PSEUDOMASTER EQUATION FOR THE NO-COUNT PROCESS IN A CONTINUOUS PHOTODETECTION*

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The detection of cavity radiation with the detector placed outside the cavity is studied. Each leaked photon has a certain probability of propagating away without being detected. It is viewed as a continuous quantum measurement in which the density matrix is continuously revised according to the readout of the detector. The concept of pseudomaster equation for the no-count process is introduced; its solution leads to the discovery of the superoperator for the same process. It has the potential to become the key equation for continuous measurement process.

Quantum theory of measurement has been a highly controversial topic ever since the pioneer work of von Neumann. The process discussed in this original work is now classified as the first-kind (nondestructive) measurement, which is described by an instantaneous projection with projection operators as its principal mathematical tools. On the other hand, photodetection is a second-kind (destructive) measurement. To deal with photodetection the concept of quantum measurement must be extended to include continuous measurement process in which the quantum state of the subject system must be continuously revised to reflect our most up-to-date knowledge about the system resulting from the ongoing measurement process. Superoperators have been the principal mathematical tools for such study. It is the purpose of this paper to introduce a simple equation which has the potential to become the basic equation to describe the continuous measurement process.

There have been two different approaches to the theory of photoelectric detection. The first approach was initiated by Mandel [1] and followed by Kelley and Kleiner [2] and by Glauber [3]. The second approach was initiated by Mollow [4] and followed by Scully and Lamb [5], by She pherd [6], and by Srinivas and Davies [7]. In the first approach, light propagates in the open space, it encounters the photodetector, and any unabsorbed photons propagate away. In the second approach, both the radiation and the photodetector are enclosed in a cavity, and any photons not absorbed by the detector at one time are available for detection at later times. For obvious reason these two different approaches are classified by Mandel [8] as *open-system model* for the former and *closed-system model* for the latter.

The formula for the photoelectron distribution obtained in the open-system model was criticized by Srinivas and Davies [7] as being a short-time approximation only, it may lead to unphysical and meaningless results when time is large. In his response, Mandel [8] pointed out that, for the open-system model, it should be understood that the normalization volume must be large enough to satisfy the condition $L \gg ct$ at all times. He also pointed out that the circumstances modelled by the closed system are much less commonly encountered in practice.

In this paper we introduce a new model that is the hybrid of the two previous approaches. The radiation source is kept inside a cavity but the photodetector is taken outside of the cavity to detect only those photons leaked out of one end of the cavity, and any unabsorbed photons propagate away. We believe that this hybrid model not only keeps the merits of both previous models but also is a closer simulation of the circumstances more frequently encountered in practice. For example, if we want to study the statistical properties of photons in the output of a laser system, this hybrid model will describe the situation most closely.

In 1981 Srinivas and Davies [7] first considered the closed-system photodetection as a continuous quantum measurement. The information provided by the readout of the detector is used to adjust the density matrix continuously so that it always represents our most current knowledge about the cavity radiation. The counting process consists of *no-count* and *one-count* processes, the former lasts a finite duration while the latter occurs instantaneously. The effects of the two processes on the density matrix can be represented by two *superoperators* postulated to be as follows: (i) The one-count process is described by the superoperator $\mathcal{J}$ such that

$$\hat{\rho}(t^+) = \mathcal{J}\hat{\rho}(t) \equiv \frac{\hat{a}\hat{\rho}(t)\hat{a}^\dagger}{\text{Tr}\left[\hat{\rho}(t)\hat{a}^\dagger\hat{a}\right]}. \tag{1}$$

where $\hat{\rho}(t)$ and $\hat{\rho}(t^+)$ are the density matrix for the radiation field immediately before and after the detection of a photoelectron while $\hat{a}$ and $\hat{a}^\dagger$ are the annihilation and creation operators, respectively, of a photon. (ii) The no-count process lasting for a duration τ is described by the superoperator $\mathcal{S}_\tau$ such that

$$\hat{\rho}(t + \tau) = \mathcal{S}_\tau\hat{\rho}(t) \equiv \frac{\exp\left(-\frac{\lambda}{2}\hat{a}^\dagger\hat{a}\tau\right)\hat{\rho}(t)\exp\left(-\frac{\lambda}{2}\hat{a}^\dagger\hat{a}\tau\right)}{\text{Tr}\left[\hat{\rho}(t)\exp\left(-\lambda\hat{a}^\dagger\hat{a}\tau\right)\right]}, \tag{2}$$

where λ is the coupling constant. It is obvious from Eqs. (1) and (2) that the evolution of the quantum state for the cavity radiation under continuous photon counting is nonunitary. These two postulates were verified very recently by Imoto *et al.* [9] by using some specific microscopic model. The system consists of a cavity radiation and a stream of two-level atoms described by the well-known Jaynes-Cumming Hamiltonian. More detailed theory has been further developed by Ueda *et al.* [10].

