N95- 13935

THE STOKES LINE WIDTH AND UNCERTAINTY RELATIONS

A. I. Nikishov, V. I. Ritus
Lebedev Physical Institute, Moscow 117924

Abstract

For a function given by contour integral the two types (conventions) of asymptotic representations are considered: the usual representation by asymptotic series in inverse powers of large parameter and the special division of contour integral in contributions of high and low saddle points. It is shown that the width of the recessive term formation zone (Stokes strip) in the second convention is determined by uncertainty relation and is much less than the zone width in the first convention. The reasons of such a difference is clarified. The results of the work are useful for understanding of formation region of the exponentially small process arising on the background of the strong one.

1 Introduction

Many physical quantities are represented by contour integrals depending on two (or more) real parameters:

$$F(z) = A \int_C dt e^{f(t,z)}, \tag{1}$$

z is one complex or a couple of two real parameters, ν and α. The asymptotic representation of these integrals are considered when one of the parameters, ν, tends to infinity where the integral has essential singularity, while another parameter, α, is near the Stokes line [1] where $\alpha = 0$. We restrict ourselves to the case when $f(t, z)$ in (1) has only two saddle points t_1, t_2 where $f'(t_{1,2}, z) = 0$, and denote

$$f_{1,2}(z) = f(t_{1,2}, z), \qquad f''_{1,2}(z) = f''(t_{1,2}, z). \tag{2}$$

Then the asymptotic representation consists of two terms:

$$F = D + R, \quad D \sim e^{f_2(z)}, \quad R \sim ig(z)e^{f_1(z)}, \quad \nu \gg 1. \tag{3}$$

The main (dominant) one $\sim e^{f_2}$ and the exponentially small relatively to it (recessive) one $\sim ige^{f_1}$, $\mathrm{Re}(f_2 - f_1) \gg 1$ when $\nu \gg 1$.

Qualitative distinction of the two terms lies in the different rates of change of their phases $\mathrm{Im} f_{1,2}(\alpha)$ with α:

$$\omega_{1,2}(\alpha) = -\frac{\partial \mathrm{Im} f_{1,2}(\alpha)}{\partial \alpha}, \qquad \omega_1(0) \neq \omega_2(0). \tag{4}$$

Then from physical point of view D and R are the dominant and recessive waves with different frequencies ω_2 and ω_1 . Another qualitative distinction is the appearance (or disappearance) of R when α crosses the Stokes line $\alpha = 0$. This appearance takes place in a certain interval $\Delta\alpha$ which may be called the Stokes line width [2,3,4]. According to these authors the switching function of recessive term coincides with error-function

$$g(\alpha) = \frac{1}{2}\text{erfc}(w) = \frac{1}{\sqrt{\pi}}\int_w^\infty dx e^{-x^2}, \tag{5}$$

where $w = w(\alpha)$ is a certain odd function of α, depending on convention [4] about dominant

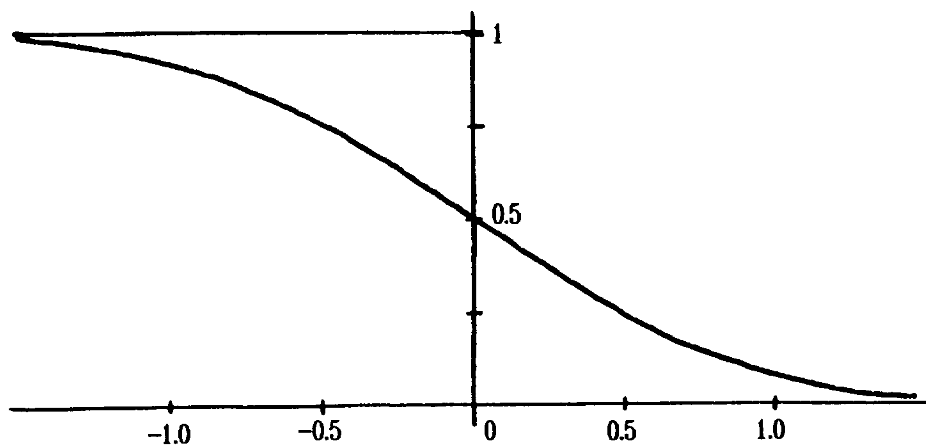

Figure 1:

term. The interval $\Delta\alpha$ defined by the condition

$$|w(\alpha)| \leq 1 \rightarrow \Delta\alpha \tag{6}$$

may be called the Stokes line width.

