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Coherent States on the m-sheeted Complex Plane as m-photon states

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Abstract

Coherent states on the m-sheeted complex plane are introduced and properties like overcompleteness and resolution of the identity are studied. They are eigenstates of the operators a_m^+, a_m which create and annihilate clusters of m-particles. Applications of this formalism in the study of Hamiltonians that describe m-particle clustering are also considered.

1. Introduction

Apart from the original (Glauber) coherent states which are associated with the Weyl group, other types of coherent states associated with other groups (e.g. SU(2), SU(1,1) etc.) have also been studied. In a recent publication [1] we extended these ideas in a different direction and introduced coherent states on the m-sheeted covering group of SU(1,1). From a physical point of view it can be used for the description of m-particle clustering. Here we extend the Glauber coherent states into coherent states

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in the m-sheeted complex plane. The properties of these states (overcompleteness, resolution of the identity etc.) are explicitly considered. Using these states we extend the Bargmann [2] analytic representation into a new formalism that we call Bargmann analytic representation in the m-sheeted complex plane. Using this representation we introduce new creation and annihilation operators a_m^+, a_m which create and annihilate clusters of m particles and show that the properties of our coherent states with respect to them, are similar to the properties of the ordinary (Glauber) coherent states with respect to the usual creation and annihilation operators a^+, a.

The above ideas are used in the description of m-particle clustering. which is a generalisation of the concept of pairing. They could be used to generalise two-photon states into m-photon states with even better properties. Some work in this direction but from a different point of view has already been presented [3, 4]

2. Coherent states on the m-sheeted complex plane

We consider the Riemann surface

$$R_m = C^* | Z_m \tag{1}$$

where C^* is the punctured complex plane

$$C^* = C - \{o\} \tag{2}$$

and Z_m is the discrete group of the integers modulo m. The punctured complex plane C^* is the m-sheeted covering surface of the Riemann surface R_m.

The sheet number s(z) of a complex number z in C^* is defined as

$$s(z) = IP \left[\frac{m\ Arg(z)}{2\pi}\right] \tag{3}$$

where IP stands for the integer part of the number. s(z) takes integer values from 0 to m - 1 (modulo m).

We also consider the harmonic oscillator Hilbert space H and express it as

$$H = \sum_{\ell=0}^{m-1} H_\ell \tag{4}$$

H_ℓ is an infinite-dimensional subspace spanned by the number eigenstates

$$H_\ell = \{ \,|\, Nm + \ell > \ ; \ N = 0, 1, 2, \ldots\ldots \} \tag{5}$$

We call π_ℓ the projection operators on H_ℓ

$$\pi_\ell = \sum_{N=0}^{\infty} |\, Nm + \ell > < Nm + \ell \,|$$

$$\pi_\ell \pi_j = \delta_{\ell j} \pi_\ell$$

$$\sum \pi_\ell = 1 \tag{6}$$

We now introduce the states

$$|z;m\rangle = \exp\left(-\frac{1}{2}\ |z|^{2m}\right) \sum_{N=0}^{\infty} z^{MN}\ (N!)^{-\frac{1}{2}}\ |\ mN + s(z)\ \rangle \tag{7}$$

where $s(z)$ is the sheet number of z defined in equ.(3). It is clear that if z belongs to the zero sheet the states $|mN\rangle$ should be used; if z belongs to the first sheet the states $|mN+1\rangle$ should be used; etc. The state $|z\rangle$ belongs to the Hilbert space $H_{s(z)}$. We refer to the states (7) as coherent states on the m-sheeted complex plane. We can prove

$$\langle z_1;m|\ z_2;m\rangle = \delta\ (s(z_1),\ s(z_2))\ \exp\left[-\frac{1}{2}\ |z_1|^{2m} - \frac{1}{2}\ |z_2|^{2m} + (z_1^{*} z_2)^{m}\right] \tag{8}$$

where δ is the kronecker delta.

In order to give a resolution of the identity for these states we first prove a resolution of the identity within the Hilbert space H_ℓ:

$$\int_{s_\ell} |z;m\rangle\ \langle z;m|\ d\mu_m(z) = \pi_\ell \tag{10}$$

$$d\mu_m\ (z) = \pi^{-1}\ m^2\ |z|^{2(m-1)}\ d^2z \tag{11}$$

s_ℓ is the ℓ-sheet and π_ℓ is the projection operator (6). Summation over ℓ gives the resolution of the identity:

$$\int_{c} |z;m\rangle\ \langle z;m|\ d\mu_m(z) = 1 \tag{12}$$

The states $|z;m\rangle$ with z in the sheet S_ℓ, form an overcomplete set within the Hilbert space H_ℓ.