These two superoperators are valid for the closed-system model. To find the corresponding superoperators for the hybrid model is the initial motivation of the present study. We have discovered a general method to solve this type of problems, and we shall call it the method of *pseudomaster equation.*

For simplicity, we assume that the radiation inside the cavity is kept inactive except for possible leakage through one side of the cavity due to the less than perfect reflectivity of the mirror. Let μ be the rate of photon leakage or the inverse of the decay time. It can be expressed in terms of the speed of light c, the reflectivity of one end mirror R and the distance between the two end mirrors d as [11]

$$\mu = -c\ln R/2d. \tag{3}$$

Let $p(n, t)$ be the probability that n photons remain inside the cavity at time t. Except for the difference in the physical meaning of the constant, the master equation for $p(n, t)$ in the hybrid model is exactly the same as the one first derived by Scully and Lamb [5] for the closed-system model, *i.e.,*

$$\frac{d}{dt}p(n, t) = -n\mu p(n, t) + (n + 1)\mu p(n + 1, t), \tag{4}$$

which describes the so-called *non-referring measurement process*, namely, the "free" evolution of the photon-number distribution without detection or without any adjustment of the quantum state according to the readout of the detector.

Assuming each leaked photon has the probability ζ of being detected, then we believe the no-count process can be described by the following difference-differential equation

$$\frac{d}{dt}q(n, t) = -n\mu q(n, t) + (n + 1)\mu(1 - \zeta)q(n + 1, t), \tag{5}$$

where $(1 - \zeta)$ is the probability that a leaked photon propagates away without been detected. Equation (5) is a modification of Eq. (4). One important characteristic of Eq. (5) is that it does not conserve the total probability. Because of this characteristic we call it pseudomaster equation. The physical meaning of $q(n, t)$ will become clear in later development.

By the way, the pseudomaster equation for the closed-system model is

$$\frac{d}{dt} q(n, t) = -\lambda n q(n, t),\tag{6}$$

which is quite trivial; so we shall focus our attention on Eq. (5) for the hybrid model.

The solution to Eq. (5) can be easily obtained through Laplace transformation. Let the no-count period begin at $t = t_1$ and let us make the identification

$$q(n, t_1) = p(n, t_1)\tag{7}$$

as the initial condition, then the solution to Eq. (5) at a later time $t = t_2$ can be written as

$$q(n, t_2) = \sum_{\ell=n}^{\infty} p(\ell, t_1) \binom{\ell}{n} \left\{ (1 - \zeta)\left[1 - e^{-\mu(t_2 - t_1)}\right] \right\}^{\ell-n} e^{-n\mu(t_2 - t_1)}.\tag{8}$$

The first physical meaning of $q(n, t)$ is that it is the unnormalized form of $p(n, t)$ in no-count process, namely,

$$p(n, t) = q(n, t) \left/ \sum_{n=0}^{\infty} q(n, t).\right.\tag{9}$$

Let $P_m(t_1, t_2)$ be the probability that m photoelectrons are detected during the period (t_1, t_2). We are particularly interested in the special case $m = 0$. We can obtain from Eq. (8) the probability that no photoelectron is detected during the period (t_1, t_2) as

$$P_0(t_1, t_2) = \sum_{n=0}^{\infty} q(n, t_2) = \sum_{\ell=0}^{\infty} p(\ell, t_1)\left[1 - \zeta + \zeta e^{-\mu(t_2 - t_1)}\right]^{\ell}.\tag{10}$$

This is the second physical meaning of $q(n, t)$.