2 First convention about dominant and recessive terms

The 1-st convention [2,4] is based on asymptotic series expansion. Consider it on the example of standard Airy function expansion [5]

$$\begin{aligned} \text{Ai}(z) = \frac{1}{2}\int_C dt e^{-i(zt+t^3/3)} &\sim \frac{\sqrt{\pi}e^{-\zeta}}{2z^{1/4}}\sum_{k=0}^\infty c_k(-\zeta)^{-k}, \quad |\arg z| < \frac{2\pi}{3}, \\ &\sim \frac{\sqrt{\pi}e^{-\zeta}}{2z^{1/4}}\sum_{k=0}^\infty c_k(-\zeta)^{-k} + i\frac{\sqrt{\pi}e^{+\zeta}}{2z^{1/4}}\sum_{k=0}^\infty c_k\zeta^{-k}, \quad \frac{2\pi}{3} < \arg z < \frac{4\pi}{3}, \\ c_k = \frac{\Gamma(k+\frac{1}{6})\Gamma(k+\frac{5}{6})}{\pi 2^{k+1}k!}&, \qquad \zeta = \frac{2}{3}z^{3/2}. \end{aligned} \tag{7}$$

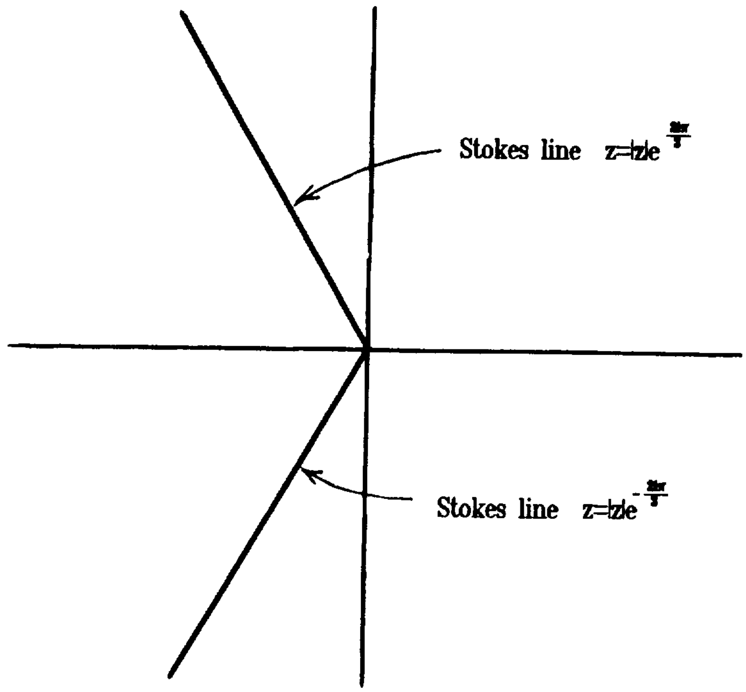

Figure 2:

Near the upper Stokes line $z{=}|z|e^{i(2\pi/3+\alpha)}$, $-\zeta = \frac{2}{3}|z|^{3/2}e^{i3\alpha/2}$. According to the 1-st convention the dominant wave S_m is formed by asymptotic series trancated near its least term, while the recessive wave R_m is represented by the remainder:

$$\mathrm{Ai}(z) = S_m(z) + R_m(z) =$$
$$= \frac{\sqrt{\pi}e^{-\zeta}}{2z^{1/4}}\sum_{k=0}^{m-1} c_k(-\zeta)^{-k} + \frac{2e^{-\frac{i\pi}{6}}}{\pi}\int_z^{\infty} dt f_m(t)\left[\mathrm{Ai}(z)\mathrm{Ai}(te^{-\frac{2\pi}{3}})-\right.$$
$$\left.-\mathrm{Ai}(t)\mathrm{Ai}(ze^{-\frac{2\pi}{3}})\right], \tag{8}$$

