BERRY PHASE IN HEISENBERG REPRESENTATION

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Abstract

We define the Berry phase for the Heisenberg operators. This definition is motivated by the calculation of the phase shifts by different techniques. These techniques are: the solution of the Heisenberg equations of motion, the solution of the Schrödinger equation in coherent-state representation, and the direct computation of the evolution operator. Our definition of the Berry phase in the Heisenberg representation is consistent with the underlying super-symmetry of the model in the following sense. The structural blocks of the Hamiltonians of supersymmetrical quantum mechanics ("superpairs") [1,2] are connected by transformations which conserve the similarity in structure of the energy levels of superpairs. These transformations include transformation of phase of the creation-annihilation operators, which are generated by adiabatic cyclic evolution of the parameters of the system.

1 INTRODUCTION

The equivalence of the Schrödinger and Heisenberg pictures in quantum mechanics is something that is taken for granted. The specific choice of the Schrödinger, Heisenberg or interaction picture is usually regarded as a matter of convenience. Berry phase was defined initially and mostly for the Schrödinger picture as nonintegrable phase factor appearing in the wave-function after (adiabatic) evolution of the system's parameters. Here we investigate how one should define the analog of the Berry phase in the Heisenberg representation.

The traditional introduction to Berry phase includes the following construction. The Hamiltonian for the particular system $H(\vec{\lambda})$ is introduced, where the set of parameters, $\vec{\lambda}$, is considered to be changing adiabatically with time. Until now, most of the applications had in mind the discrete, though maybe degenerate, spectrum of the Hamiltonian, for all values of the paramters. Then, after the adiabatic cyclic evolution of the parameters, each eigenstate ψ_n of a corresponding stationary problem

$$H(\vec{\lambda})\psi_n = E_n(\vec{\lambda})\psi_n \tag{1}$$

acquires a nonintegrable phase: $\psi_n \rightarrow e^{i\phi_n}\psi_n$. This phase usually admits geometric interpretation in terms of some contour in parameter space.

In the Heisenberg representation, the operators of observables, and not the eigenstates, do evolve in time. One can assume, that for particular observable, $a(t)$, the cyclic evolution of parameters results in the following relation after the period, T:

$$a(T) = S^\dagger a(0) S, \tag{2}$$

where S is a unitary operator.

One supposes, that the operator, $S^\dagger$, in a basis of the eigenfunctions of the Eq. (1) has the form:

$$S^\dagger = \begin{pmatrix} e^{i\phi_0} & 0 & 0 & \cdots \\ 0 & e^{i\phi_1} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & e^{i\phi_n} & \cdots \end{pmatrix} \tag{3}$$

It represents what can be naturally called matrix of a Berry phase in Heisenberg picture, although now it is a *unitary operator*, not a number. The rest of this paper is dedicated to demonstrating the usefulness of this notion.

We shall demonstrate for a simple exactly solvable model that this operator can be defined consistent with the (super)symmetry of a model. We cannot prove it in a general way, though we believe that the following proposition is true for any Hamiltonian H, for which Berry phase could be defined. Namely, if there exists the Hamiltonian, $H_1(\vec{\lambda}(t))$, after adiabatic cyclic transformation of its parameters it will be transformed into new Hamiltonian, H_2, which is a *superpair* to the initial Hamiltonian, H_1. The adiabatic theorem reappears in this approach as the identity of the eigenvalues of H_1, and H_2 (however, the degeneracy of zero eigenvalue, in general, may change). The wavefunctions, may, however undergo some unitary transformation.

This is in accord with general ideology of supersymmetric quantum mechanics, which usually includes the compound Hamiltonian, $\mathbf{H}$, formed by two simpler Hamiltonians, H_1 and H_2 [1,2,3,4],

$$\mathbf{H} = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} \tag{4}$$

The properties of the Hamiltonians, H_1 and H_2, are closely related. Normally, these Hamiltonians have identical spectral structure, but, perhaps, only a finite number of energy levels. Consequently, almost all of the levels of the Hamiltonian $\mathbf{H}$ are doubly degenerate.

