U}(t,0)\hat{\rho}\hat{U}(0,t)\} = f_{\Delta_t}(q_t,p_t;\,\hat{\rho}(0)), \quad \hat{\Delta}_t \equiv \hat{U}_2(0,t)\hat{\Delta}\hat{U}_2(t,0) \tag{14}$$

One sees that the function $f_\phi$ evolves classically iff $|\phi_t\rangle$ is stationary, i.e., if $\hat{H}_2$ is time-independent, and $|\phi\rangle = |E_n\rangle$ is an eigenket of it. For instance, if $\hat{H}_2$ is a harmonic oscillator, and $|\phi\rangle = |E_0\rangle$ is the ground state, then $|\phi_{qp}\rangle$ are coherent states, and

$$f_\phi(q,p;\,\psi(t)) = e^{-i\chi(q,p,t)+itE_0}\,f_\phi(q_t,p_t;\,\psi(0)) \tag{15}$$

is the Husimi function [2], which is well known to evolve classically under *that* $\hat{H}_2$ of which $|\phi\rangle$ is an eigenket. Clearly, no function $f_\phi$ linear in state *vectors* can possibly evolve classically if $\hat{H}_2$ is time-dependent. Consider now (14): Again, $f_\Delta$ evolves classically iff $\hat{\Delta}_t$ is stationary: For instance, if $\hat{H}_2$ is time-independent, and $\hat{\Delta} = |E_n\rangle\langle E_{n'}|$ or $\hat{\Delta} = g(\hat{H}_2)$, then the evolution is classical - but only for *that* specific time-independent $\hat{H}_2$. Is there an operator $\hat{\Delta}$ (apart from the unit operator) such that

$$\hat{U}_2(0,t)\,\hat{\Delta}\,\hat{U}_2(t,0) = e^{-i\varphi(t)}\,\hat{\Delta} \quad \Leftrightarrow \quad [\hat{H}_2(t),\,\hat{\Delta}] = \dot{\varphi}(t)\,\hat{\Delta} \qquad (\,\dot{\varphi}(t)=\partial\varphi/\partial t) \tag{16}$$

($\varphi$ a real phase) for time-dependent $\hat{H}_2(t)$? Yes, the parity operator $\hat{\Pi}$, since $\hat{q}\hat{\Pi} = -\hat{\Pi}\hat{q}$, $\hat{p}\hat{\Pi} = -\hat{\Pi}\hat{p}$ imply $[\hat{H}_2, \hat{\Pi}]=0$. Setting $\hat{\Delta} = \hat{\Pi}$ in (14) yields the *Wigner function* [1,3]

$$f_w(q,p;\,\hat{\rho}) = \pi^{-1}\mathrm{Tr}\{\hat{D}_{qp}\hat{\Pi}\,\hat{D}_{qp}{}^{-1}\,\hat{\rho}\}, \qquad f_w(q,p;\,\hat{\rho}(t)) = f_w(q_t,p_t;\,\hat{\rho}(0)) \tag{17}$$

The Wigner function is the only phase space function which evolves classically under (any) time-dependent $\hat{H}_2(t)$. One can see this as follows: Represent any $\hat{\Delta}$ by its Weyl symbol $\Delta_w(q,p)$, where the Weyl symbol of an operator $\hat{A}$ is defined as [3,4]

$$A_w(q,p) = 2\mathrm{Tr}\{\hat{D}_{qp}\hat{\Pi}\,\hat{D}_{qp}^{-1}\,\hat{A}\} \tag{18}$$

We want $\hat{\Delta}_t$, hence its Weyl symbol, to be stationary for any $\hat{H}_2(t)$. Now, just as the Weyl symbol (17) of $\hat{\rho}$ evolves classically under $\hat{H}_2(t)$, so does that of $\hat{\Delta}_t$. The only possible way for $(\hat{\Delta}_t)_w(q,p)$ to be stationary under classical evolution with different $H_2$ 's (e.g., with orbits which are ellipses or hyperbolas of different eccentricities) is that it be be concentrated at the origin: We must thus have $\Delta_w(q,p) = (const)\delta(q)\delta(p)$; this implies [5b] that $\hat{\Delta} = (const)\hat{\Pi}$.