$f_m(t) = (-1)^m\sqrt{\frac{3\pi}{2}}mc_m\zeta^{-m-1/2}e^{-\zeta}$, $m = m(z)$ - number of the least term. For $z \gg 1$ the $m = m(z) \gg 1$ and it is possible to find the asymptotic expression for $R_m(z)$. The investigation shows that for asymptotic series whose terms behave with number k as

$$\Gamma(ak+b)(cz)^{-dk} \tag{9}$$

the ratio

$$\frac{R_m(z)}{S_m(z)} \sim i\frac{1}{2}\mathrm{erfc}(\xi)e^{f_1-f_2}, \tag{10}$$

where

$$\xi(z) \approx \frac{\mathrm{Im}(f_1 - f_2)}{\sqrt{2\mathrm{Re}(f_2 - f_1)}}. \tag{11}$$

So the recessive wave is switched on when $|\xi| \geq 1$, or when the phase difference of dominant and recessive waves becomes large:

$$|\mathrm{Im}(f_1 - f_2)| \geq \sqrt{2\mathrm{Re}(f_2 - f_1)} \gg 1. \tag{12}$$

3 Second convention and uncertainty relation

The 2-nd convention [4] based on contour integral representation and dividing contour integral at the height of recessive saddle. The dominant and recessive terms of $F(z)$ are nothing else than contributions of high and low saddles of the integrand. If t_2 and t_1 are the high and low saddle points and z is near the Stokes line then the steepest decent lines going over the t_2 and t_1 (SDL_2 and SDL_1) on the complex t-plane are represented on the fig.3 together with the level line (LL_1) of low saddle point t_1.

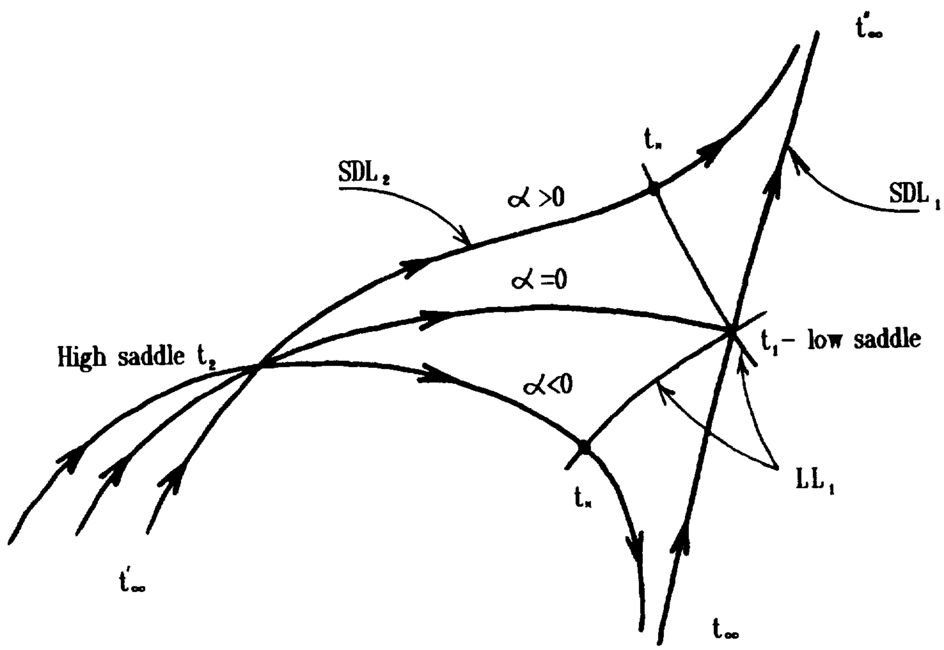

Figure 3:

The point $t_*{=}t_*(z)$ on the intersection of SDL_2 and LL_1 is a root of eqs.:

$$\begin{aligned} \mathrm{Im}f(t,z) &= \mathrm{Im}f_2, \\ \mathrm{Re}f(t,z) &= \mathrm{Re}f_1. \end{aligned} \tag{13}$$

The 2-nd convention defines the dominant D and recessive R terms of $F = D + R$ as integrals

$$D(z) = A\int_{t'\infty}^{t_*} dt e^{f(t,z)}, \qquad R(z) = A\int_{t_*}^{t''\infty} dt e^{f(t,z)} \tag{14}$$

Using in R the Taylor expansion for $f(t,z)$ near saddle point t_1

$$f(t,z) = f_1 + \frac{1}{2} f_1''(t-t_1)^2 + \dots \tag{15}$$

we obtain

$$R(z) \approx A\sqrt{\frac{-2\pi}{f_1''}} e^{f_1} \cdot \frac{1}{2}\mathrm{erfc}(w), \tag{16}$$

where

$$w(\alpha) = \sqrt{i\ \mathrm{Im}(f_1 - f_2)} \tag{17}$$

is a complex function of α. Hence, the switching function $g(\alpha)$ is complex and is given by a Fresnel type integral. The recessive wave switches on or off when $|w| \geq 1$ or

$$|\mathrm{Im}(f_1 - f_2)| \geq 1. \tag{18}$$

It is very natural condition: the phase difference of dominant and recessive waves is of the order of 1 or greater.

Near the Stokes line the phase is linear function of α:

$$\mathrm{Im} f_{1,2}(\alpha) = \mathrm{Im} f_{1,2}(0) - \omega_{1,2}(0) \cdot \alpha + \dots, \tag{19}$$

and

$$\mathrm{Im}(f_1 - f_2) = \Delta\omega \cdot \alpha + \dots, \qquad \Delta\omega = \omega_2(0) - \omega_1(0), \tag{20}$$

as on the Stokes line $\mathrm{Im} f_1(0) = \mathrm{Im} f_2(0)$. Two waves may be distinguished only outside of the Stokes line (Stokes strip)

$$|\mathrm{Im}(f_1 - f_2)| \geq 1 \quad \text{or} \quad \Delta\omega \cdot \Delta\alpha \geq 1, \tag{21}$$

when uncertainty relation is fulfilled. That is why the Stokes line width

$$\Delta\alpha \sim \frac{1}{\Delta\omega} \tag{22}$$

may be called natural.

In the first convention, where

$$F = S_m + R_m, \tag{23}$$

due to condition(12) or

$$\Delta\omega \cdot \Delta\alpha \geq \sqrt{2\mathrm{Re}(f_2 - f_1)} \gg 1 \tag{24}$$

the Stokes line width is much greater than natural

$$\Delta\alpha \sim \frac{\sqrt{2\mathrm{Re}(f_2 - f_1)}}{\Delta\omega} \gg \frac{1}{\Delta\omega}. \tag{25}$$

The slower formation of wave R_m in comparison with R is caused by the fact that the last $\sqrt{m} \sim \sqrt{2\mathrm{Re}(f_2 - f_1)} \gg 1$ terms in S_m are coherent to R_m and the disentanglement occurs slowly. This coherence disappears when α goes out of the more wider interval $\Delta\alpha \sim \sqrt{m}/\Delta\omega$ than natural one and then all recessive properties are concentrated in R_m.

4 T-parity and asymptotic expansions

T-transformation consists of the change $\alpha \to -\alpha$ and the complex conjugation. We consider here only the important case when functions $f_{1,2}$ in (2) satisfy the conditions

$$f_{1,2}(\alpha) \to f^*_{1,2}(-\alpha) = f_{1,2}(\alpha), \quad w(\alpha) \to w^*(-\alpha) = -w(\alpha), \tag{26}$$

and dominant wave D goes into itself, $D \to D + \delta$, up to unimportant phase factor and exponentially small additional term δ. One can say conditionally that D has positive T-parity. At the same time

$$R \sim ig(w)e^{if_1} \to -ig(-w)e^{if_1} \tag{27}$$

and does not have definite T-parity inside the Stokes strip because

$$g(-w) = 1 - g(w). \tag{28}$$

Yet outside the Stokes strip, when $|w| \gg 1$,

$$ig(w)e^{f_1} \approx \begin{cases} ie^{f_1 - w^2}/(2\sqrt{\pi}w), & \mathrm{Re}w \gg 1, \\ ie^{f_1} + ie^{f_1 - w^2}/(2\sqrt{\pi}w), & \mathrm{Re}w \ll -1. \end{cases} \tag{29}$$