Previously [5], in this context, we studied the supersymmetric structure of the Jaynes-Cummings Model (JCM). The formal introduction of the supersymmetry in the JCM includes the replication of the JCM with a trivial phase transformation performed on the creation-annihilation operators of the boson:

$$a \rightarrow e^{i\phi}a, a^\dagger \rightarrow a^\dagger e^{-i\phi} \tag{5}$$

When ϕ is a real function of time, the Hamiltonians, H_1 and H_2, have identical spectra, and all the energy levels of the Hamiltonian H are doubly degenerate.

The physical meaning of the phase shift, ϕ, is not clear. It would be desirable to interpret it, analogous to Berry's interpretation of the Aharonov-Bohm effect [6], as a manifestation of a Berry

phase. Indeed, the JCM has a nontrivial Berry phase with potentially observable ramifications [7,8]. The interpretation of the phase shift (5) as Berry phase for some cyclic evolution in the JCM is not yet proven, however we demonstrate that this is true for somewhat simpler model of Section 3.

Thus, for our purposes we seek the transformations of phases of the Hermitian operators rather than wavefunctions. Therefore, we calculate Berry phase for our model using both the Schrödinger formalism for the wavefunctions and the Heisenberg formalism for the operators. Also, we calculate the phase shifts by an explicit expression of the evolution operator through SU(1,1) group operators. Both approaches are shown to be equivalent as the result of our analysis.

Because performing the adiabatic cyclic evolution of the system can be, under certain conditions, a valid quantum measurement, the result of the paper can be put in other form. Namely, the distinguishability of the systems described by the Hamiltonians forming a superpair is equivalent to the nontriviality of the Berry phase obtained during a cyclic evolution of the system's parameters. The separate measurements of the Berry phase of those two systems during the evolution distinguishes these systems.

The structure of paper is as follows. In Section II, we make a general definition of the cyclic evolution in the Heisenberg representation. In Section III, the supersymmetric family of Hamiltonians, similar to the Hamiltonian for degenerate parametric two-photon optical interaction, is considered. The calculation of the evolution of the Heisenberg operators shows that besides the dynamic phase shift (called self-modulation of phase in nonlinear optics) and amplitude change (amplification and de-amplification), there is an additional term. Unlike the previous ones this term has a nonzero value in the case of adiabatic cyclic evolution. However, the explicit proof of the identity of this phase to the phase, which is obtained by the wavefunction in course of adiabatic cyclic evolution is required. This is done by obtaining the explicit WKB solution of the Schrödinger equation in the coherent state formalism and comparing the results (Section IV). Following the recent tradition supported by the papers of Aharonov and Anandan, and Samuel and Bhandari [9,10] we identify the "slow" physical time of adiabatic evolution as a parameter of a closed contour. This definition is supported by the calculation of the Berry phase from Heisenberg equation of motion and through the explicit expression of the evolution operator in Section V.

2 BERRY PHASE IN THE HEISENBERG PRESENTATION

We consider the Hamiltonian, H, with time-dependent parameters, $\vec{\lambda}(t)$:

$$H = H(\vec{\lambda}(t)). \tag{6}$$

We assume that periodicity and adiabaticity of the evolution of the Hamiltonian can be represented in two ways. First, one can follow the evolution of $H(t)$ as a function of the time-dependent parameters. In this case, the Hamiltonian will undergo trivial transformation after the evolution.

$$H(0) = H(T) \tag{7}$$

The second way is to consider the change of the Hamiltonian as an observable with physical meaning of energy under the action of the evolution operator $U(t)$ which is engendered by the

Hamiltonian $H(t)$

$$\tilde{H}(t) = U^{-1}(t)H(0)U(t) \qquad (8)$$

$$U(t) = Texp(i \int^t H(t)dt) \qquad (9)$$

In this case the transformed Hamiltonian $\tilde{H}(T)$ can be different, unitarily equivalent to the initial Hamiltonian.

$$\tilde{H}(T) = U^{-1}(T)H(T)U(T) \qquad (10)$$

The spectra of these Hamiltonians are therefore identical. However, other features can be different. This distinction can be extremely important if there are lines of singularity or other topological complications in the space of parameters.