## 3 Squeezing by a Periodically Modulated Harmonic Oscillator

The classical evolution of the Wigner function under quadratic $\hat{H}(t)$ is very useful for understanding the quantum in terms of the classical evolution. As an example, consider a harmonic oscillator

$$\hat{H} = \tfrac{1}{2}m^{-1}\hat{p}^2 + \tfrac{1}{2}m\omega^2\hat{q}^2 = \tfrac{1}{2}\omega(\alpha\hat{p}^2 + \hat{q}^2/\alpha), \qquad \alpha = (m\omega)^{-1} \tag{19}$$

If $\alpha$=1 (in suitable units), the classical orbits $H(q,p) =$ *constant* are circles in the phase plane (q,p). If $\alpha \neq 1$, the orbits are ellipses (Fig.1), the ratio of the q semi-axis to the p semi-axis being $\alpha$. We will now let $\alpha$ alternate between two values, $\gamma$ and $\gamma^{-1}$ (where $\gamma > 1$), at every quarter of a period $2\pi/\omega$, while keeping $\omega$ fixed (i.e., only m changes). Let $\alpha = \gamma^{-1} < 1$ for the first quarter period: During that time interval, a point initially on the positive q axis (beginning of trajectory 1 on Fig.2) moves to the negative p axis, while receding away from the origin by a factor $\gamma$. Then let $\alpha = \gamma > 1$ during the next quarter period: The point moves to the negative q axis, receding away from the origin by another factor $\gamma$. And so on. One here has parametric amplification. On the other hand, points initially on the p axis (trajectory 2) close in on the origin. Thus, classically, the phase plane gets squeezed into the rotating q axis, by a factor $\gamma$ at each quarter period. Whence a corresponding squeezing of quantum states [5a,6].

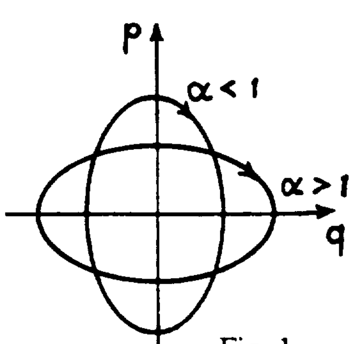

Fig. 1

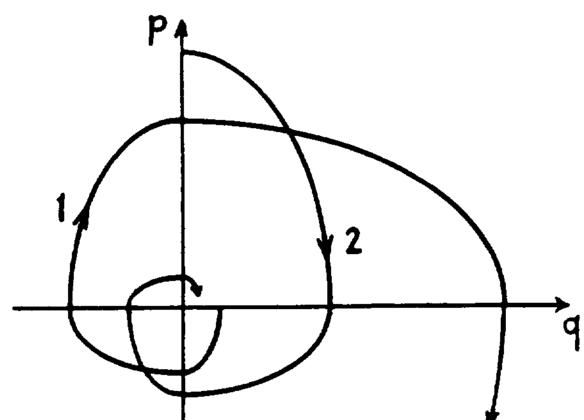

Fig. 2

## 4 Ehrenfest Theorem for the Quantum Uncertainties

The above concerned quadratic Hamiltonians. It will now be indicated that the Wigner-Weyl (WW) representation allows to reinterpret Ehrenfest's theorem, and to extend it to quadratic observables, hence to the quantum uncertainties. This makes it possible to deduce features of the quantum evolution from the classical evolution even in the case of arbitrary Hamiltonians. Let us first recall that in the WW representation, expectation values of operators $\hat{A}$ have the "classical" form [1,7]

$$\langle\hat{A}\rangle \equiv \mathrm{Tr}\{\hat{A}\,\hat{\rho}\} = \int dqdp\; A_w(q,p)\, f_w(q,p) \tag{20}$$

In particular, the quantum expectations of $\hat{q}$ and $\hat{p}$ are

$$\langle\hat{q}\rangle = \int dqdp\; q\, f_w(q,p), \qquad \langle\hat{p}\rangle = \int dqdp\; p\, f_w(q,p) \tag{21}$$

and the uncertainty matrix is

$$\begin{aligned} C_{qq} &= \langle(\hat{q} - \langle\hat{q}\rangle)^2\rangle = \int dqdp\; (q - \langle q\rangle)^2 f_w(q,p) \\ C_{qp} = C_{pq} &= \langle\tfrac{1}{2}(\hat{q}\hat{p}+\hat{p}\hat{q}) - \langle\hat{q}\rangle\langle\hat{p}\rangle\rangle = \int dqdp\; (q-\langle q\rangle)(p-\langle p\rangle)\, f_w(q,p) \\ C_{pp} &= \langle(\hat{p} - \langle\hat{p}\rangle)^2\rangle = \int dqdp\; (p - \langle p\rangle)^2 f_w(q,p) \end{aligned} \tag{22}$$

A quantum state (wave packet) may be roughly represented in phase space by an uncertainty ellipse

$$(u - \langle u\rangle)\cdot \mathbf{C}^{-1} (u - \langle u\rangle) = 1 \quad \text{where} \quad u = \begin{pmatrix} q \\ p \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} C_{qq} & C_{qp} \\ C_{pq} & C_{pp} \end{pmatrix} \tag{23}$$