Then before Stokes strip the R is $2\sqrt{\pi}|w|$ times less than its value R_S on the Stokes line, does not change at T-inversion and has the phase of dominant wave shifted by the $\arg(iw^{-1})$, as $f_1 - w^2 = \mathrm{Re}f_1 + i\mathrm{Im}f_2$. After the Stokes strip the wave $R \approx 2R_S$, has the proper phase $\mathrm{Im}f_1 + \pi/2$, changes its sign at T-inversion and is accompanied by small additional term of the same type as R itself was before the Stokes strip.

Therefore the Stokes strip is the forming region for recessive wave with frequency $\omega_1 \neq \omega_2$ and negative T-parity.

As to behaviour of dominant and recessive waves at T-transformation in the representation $F = S_m + R_m$, then for the examples considered in [4] the S_m transforms into itself up to the same factor as for D but without any additional term δ,i.e. $S_m \to S_m$, while the R_m behaves according to (29) with the change of w by real ξ, see (11). Therefore the phase of forming wave R_m equals to $\mathrm{Im}f_1 + \pi/2$ and its T-parity changes from positive to negative value being indefinite inside the Stokes strip. As $D = S_m + R_m - R$, the additional term δ having indefinite T-parity inside the wide Stokes strip vanishes outside it as $\delta \sim ie^{-\xi^2 + f_1}/2\sqrt{\pi}\xi$.

5 Stokes line width and the method of osculating parameters

It is instructive to see the appearance of Stokes width in the method of osculating parameters. According to this method the particular solution $y(\theta)$ of the differential equation of the second order with large parameter ν is sought as a superposition of quasiclassical solutions ${}^{\pm}f(\theta)$ with the correcting coefficient functions $a_{\pm}(\theta)$ defined by the relations

$$y(\theta) = a_+(\theta)\,{}^+f(\theta) + a_-(\theta)\,{}^-f(\theta), \tag{30}$$

$$y'(\theta) = a_+(\theta)\,{}^+f'(\theta) + a_-(\theta)\,{}^-f'(\theta), \tag{31}$$

and boundary condition

$$a_+(-\infty) = 1, \qquad a_-(-\infty) = 0. \tag{32}$$

The latter means that the solution in question is ${}_+y(\theta)$. As $a_{\pm}(\theta)$ are not differentiated in eq.(31) the differential equation of the second order is reduced to the system of two differential equations of the first order. This is sometimes useful for seeking out the appropriate approximation.

In physical literature there is a tendency to treat the two terms on the r.h.s. of (30) as two waves with $\pm$ frequencies for arbitrary θ and not only for $\theta \to +\infty$ (see [6] and references therein). This is done on the ground that quasiclassical solutions ${}^{\pm}f$ conserve the sign of frequency and the factors $a_{\pm}(\theta)$ should only correct the solutions. Yet this is true only in the case when ${}^+f(\theta)$, describing the strong wave, is taken with the accuracy up to the amplitude $a_-(\theta)$ of the weak wave $a_-(\theta)\,{}^-f(\theta)$, which under considered condition is exponentially small, for example $a_-(\theta) \sim e^{-\pi\nu}$.

To see this we note preliminarily that as follows from (31,32)

$$\begin{aligned} a_{\pm}(\theta) &= \pm\left(y(\theta)\,{}^{\mp}f'(\theta) - y'(\theta)\,{}^{\mp}f(\theta)\right)/D, \\ D &= {}^+f\,{}^-f' - {}^-f\,{}^+f'. \end{aligned} \tag{33}$$

We use now as an example the parabolic cylinder function $y(\theta)=CD_{i\nu-1/2}(-e^{-i\pi/4}2\sqrt{\nu}\theta)$. The constant C is fixed by the condition $a_+(-\infty) = 1$. The first terms of the asymptotic expansion of $y(\theta)$ in power series in ν^{-1} can be obtained by Darwin method [7].