In the case of adiabatic cyclic evolution, this operator doesn't depend on particular law of evolution of the system's parameters and we shall define this unitary operator as Berry phase in Heisenberg representation. It implies, that the difference from $\tilde{H}(T)$ and $H(T)$ can be expressed in terms of certain phase factors (eigenvalues of the operator $U(T)$). As a rule, they appear for a certain components of the Hamiltonian H. These phases are closely connected with the Berry phases in Schrödinger representation, indeed, in the example of the Section III they are identical.

The fact that the Hamiltonians $H(t)$ and $\tilde{H}(T)$ are forming a superpair is of primary importance. One of the simplest ways of representation follows from Eq. (7), which connects $H(T)$ and $\tilde{H}(T)$

$$H = \begin{pmatrix} H(T) & 0 \\ 0 & \tilde{H}(T) \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{H}(0)U(t) \\ U^{-1}(T)\sqrt{H}(0) & 0 \end{pmatrix}^2 \qquad (11)$$

One can then demonstrate using the formalism of that relationship implies superstructure [2,3]. This relation holds only in the case of nonnegative spectra of the Hamiltonians. Thus, we restrict ourselves by the Hamiltonians bounded from below, in which case we shift the zero level of energy to avoid negative eigenvalues.

The superstructure could be introduced in more sophisticated ways. A particular example would be demonstrated as well. The paper is dedicated to the connection between Berry phase and supersymmetry on the example of the simple model. More complicated examples such as the Jaynes-Cummings model (JCM) will be studied elsewhere.

In the paper [11], the authors proposed the generalization of Berry's concept, interpreting their phase, a factor similar to our symbol V (Eqs. (20)-(21)), as a gauge transformation of the wavefunction. It is induced by the reparametrization of the Hamiltonian. The main result of this paper can be formulated in our language in the following way. The gauge transformation of Wilzcek and Zee while being applied to the Hamiltonian, regarded as observable, can result in other Hamiltonian, even if the evolution is adiabatic and cyclic. However, both Hamiltonians are unitary equivalent and the observable quantities are identical for both of them. The adiabatic theorem in the general quantum mechanics formalism is represented by supersymmetry in our paper and by the gauge invariance in their formalism.

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3 BOSONIC FIELDS WITH SELF-ACTION: THE SUPERSYMMETRY OF THE SQUEEZING HAMILTONIANS

The problem is formulated for the simple model of two identical noninteracting bosonic fields described by the Hamiltonian

$$H_1 = \omega a^\dagger a \hat{I} + |\lambda| a^2 \begin{pmatrix} e^{i(\phi+\Theta)} & 0 \\ 0 & e^{i(\tilde{\phi}+\tilde{\Theta})} \end{pmatrix}$$

$$+ |\lambda| a^{\dagger 2} \begin{pmatrix} e^{i(\phi+\Theta)} & 0 \\ 0 & e^{-i(\tilde{\phi}+\tilde{\Theta})} \end{pmatrix} \begin{pmatrix} 1-c & 0 \\ 0 & c \end{pmatrix} \omega \tag{12}$$

The superpair of this Hamiltonian is

$$H_2 = \omega a^\dagger a \hat{I} + |\lambda| a^2 \begin{pmatrix} e^{i(\tilde{\phi}+\tilde{\Theta})} & 0 \\ 0 & e^{i(\phi+\Theta)} \end{pmatrix}$$

$$+ |\lambda| a^{\dagger 2} \begin{pmatrix} e^{-i(\tilde{\phi}+\tilde{\Theta})} & 0 \\ 0 & e^{-i(\phi+\Theta)} \end{pmatrix} + \begin{pmatrix} 1-c & 0 \\ 0 & c \end{pmatrix} \omega \tag{13}$$

The constant c is defined from the equation

$$c(1-c) = \frac{|\lambda|^2}{\omega^2}$$

The system described by the Hamiltonian $\mathbf{H}$ in the form (4), where H_1 and H_2 have the form (12) and (13), respectively, are supersymmetric. This supersymmetry is generated by the following supercharges [5]:

$$Q = \begin{pmatrix} 0 & q_1 \\ q_2 & 0 \end{pmatrix}, Q^2 = H \tag{14}$$

$$q_1 = \begin{pmatrix} 0 & e^{i\phi} \\ |\lambda| e^{i\tilde{\Theta}} & 0 \end{pmatrix} a + \begin{pmatrix} 0 & \frac{c\omega}{|\lambda|} e^{-i\Theta} \\ (1-c)\omega e^{-i\phi} & 0 \end{pmatrix} a^\dagger \tag{15}$$