To be consistent, the no-count probability must satisfy the condition

$$P_0(t_1, t_2) P_0(t_2, t_3) = P_0(t_1, t_3).\tag{11}$$

Using Eqs. (8) and (10) in Eq. (9) we obtain

$$p(n, t_2) = \frac{1}{P_0(t_1, t_2)} \sum_{\ell=n}^{\infty} p(\ell, t_1) \binom{\ell}{n} \left[(1 - \zeta)\left(1 - e^{-\mu(t_2 - t_1)}\right)\right]^{\ell-n} e^{-n\mu(t_2 - t_1)}.\tag{12}$$

Replacing t_1 and t_2 in Eq. (10) by t_2 and t_3, respectively, and using Eq. (12), we obtain

$$P_0(t_2, t_3) = \sum_{\ell=0}^{\infty} p(\ell, t_2)\left[1 - \zeta + \zeta e^{-\mu(t_3 - t_2)}\right]^{\ell}$$

$$= \frac{1}{P_0(t_1, t_2)} \sum_{\ell=0}^{\infty} p(\ell, t_1)\left[1 - \zeta + \zeta e^{-\mu(t_3 - t_1)}\right]^{\ell} = \frac{P_0(t_1, t_3)}{P_0(t_1, t_2)};\tag{13}$$

so the condition of Eq. (11) is satisfied.

There are two kinds of waiting times, conditional and unconditional. We pick an arbitrary instant and ask: "How long do we have to wait until we detect a photoelectron?" This is the unconditional waiting time. If the waiting begins immediately after the detection of a photoelectron, then it is called conditional waiting time. From the expression for $P_0(t_1, t_2)$ we can also derive the unnormalized distributions for the two different waiting times according to [12]

$$W(t_1, t_2) = -\frac{\partial}{\partial t_2} P_0(t_1, t_2) = \mu\zeta e^{-\mu(t_2-t_1)} \sum_{n=1}^{\infty} p(n, t_1)n\left[1 - \zeta + \zeta e^{-\mu(t_2-t_1)}\right]^{n-1} \quad (14)$$

and

$$V(t_1, t_2) = -\frac{1}{\langle n\rangle_{t_1}} \frac{\partial^2}{\partial t_1 \partial t_2} P_0(t_1, t_2)$$
$$= \frac{\mu\zeta}{\langle n\rangle_{t_1}} e^{-\mu(t_2-t_1)} \sum_{n=2}^{\infty} p(n, t_1)n(n-1)\left[1 - \zeta + \zeta e^{-\mu(t_2-t_1)}\right]^{n-2}, \quad (15)$$

where $\langle n\rangle_t$ is the average number of photons remaining at time t.

The difference between unconditional and condition waiting times is that the latter begins immediately after the detection of a photons. Therefore we can see the effect on the quantum state of the radiation due to the detection of a photon by comparing Eqs. (14) and (15). If we replace $p(n, t_1)$ in Eq. (14) by

$$p(n, t_1^+) = \frac{n+1}{\langle n\rangle_{t_1}} p(n+1, t_1), \quad (16)$$

we obtain Eq. (15). On the other hand, if we try to use Eq. (1), we also have

$$p(n, t_1^+) = \langle n|\hat{\rho}(t_1^+)|n\rangle = \frac{\langle n|\hat{a}\hat{\rho}(t_1)\hat{a}^\dagger|n\rangle}{\langle n\rangle_{t_1}} = \frac{n+1}{\langle n\rangle_{t_1}} p(n+1, t_1), \quad (17)$$

which is identical to Eq. (16). So we conclude that the effect on the quantum state due to the one-count process in the hybrid model is the same as that in the closed-system model, and the superoperator for the one-count process in hybrid model remains the same as given in Eq. (1).

We now try to find the superoperator for the no-count process. Equation (12) provides the clue for this search. It turns out to be quite sophisticated because it involves the exponential of a superoperator. First let us define a superoperator $\mathcal{R}(\tau, \mu, \zeta)$ such that

$$\mathcal{R}(\tau, \mu, \zeta)\hat{\rho}(t) \equiv (1 - \zeta)(1 - e^{-\mu\tau})\hat{a}\hat{\rho}(t)\hat{a}^\dagger. \quad (18)$$

Then we can define the exponential of this superoperator as

$$\exp\left[\mathcal{R}(\tau, \mu, \zeta)\right]\hat{\rho}(t) \equiv \sum_{\ell=0}^{\infty} \frac{\mathcal{R}^\ell(\tau, \mu, \zeta)}{\ell!}\hat{\rho}(t) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!}[(1 - \zeta)(1 - e^{-\mu\tau})]^\ell(\hat{a})^\ell\hat{\rho}(t)\left(\hat{a}^\dagger\right)^\ell, \quad (19)$$

which must be applied to both sides of the operand in lock step and cannot be split up into two separate parts, with one being the Hermitian conjugate of the other. Let us define another superoperator $\mathcal{S}(\tau, \mu)$ such that