3. Extended Bargman representation on the m-sheeted complex plane

Bargmann [2] introduced analytic representations in the complex plane which are based on ordinary coherent states. In refs. [5] analytic representations in the unit disc which are based on the SU(1,1) Perelomov coherent states, have been studied. This formalism has been extended in ref.[1] into analytic representations in the m-sheeted unit disc. In this section we study an analogous extension from the Bargmann representation in the complex plane into an analytic representation in the m-sheeted complex plane.

We generalise the Bargmann representation by representing the arbitrary (normalised) state $|f\rangle$ with the function

$$|f\rangle = \sum f_N \mid N\rangle$$

$$f(z;m) = \exp\left(\frac{1}{2}|z|^{2m}\right)\langle z^*;m|f\rangle = \sum_{N=0}^{\infty} f_{mN+s(z)}\, z^{mN}\,(N!)^{-\frac{1}{2}} \qquad (13)$$

where z takes values in C^* and s(z) is the sheet number of z (equ.(3)). The f(z;m) is analytic in the interior of each sheet and has discontinuities across the cuts C_ℓ.

As an example we consider the number eigenstates $|M = mN+\ell\rangle$ where N, ℓ are the integer part and remainder of M divided by m, correspondingly. They are represented by the function

$$f(z;m) = \delta(\ell,s(z))\ z^{mN}\ (N!)^{-\frac{1}{2}} \tag{14}$$

The kronecker $\delta(\ell,s(z))$ ensures that this function is non-zero only in the ℓ sheet.

As a second example we consider the states $|z_o;m\rangle$ of equ.(7) which are represented by the function

$$f(z;m;z_o) = \exp\left(\frac{1}{2}\ |z|^{2m}\right) \langle z^*;m|z_o;m\rangle =$$

$$\delta(s(z);s(z_o))\ \exp\left[-\frac{1}{2}\ |z_o|^{2m} + (zz_o)^m\right] \tag{15}$$

We next consider the operators

$$a_m = m^{-1}\ z^{1-m}\ \partial_z \tag{16}$$

$$a_m^+ = z^m \tag{17}$$

$$[a_m,\ a_m^+] = 1 \tag{18}$$

$$a_m^+\ |\ mN + \ell\ \rangle = (N+1)^{\frac{1}{2}}\ |\ m(N+1) + \ell\ \rangle \tag{19}$$

$$a_m\ |\ mN + \ell\ \rangle = N^{\frac{1}{2}}\ |\ m(N-1) + \ell\ \rangle \tag{20}$$

It is seen that they act as creation and annihilation operator within each one of the Hilbert spaces H_ℓ.

The operator

$$a_m^+ a_m = m^{-1} z\partial_z$$

$$a_m^+ a_m \mid mN + \ell > = N \mid mN + \ell > \quad (21)$$

can be considered as a number operator within the Hilbert space H_ℓ.

We also consider the operator

$$R = a^+a - m\, a_m^+ a_m \quad (22)$$

$$R\ |mN+\ell> = \ell\ |mN+\ell> \quad (23)$$

which we call "remainder operator" or "number modulo m operator". Its eigenstates are the number states and its eigenvalues the remainders ℓ of the division of the number of the state over m. The operator R commutes with the operators a_m, a_m^+:

$$[R,\ a_m] = [R,\ a_m^+] = 0 \quad (24)$$

Note that the operators a_m, a_m^+ commute with the projection operators π_ℓ of equ.(7):

$$[a_m,\ \pi_\ell] = [a_m^+,\ \pi_\ell] = 0 \quad (25)$$

A consequence of that is that an "arbitrary" function of a_m, a_m^+ leaves each of the Hilbert spaces H_ℓ invariant in the sense that when it acts on a state which belongs in H_ℓ it produces another state which also belongs in the same space.