$$q_2 = \begin{pmatrix} 0 & e^{i\tilde{\phi}} \\ |\lambda| e^{i\Theta} & 0 \end{pmatrix} a + \begin{pmatrix} 0 & \frac{c\omega}{|\lambda|} e^{-i\tilde{\Theta}} \\ (1-c)\omega e^{-i\phi} & 0 \end{pmatrix} a^\dagger \tag{16}$$

$$H_1 = q_1 q_2 \quad H_2 = q_2 q_1 \tag{17}$$

The second supercharge has the form

$$Q_2 = \begin{pmatrix} 0 & -iq_1 \\ iq_2 & 0 \end{pmatrix} \tag{18}$$

This finishes the formal description of supersymmetry. The procedure of assigning a supersymmetric structure for an initial model remind our prescription for the JCM [5]. One should make a constant phase shift of the creation and annihilation operators and consider a replica of the initial system with transformed operators as a superpair for the initial system.

The described procedure has an obscure physical meaning. To clarify it we explicitly describe the unitary operator, which connects the components of the Hamiltonian in subsequent sections.

4 THE OPERATOR BERRY PHASE

Now we are prepared to study the Berry phase of the Heisenberg operators. We suppose the three parameters $\omega, Re\lambda, Im\lambda^*$ to undergo adiabatic periodic change in time. First, we calculate the nonintegrable phase attained by the wavefunction of the model, described in previous Sections, in coherent-state representation. Below, we shall demonstrate, that all the components of the Hamiltonian $\mathbf{H}$, defined by Eqs. (12) and (13) can be obtained from one component of this Hamiltonian by applying the adiabatic cyclic evolution of the parameters of the system. Thus, the superpairs are connected by a unitary operator, which we call the Berry phase in Heisenberg representation. A single component of the Hamiltonian $\mathbf{H}$, corresponds to the Schrödinger equation

$$i\frac{d}{dt}U = [\omega a^\dagger a + \lambda a^2 + \lambda^* a^{\dagger 2} + h_0]U \tag{19}$$

The contribution of the scalar term $h_0(t)$ is removed by the transformation

$$U = V exp(i\int^t h_0(\tau)d\tau) \tag{20}$$

The wavefunction V is represented in the coherent state formalism by

$$V(\alpha*,\beta,t) = \langle\alpha|Texp(i\int^t (h - h_0)d\tau)|\beta\rangle \tag{21}$$

where T is time ordering and $|\alpha\rangle, |\beta\rangle$ are coherent states:

$$a|\alpha\rangle = \alpha|\alpha\rangle \quad a|\beta\rangle = \langle\alpha|a^\dagger = \langle\alpha|\alpha^* \tag{22}$$

For the quadratic Hamiltonian (5) the calculations can be done explicitly. The solution $V(\alpha^*,\beta,t)$ of the Eq. (19) in the coherent-state formalism has the form

$$V(\alpha^*,\beta,t) = (\xi(t))^{-1/2}exp[\frac{1}{2}\alpha^{*2}\frac{\eta}{\xi} + \alpha^*\beta\frac{1}{\xi} + \frac{1}{2}\beta^2\frac{\eta*}{\xi}] \tag{23}$$

where a,b are the parameters of the initial state. As usual for coherent-state representations [12], the solution is expressed in terms of auxiliary functions $\xi(t), \eta(t)$ which satisfy the following system of the equations [13]:

$$\begin{aligned}\dot{\xi}(t) &= i(\xi\eta - 2\eta\lambda) \\ \dot{\eta}(t) &= i(\lambda^*\xi - \eta\omega)\end{aligned} \tag{24}$$

with the initial conditions $\xi(0) = 1, \eta(0) = 0$.