$$\mathcal{S}(\tau, \mu)\hat{\rho}(t) \equiv \exp\left(-\frac{\mu}{2}\hat{a}^\dagger\hat{a}\tau\right)\hat{\rho}(t)\exp\left(-\frac{\mu}{2}\hat{a}^\dagger\hat{a}\tau\right), \quad (20)$$

which is slightly different from that defined in Eq. (2) in the physical meaning of the constant μ and in that it does not include the renormalization. We are now ready to present the superoperator $\mathcal{Q}(\tau, \zeta)$ for the no-count process such that

$$\mathcal{Q}(\tau, \mu, \zeta)\hat{\rho}(t) \equiv \mathcal{S}(\tau, \mu) \exp\left[\mathcal{R}(\tau, \mu, \zeta)\right]\hat{\rho}(t). \tag{21}$$

Then the density operator at the end of the no-count process can be written as

$$\hat{\rho}(n, t + \tau) = \mathcal{Q}_\mathcal{N}(\tau, \mu, \zeta)\hat{\rho}(t) \equiv \frac{\mathcal{Q}(\tau, \mu, \zeta)\hat{\rho}(t)}{\mathrm{Tr}\{\mathcal{Q}(\tau, \mu, \zeta)\hat{\rho}(t)\}}, \tag{22}$$

where the difference between the two superoperators $\mathcal{Q}(\tau, \mu, \zeta)$ and $\mathcal{Q}_\mathcal{N}(\tau, \mu, \zeta)$ is that the later includes the renormalization.

It should be pointed out that this superoperator must satisfy the following conditions:

$$q(n, t + \tau) = \langle n|\mathcal{Q}(\tau, \mu, \zeta)\hat{\rho}(t)|n\rangle, \tag{23}$$

$$P_0(t, t + \tau) = \mathrm{Tr}\{\mathcal{Q}(\tau, \mu, \zeta)\hat{\rho}(t)\}, \tag{24}$$

and

$$\mathcal{Q}(\tau_1 + \tau_2, \mu, \zeta) = \mathcal{Q}(\tau_2, \mu, \zeta)\mathcal{Q}(\tau_1, \mu, \zeta). \tag{25}$$

The last semigroup condition is very critical because we have found at least two less sophisticated expressions for $\mathcal{Q}(\tau, \mu, \zeta)$, which satisfy the first two conditions, but must be abandoned because they do not satisfy the last condition. It can be easily verify that the expression given in Eq. (21) satisfy the first two condition. That it also satisfies the last condition can be shown as follows:

$$\mathcal{Q}(\tau_2, \mu, \zeta)\mathcal{Q}(\tau_1, \mu, \zeta)\hat{\rho}(t)$$

$$= \sum_{k=0}^\infty \sum_{\ell=0}^\infty \frac{(1-\zeta)^{k+\ell}}{k!\ell!}\left(1 - e^{-\mu\tau_2}\right)^k\left(1 - e^{-\mu\tau_1}\right)^\ell e^{-\mu\hat{a}^\dagger\hat{a}\tau_2/2}(\hat{a})^k e^{-\mu\hat{a}^\dagger\hat{a}\tau_1/2}(\hat{a})^\ell$$

$$\times \hat{\rho}(t)\left(\hat{a}^\dagger\right)^\ell e^{-\mu\hat{a}^\dagger\hat{a}\tau_1/2}\left(\hat{a}^\dagger\right)^k e^{-\mu\hat{a}^\dagger\hat{a}\tau_2/2}$$

$$= \sum_{k=0}^\infty \frac{(1-\zeta)^k}{k!}\left[1 - e^{-\mu(\tau_1+\tau_2)}\right]^k e^{-\mu\hat{a}^\dagger\hat{a}(\tau_1+\tau_2)/2}(\hat{a})^k \hat{\rho}(t)\left(\hat{a}^\dagger\right)^k e^{-\mu\hat{a}^\dagger\hat{a}(\tau_1+\tau_2)/2}$$

$$= \mathcal{Q}(\tau_1 + \tau_2, \mu, \zeta)\hat{\rho}(t), \tag{26}$$

where we used the operator identity [13] $e^{x\hat{a}^\dagger\hat{a}}f(\hat{a}, \hat{a}^\dagger)e^{-x\hat{a}^\dagger\hat{a}} = f(\hat{a}e^{-x}, \hat{a}^\dagger e^x)$ to change the orders of some operators and used the summation identity $\sum_{k=0}^\infty \sum_{\ell=0}^\infty A(k, \ell) = \sum_{k=0}^\infty \sum_{\ell=0}^k A(k - \ell, \ell)$ to carry out one summation.