We next consider the displacement operators with respect to the a_m^+, a_m

$$D_m(z_o) = \exp\, [z_o\, a_m^+ - z_o^*\, a_m] \tag{26}$$

They displace the operators a_m, a_m^+ by a constant and they commute with the operator R or equ.(22)

$$D_m(z_o)\ a_m\ D_m^+\ (z_o) = a_m - z_o \tag{27}$$

$$D_m(z_o)\ a_m^+\ D_m^+\ (z_o) = a_m^+ - z_o^* \tag{28}$$

$$[D_m\ (z_o),\ R] = 0 \tag{29}$$

We now act with the operators $D_m(z_o)$ on the number eigenstates $|\ell\rangle$ ($0 \le \ell \le m-1$) and get the coherent states on the m-sheeted complex plane (7):

$$|z = (z_o)_\ell^{1/m}\ ;\ m\rangle = D_m\ (z_o)\ |\ \ell\ \rangle \tag{30}$$

The subscript ℓ indicates that among the m roots, the one which belongs to the sheet S_ℓ should be chosen. Equ.(30) can also be writen as

$$|z;m\rangle = D_m\ (z^m)\ |s(z)\rangle \tag{31}$$

where $|s(z)\rangle$ is number eigenstate and s(z) the sheet number of z (equ.(3)).

Using equs.(27), (31) we prove that the $|z;m\rangle$ are eigenstates of a_m

$$a_m\ |z;m\rangle = z^m|z;m\rangle \tag{32}$$

It should be pointed out that operators similar to a_m^+, a_m have been considered in [6] and used in refs. [4] where the following states have been studied

$$\exp\left(-\frac{1}{2}|z|^2\right)\sum_{N=0}^{\infty} z^N (N!)^{-\frac{1}{2}} |mN\rangle \tag{33}$$

They are a subset of our coherent states associated with the Hilbert space H_ℓ with $\ell = 0$; or equivalently with the zero sheet of our m-sheeted complex plane. It has been shown in [4] that the states (33) have very interesting quantum statistical properties.

4. m-photon states

We consider the Hamiltonian

$$H = \Omega a^+ a + \tau a_m^+ + \tau^* a_m = \Omega R + H_1 \tag{34}$$

$$H_1 = \Omega m a_m^+ a_m + \tau a_m^+ + \tau^* a_m \tag{35}$$

$$[R, H_1] = 0 \tag{36}$$

and express H as:

$$H = D_m\left(-\frac{\tau}{\Omega m}\right)\left[\Omega m a_m^+ a_m\right] D_m^+\left(-\frac{\tau}{\Omega m}\right) + \Omega R - \frac{|\tau|^2}{\Omega m} \tag{37}$$

We easily see that the eigenvectors and eigenvalues of H are:

$$H D_m\left(-\frac{\tau}{\Omega m}\right)|mN+\ell\rangle = \left[\Omega(mN+\ell) - \frac{|\tau|^2}{\Omega m}\right] D_m\left(-\frac{\tau}{\Omega m}\right)|mN+\ell\rangle \tag{38}$$

The physical significance of this Hamiltonian lies in the fact that the operators a_m^+, a_m create and annihilate clusters of m particles. Pairing of particles plays a very important role in various contexts in Physics. Generalisation from pairing into m-particle clustering can be described with various Hamiltonians. The obvious choice is

$$H = \Omega\, a^+ a + r\, (a^+)^m + r^* a^m \tag{39}$$

There are certain difficulties associated with this Hamiltonian [3] and in any case it is useful to explore alternative models, especially if they are based on some symmetry which can be exploited to handle these highly non-linear terms. In ref. [1] a Hamiltonian associated with the SU(1,1) group, which describes m-particle clustering has been studied. In this section the Hamiltonian (34) which is associated with the Weyl group and which also describes m-particle clustering, has been studied.

5. Discussion

Coherent states on the m-sheeted complex plane have been introduced in equ.(7). The Hilbert space has been split into m subspaces (equ.(4)) and operators a_m^+, a_m which play the role of creation and annihilation operators in each subspace, have been introduced. The a_m^+, a_m are different from the usual creation and annihilation operators a^+, a. It has been shown that our coherent states have the usual properties of coherent states with respect to the a_m^+, a_m. They are eigenstates of a_m (equ.(32)); and they can be expressed as the product of the displacement operator times the lowest state (equ.(31)).

All these ideas can be used for the description of m-particle clustering. This is a generalisation of the concept of pairing which plays an important role in areas like squeezing in quantum optics, superconductivity, superfluidity, phase transistions etc. Consequently this formalism might be used for generalisations in all these areas. A Hamiltonian that describes m-particle clustering has been considered in equ.(34) and its eigenvalues and eigenfunctions have been calculated.

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