The following system of equations is valid for the functions ξ, η

$$\ddot{\eta} - \frac{\dot{\lambda}^*}{\lambda^*}\dot{\eta} + [i\dot{\omega} - i\omega\frac{\dot{\lambda}^*}{\lambda^*} + \omega^2 - 4|\lambda|^2]\eta = 0.$$

$$\ddot{\xi} - \frac{\dot{\lambda}}{\lambda}\dot{\xi} + [i\dot{\omega} + i\omega\frac{\dot{\lambda}}{\lambda} + \omega^2 - 4|\lambda|^2]\xi = 0. \tag{25}$$

Below we shall demonstrate that WKB-solution of these equations will provide a multiplier with a nonintegrable phase.

The Hamiltonians H_1, H_2 include the creation-annihilation operators distinguished by some deterministic phase shift. For a description of the evolution of these operators we use one of the four components of the Hamiltonian $\mathbf{H}$, because all four Hamiltonians are independent and unitary-equivalent to each other,

$$h = \omega a^\dagger a + \lambda a^2 + \lambda^* a^{\dagger 2} + h_0,$$
$$\lambda = |\lambda|e^{i(\phi+\Theta)} \tag{26}$$

The Heisenberg equations have the form [14],

$$\begin{aligned}
\dot{a} &= i[h, a] \\
\dot{a}^\dagger &= i[h, a^\dagger] \\
\dot{a} &= -i(\omega a + 2\lambda^* a^\dagger) \\
\dot{a}^\dagger &= i(\omega a^\dagger + 2\lambda a).
\end{aligned} \tag{27}$$

It is again assumed that the parameters $\omega, \lambda, \lambda^*$ are slowly varying periodic functions of time. The operators $a, a^\dagger$ are satisfying the following equations:

$$\ddot{a} - \frac{\dot{\lambda}^*}{\lambda^*}\dot{a} + [i\dot{\omega} - i\omega\frac{\dot{\lambda}^*}{\lambda^*} + \omega^2 - 4|\lambda|^2]a = 0,$$

$$\ddot{a}^\dagger - \frac{\dot{\lambda}}{\lambda}\dot{a}^\dagger + [-i\dot{\omega} + i\omega\frac{\dot{\lambda}}{\lambda} + \omega^2 - 4|\lambda|^2]a^\dagger = 0. \tag{28}$$

One observes that equations (25), for the functions ξ, η are identical to the system of equations (28) for the operators $a^\dagger, a$. By substitution

$$a = b \exp(\frac{1}{2} \int^t \frac{\dot{\lambda}^*}{\lambda^*} d\tau) \tag{29}$$

one obtains for operator b and its Hermitian conjugate, if the terms $(\dot{\lambda}^*/\lambda^*)^2, \frac{d}{dt}(\dot{\lambda}^*/\lambda^*)$ in the equation for b are neglected

$$\ddot{b} + [i\dot{\omega} - i\omega\frac{\dot{\lambda}^*}{\lambda^*} + \omega^2 - 4|\lambda|^2]b = 0,$$

$$\ddot{b}^\dagger + [-i\dot{\omega} + i\omega\frac{\dot{\lambda}}{\lambda} + \omega^2 - 4|\lambda|^2]b^\dagger = 0. \tag{30}$$

The WKB approximation for the solution of Eqs. (28) yields

$$b = \frac{b_1(0)}{\sqrt{\Omega}}exp(\int_0^t (f(\tau) + i\psi(\tau))d\tau) + \frac{b_2(0)}{\sqrt{\Omega}}exp(\int_0^t (f(\tau) - i\psi(\tau)d\tau), \tag{31}$$

with

$$
\begin{aligned}
\Omega^2 &= \omega^2 - 4|\lambda|^2 \\
f(t) &= \frac{\dot{\omega} - \omega\frac{\dot{\rho}}{\rho}}{4(\omega^2 - 4\rho^2)^{1/2}} = \frac{\frac{d}{dt}(\frac{\omega}{\rho})}{4(\frac{\omega}{\rho} - 4)^{1/2}}, \rho = |\lambda|, \\
\psi(t) &= \omega\frac{(\dot{\phi} + \dot{\Theta})}{4\Omega} = \frac{\omega\dot{k}}{4\Omega}dt,
\end{aligned}
\tag{32}
$$

For cyclical evolution of the parameters the contribution of the function $f(t)$ is zero, because it is an exact differential. However, the integral of the function $\psi(t)$ yields the Berry phase.