For a perfect detector with $\zeta = 1$, we have $\mathcal{Q}_\mathcal{N}(\tau, \lambda, 1) = \mathcal{S}_\tau$ as given in Eq. (2) for the closed-system model. This means the present model with perfect detector is identical to the closed-system model as far as mathematics is concerned. On the other hand, when $\zeta = 0$ we obtain the superoperator for the non-referring measurement process, which always preserves the total probability, i.e., $\mathrm{Tr}[\mathcal{Q}(\tau, \mu, 0)\hat{\rho}(t)] = 1$; so no renormalization is necessary.

We can also consider a more general "closed-system" model with the detector inside the cavity but the cavity has some leakage. The pseudomaster equation for the no-count process of such a system can be written as

$$\frac{d}{dt}q(n, t) = -n(\lambda + \mu)q(n, t) + (n + 1)\mu q(n + 1, t), \tag{27}$$

where λ is the coupling constant between the radiation field and the cavity and μ denotes the rate of photon leakage. Except for a slight change of constants, the calculations of this generalized closed-system model are almost exactly the same as the hybrid model. We just list the most significant results as follows:

$$q(n, t_2) = \sum_{\ell=n}^{\infty} p(\ell, t_1) \binom{\ell}{n} \left\{ \frac{\mu}{\lambda + \mu} \left[1 - e^{-(\lambda+\mu)(t_2-t_1)} \right] \right\}^{\ell-n} e^{-n(\lambda+\mu)(t_2-t_1)}, \quad (28)$$

$$P_0(t_1, t_2) = \sum_{\ell=0}^{\infty} p(\ell, t_1) \left[\frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} e^{-(\lambda+\mu)(t_2-t_1)} \right]^{\ell}, \quad (29)$$

and

$$Q(\tau, \lambda, \mu) = S(\tau, \lambda + \mu) \exp\left[\mathcal{R}(\tau, \lambda + \mu, \lambda/(\lambda + \mu)) \right]. \quad (30)$$

In conclusion, we have introduced a hybrid model for photodetection which is the mixture of the previous open-system and closed-system models. It is without the defects of the former and more realistic than the latter. We have also introduced the concept of pseudomaster equation, the solution of which provides the clue to discovering the superoperator for the no-count process. It is obvious that, comparing with the superoperator approach, the method of pseudomaster equation is simpler, easier to see how to write it down and easier to handle. So we believe it has the potential to become the fundamental equation for analyzing various models of photodetections as continuous quantum measurement process.

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[1] L. Mandel, Proc. Roy. Soc. London 72, 1037 (1958); ibid. 74, 233 (1959).
[2] P. C. Kelley and W. H. Kleiner, Phys. Rev. A136, 316 (1964).
[3] R. J. Glauber, in: Quantum Optics and Electronics, ed. by C. deWitt, A. Blandin, and C. Cohen-Tannoudji, Gordon and Breach, New York, 1965, p. 63.
[4] B. R. Mollow, Phys. Rev. 168, 1896 (1968).
[5] M. O. Scully and W. E. Lamb, Phys. Rev. 179, 368 (1969).
[6] T. J. Shepherd, Optica Acta 28, 567 (1981); ibid. 31, 1399 (1984).
[7] M. D. Srinivas and E. B. Davies, Optica Acta 28, 981 (1981); ibid. 29, 235 (1982).
[8] L. Mandel, Optica Acta 28, 1447 (1981).
[9] N. Imoto, M. Ueda and T. Ogawa, Phys. Rev. A 41, 4127 (1990).
[10] M. Ueda, Quantum Optics 1, 131 (1981); Phys. Rev. A 41, 3891 (1990); M. Ueda, N. Imoto and T. Ogawa, ibid. 41, 3891 (1990).
[11] For a rigorous derivation of this expression see, for example, C. T. Lee, Opt. Communi. 27, 277 (1978). The expression for the decay time is given in Eq. (33) as $\tau = -2\ell/c \ln R$, but it should be pointed out that this formula is for leakage through both ends and the separation between the two ends is 2ℓ.
[12] C. T. Lee, Phys. Rev. A 46, 6100 (1992).
[13] See, for example, W. H. Louisell, Quantum Statistical Properties of Radiation, John Wiley & Sons, New York, 1973, p. 154.