For the n-th approximation we have

$$\begin{gathered} {}^+f_n(\theta) = e^{iS+\sigma_n}, \quad {}^-f = {}^+f^*, \quad S(\theta) = -\nu(\theta\sqrt{1+\theta^2} + \mathrm{Arsh}\theta), \\ \sigma_n = \sum_{k=0}^{n}(i\nu)^{-k}c_k(\theta), \quad c_0 = -\frac{1}{4}\ln(1+\theta^2). \end{gathered} \tag{34}$$

Here c_k are the real functions of θ, bounded for $k \geq 1$ together with their derivatives and satisfying the relation

$$c_k(-\theta) = (-1)^k c_k(\theta).$$

It follows from (33,34) that $a_-(\theta)\,{}^-f_n(\theta)$ consists of positive- and negative-frequency terms which have the form

$$a_-(\theta)\ {}^-f_n(\theta) = \frac{e^{iS+\sigma_0} c^{*\prime}{}_{n+1}(\theta)}{4(i\nu)^{n+2}\sqrt{1+\theta^2}} \left(1 + O(\nu^{-1})\right) -$$

$$-ie^{-\pi\nu} e^{-iS+\sigma_0} \left(1 + O(\nu^{-1})\right). \tag{35}$$

As seen from here $a_-\ {}^-f$ becomes approximately the negative-frequency wave only when the first term on the r.h.s. is much smaller than the second one:

$$\frac{c^{*\prime}{}_{n+1}(\theta)}{4\nu^{n+2}\sqrt{1+\theta^2}} \ll e^{-\pi\nu}. \tag{36}$$

In notation of [8]

$$c'_n(\theta) = -2(-i\nu)^{n+\frac{1}{2}} h_{3n} X^{-3n-2}. \tag{37}$$

One can show that for $\theta \gg 1$ the function $c'_n(\theta) \approx a_n 2^{-2n}\theta^{-2n-1}$, $a_n \approx 2^{n-1}\Gamma(n+1)$, $n \gg 1$. Then the condition (36) takes the form

$$\theta^2 \gg \frac{1}{2\nu}\left[\frac{\Gamma(n+2)}{4} e^{\pi\nu}\right]^{\frac{1}{n+2}}. \tag{38}$$

So for $n \sim \nu \gg 1$ we have $\theta^2 \gg 1$. It is seen that with each successive step in approximation for ${}^\pm f_n$ the width, in which positive- and negative-frequencies are not separated, shrinks quickly, but only at the step $n \sim \nu$ the width approaches the barrier one — a physically reasonable result.

References

[1] G. G. Stokes, Trans. Cambridge Philos. Soc., No 10, 106 (1864).

[2] M.V. Berry, Proc. R. Soc. London, Ser. A, **422**, 7 (1989), **427**, 265 (1990), **434**, 465 (1991).

[3] A. I. Nikishov, V.I. Ritus, in Sakharov Memorial Lectures in Physics. Proc. of the First Int. Sakharov Conf. on Physics, May21-31, 1991, Moscow, v.1, Nova Sci. Pub., New York.

[4] A. I. Nikishov, V.I. Ritus, Teor. Mathem. Fiz., **92**, 24 (1992).

[5] R. B. Dingle, *Asymptotic expansions: Their Derivation and Interpretation*, Academic Press, New York (1973).

[6] A. A. Grib, S. G. Mamaev, V. M. Mostepanenko, *Vacuum Quantum Effects in Strong Fields*, Energoatomizdat, Moscow (1988).

[7] M. Abramovitz, I. A. Stegun, *Handbook of Mathematical Functions with Formules, Grafs, and Mathematical Tables*, Dover (1964).

[8] J. C. P. Miller, in *Tables of Weber Parabolic Cylinder Functions*, Her Majesty's Stationary Office, London (1955).