$$\phi_B = \int_C \frac{\omega\dot{k}}{4\Omega}dt, \tag{33}$$

and $\tilde{\phi}_B$ is given by the same expression with $k \to \tilde{k} = \tilde{\phi} + \tilde{\Theta}$. The initial conditions can be chosen in such a way, that the phase transformations of the operators take place as a result of the evolution

$$a \to ae^{i\phi_B}, a^\dagger \to a^\dagger e^{-i\phi_B}, \tag{34}$$

then

$$h \to h = \omega a^\dagger a + \lambda a^2 e^{2i\phi_B} + \lambda^*(a^\dagger)^2 e^{-2i\phi_B}, \tag{35}$$

The calculation of the Berry phase for the model Hamiltonian H is now complete.

One observes that the superpairs H_1 and H_2, composing the supersymmetric Hamiltonian H, are distinguished by the Berry phase. As the Hamiltonian H_2 transforms as a result of cyclic evolution, the phases $(\phi + \Theta)$ and $(\tilde{\phi} + \tilde{\Theta})$ evolve in a different way. Berry phase is independent of the particular law of evolution, nevertheless it depends on the choice of the contour in parameter space. The contours can be chosen in such a way that the following relation is valid [15].

$$\phi + \Theta + \phi_B = \tilde{\phi} + \tilde{\Theta}, \tilde{\phi} + \tilde{\Theta} + \tilde{\phi}_B = \phi + \Theta$$

or

$$\phi_B = -\tilde{\phi}_B = (\tilde{\phi} - \phi) + (\tilde{\Theta} - \Theta) \tag{36}$$

Now we have found the geometric phase for the Hamiltonian H using the Heisenberg representation.

5 THE CALCULATION OF THE BERRY PHASE USING EVOLUTION OPERATOR

We have shown that creation-annihilation operators obtain phase shift according to Eqs. (34)-(35) as a result of adiabatic cyclic evolution which is identical with the Berry phase gained by the coherent phase. The substitution of the transformed operators in the Hamiltonian h gives

$$h \to h_p = \omega a^\dagger a + \lambda a^2 e^{2i\phi} + \lambda a^{+2} e^{-2i\phi} + h_0 \tag{37}$$

This result can be obtained directly from computation of the action of evolution operator on Hamiltonian **H**.

One can rewrite the Hamiltonian **H** in the terms of SU(1,1) operators [16,17]:

$$h = \omega K_0 + \lambda K_- + \lambda^* K_+, \tag{38}$$

where

$$K_0 = 1/2(a^\dagger a + aa^\dagger), K_- = a^2, K_+ = (a^\dagger)^2$$
$$[K_0, K_+] = +K_+, [K_0, K_-] = -K_-, [K_+, K_-] = 2K_0. \tag{39}$$

To get rid of the T-exponent in the evolution operator:

$$U = T exp(\int^t h(\tau) d\tau),$$

one can rewrite it in the form [12]:

$$U = e^{\alpha K_+} e^{\beta K_0} e^{\gamma K_-}. \tag{40}$$

This decomposition expresses the evolution operator in terms of three time-dependent functions $\alpha(t), \beta(t) and \gamma(t)$. Using the commutation relations one obtains the following equations for these functions:

$$\alpha - \alpha\beta + \alpha^2 \gamma e^{-\beta} = -i\mu^*$$
$$\dot\beta - 2\alpha\dot\gamma e^{-\beta} = -i\omega$$
$$\dot\gamma e^{-\beta} = i\mu. \tag{41}$$

The quantity $\alpha(t)$ satisfies the equation

$$\dot\alpha + i\omega\alpha + i\mu\alpha^2 = -i\mu^*, \tag{42}$$

which can be rewritten after the substitution $\alpha = (1/i\mu)(\dot\psi/\psi)$ in the form

$$\ddot\psi + \dot\psi(-\frac{\dot\mu}{\mu} + i\omega) - |\mu|^2 \psi = 0$$

$$\kappa = \psi exp[\frac{1}{2} \int^t d\tau(-\frac{\dot\mu}{\mu} + i\omega)]$$

$$\ddot\kappa + \kappa[\omega^2 - 4|\mu|^2 + i\omega\frac{\dot\mu}{\mu} - i\dot\omega] = 0. \tag{43}$$

The equations (43) coincide in the form with the equations (25), (28) which were our starting point of obtaining expression of the Berry phase. The phase given by the equation (43) is identical to the Berry phase as defined above. Now one can study the behavior of the functions α, β, γ under cyclic adiabatic evolution of the parameters. The function κ is the solution of the equation with the periodic potential. This implies that this function is quasiperiodic:

$$\kappa(t+T) = e^{i\phi_T}\kappa(t), \tag{44}$$

where ϕ_T is independent of t. Thus the ration $\dot{\kappa}/\kappa$ is a periodic function as well.