We also need the result [7]

$$[\hat{A}, \hat{B}]_w(q,p) = i\{A_w, B_w\}_{PB}(q,p) \quad \text{if} \quad \hat{A} \text{ or } \hat{B} \text{ is quadratic} \tag{24}$$

that is: The Weyl symbol of the commutator of two operators, one of which is quadratic, is equal to the Poisson bracket of the individual Weyl symbols. Let now $\hat{A}$ be an observable quadratic in $\hat{q}$ and $\hat{p}$. We then have, by (20) and (24), for any $\hat{H}(t)$:

$$\begin{aligned} \frac{\partial}{\partial t}\langle\hat{A}\rangle &= -i\mathrm{Tr}\{\hat{A}[\hat{H}, \hat{\rho}]\} = -i\mathrm{Tr}\{[\hat{A},\hat{H}]\hat{\rho}\} = \int dqdp\; \{A_w, H_w\}_{PB} f_w(q,p,t) \\ &= \int dqdp\; A_w(q,p)\{H_w, f_w\}_{PB} \qquad (\hat{A} \text{ quadratic, any } \hat{H}) \end{aligned} \tag{25}$$

where in the last line we performed an integration by parts. Eq.(25) says that the rate of change of the expectation of a quadratic observable is classical, in the sense that it is the same as if each point in

Wigner phase space instantaneously followed a classical trajectory. This does not mean that $\langle\hat{A}\rangle$ evolves classically over finite time intervals, because $f_W(q,p,t)$ in (25) is the exact *quantally* evolved Wigner function at time t, not one evolved classically during some finite time (unless $\hat{H}$ is quadratic); more specifically, the first time derivative of $\langle\hat{A}\rangle$ is classical, but not the higher order derivatives. By letting $\hat{A}$ stands for $\hat{q}$ or $\hat{p}$ in (25), one gets Ehrenfest's theorem:

$$\frac{\partial}{\partial t}\langle\hat{q}\rangle = \int dqdp\; q\{H_W, f_W\}_{PB} = \int dqdp\; \{q, H_W\}_{PB} f_W(q,p,t) = \langle\partial\hat{H}/\partial\hat{p}\rangle \tag{26a}$$

$$\frac{\partial}{\partial t}\langle\hat{p}\rangle = \int dqdp\; p\{H_W, f_W\}_{PB} = \int dqdp\; \{p, H_W\}_{PB} f_W(q,p,t) = \langle-\partial\hat{H}/\partial\hat{q}\rangle \tag{26b}$$

The last expressions in (26a,b) are the usual statement of Ehrenfest's theorem [equivalent to (1) if $\hat{H} = \frac{1}{2}\hat{p}^2/m + V(\hat{q})$], giving a formal quantum-classical analogy. The first expressions in (26a,b) tell us much more: That the rates of change of $\langle\hat{q}\rangle$ and $\langle\hat{p}\rangle$ *are classical*, but relative to a phase space distribution function. Eq.(25) generalizes Ehrenfest's theorem to quadratic observables, and thus to the uncertainties: Indeed, according to (21), (22) and (25), the uncertainty ellipse (23) evolves exactly as if each point in Wigner phase space instantaneously followed a classical trajectory. For instance, if the ellipse gets squeezed classically, during some small (infinitesimal) time interval, then so does it quantally. In general, one may expect that if the classical motion during a *finite* time interval squeezes the uncertainty ellipse, then so does the quantum evolution. The latter statement is of course rigorously true if $\hat{H}(t)$ is quadratic, in view of (17); it is also approximately true, in the case of arbitrary $\hat{H}(t)$, if $f_W(q,p,t)$ is sufficiently localized on the scale of non-harmonic variation of $H(q,p,t)$, for the evolution of $f_W(q,p,t)$ is then approximately classical [5b].

Let us mention, finally, that Ehrenfest's theorem for quadratic observables can also be written in a form corresponding to the last expressions in (26a,b), namely [5c]

$$\frac{\partial}{\partial t}\langle\hat{A}\rangle = \langle\{A_W, H_W\}_{PB}(\hat{q},\hat{p})_W\rangle \qquad (\hat{A}\text{ quadratic, any }\hat{H}) \tag{27}$$

where the subscript w on the function $\{A_W, H_W\}_{PB}(\hat{q},\hat{p})$ of the non-commuting operators $\hat{q}$ and $\hat{p}$ signifies that they are ordered according to Weyl's ordering rule [4].


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