The function $\beta = -i \int d\omega - \ln\psi$ is not periodic. According to Eq. (32) after the cyclic evolution β changes by a constant ϕ_T

$$\beta \to \beta - i\phi_T. \tag{45}$$

The function γ is not periodic as well, however, it is not very important to us here.

One can now investigate the transformation of the primary Hamiltonian h in course of evolution. Since we decomposed h in the sum of three operators K_0, K_+, K_-, we can consider the action of the evolution operator on the single terms of the sum. It is quite necessary for Eqs. (41) not to be symmetrical with respect to the functions α, β, γ. We can use the decomposition from Eq. (40) only for the study of the transformation of the operator K_- :

$$\tilde{K}_-(T) = U^{-1}(T)K_-(0)U(T). \tag{46}$$

The initial conditions of the evolution were chosen as $\alpha(0) = \beta(0) = 0$ as a consequence of periodicity it implies $\alpha(T) = 0$. This means that $e^{\alpha K_+}$ and $e^{\gamma K_-}$ do not contribute to the periodic evolution of group operators. The only term with which changes is $e^{\beta K_0}$. Taking into account their relation $\beta(T) = -\phi_T$ one has

$$\tilde{K}_-(T) = e^{2i\phi_T a^\dagger a}K_0(0)e^{-2i\phi_T a^\dagger a} = e^{-2i\phi_T}K_0(0). \tag{47}$$

Similarly, choosing a different order of multiplication in the decomposition (40) one gets the transformation property for the other operators.

$$\begin{aligned} \tilde{K}_+(T) &= e^{2i\phi_T}K_+(0) \\ \tilde{K}_0 &= K_0(0). \end{aligned} \tag{48}$$

Consequently, the Hamiltonian H is transformed under the evolution as

$$\begin{aligned} h \to h_\phi &= \omega K_0 + \lambda e^{-2i\phi_T}K_- + \lambda^* e^{2i\phi_T}K_+ \\ &= \omega a^\dagger a + \lambda e^{-2i\phi_T}a^2 + \lambda^* e^{2i\phi_T}(a^\dagger)^2 + h_0. \end{aligned} \tag{49}$$

This expression is the same one we obtained from the WKB solution of the operator equations (28). The phase shift is identical to the phase shift gained by coherent-state representation of the wavefunction. The reconciliation of all methods proved the correctness of the definition of Berry phase in Heisenberg picture. Because all of the components of the Hamiltonian **H** can be

obtained through the constant phase shift of the creation-annihilation operators or sign reversal of the phase factors Θ, ϕ we finish up with the proposition that the trivial phase shifts in the supersymmetric Hamiltonian can be regarded as the Berry phases.

6 CONCLUSION

The present paper proposes the following definition of Berry phase in Heisenberg representation. Berry phase is the unitary operator connecting the Hamiltonian of the system with the initial Hamiltonian after adiabatic cyclic evolution of the system's parameters. We study the simplest model in which this unitary operator is independent of the particular quantum numbers of the system and reduces to a certain c-number phase factor. This definition is motivated by the identification of this number to the adiabatic phase factor gained by the wavefunction of the system in coherent-state representation. Since the model is exactly soluble, the direct computation of the evolution operator is possible and it confirms the calculations made by two other different methods: the WKB solution of the Heisenberg equations and the WKB-solution of the Schrödinger equation in coherent-state representation. This definition allows us to interpret the supersymmetric compound Hamiltonian as if it has been formed by images of the initial Hamiltonian, obtained as a result of an adiabatic cyclic evolution over a different contours in a parameter space.

7 ACKNOWLEDGEMENTS

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SECTION 7

INTERACTION OF LIGHT AND MATTER