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Complementarity and Path Distinguishability: some recent results concerning photon pairs.

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Abstract

Two results concerning photon pairs, one previously reported and one new, are summarized. It was previously shown that if the two photons are prepared in a quantum state formed from $|A\rangle$ and $|A'\rangle$ for photon 1 and $|B\rangle$ and $|B'\rangle$ for photon 2, then both one- and two-particle interferometry can be studied. If v_i is the visibility of one-photon interference fringes ($i = 1, 2$) and v_{12} is the visibility of two-photon fringes (a concept which we explicitly define), then

$$v_i^2 + v_{12}^2 \leq 1.$$

The second result concerns the distinguishability of the paths of photon 2, using the known 2-photon state. A proposed measure E for path distinguishability is based upon finding an optimum strategy for betting on the outcome of a path measurement. Mandel has also proposed a measure of distinguishability P_D, defined in terms of the density operator ρ of photon 2. We show that E is greater than or equal to P_D and that $v_2 = (1 - E^2)^{1/2}$.

1 Introduction.

The idea of an entangled quantum state of a composite system – i.e., a state not factorizable into a product of one-particle states – was discovered by Schrödinger in 1926, and has been intensively studied as a result of analyses by Einstein-Podolsky-Rosen and Bell. A very convenient method for preparing entangled photon pairs by parametric down-conversion in laser-pumped nonlinear crystals was discovered by Burnham and Weinberg in 1970. Their discovery permitted the development of two-photon interferometry by Mandel and his school, Alley and Shih, Franson, Rarity and Tapster, Chiao and his school, and others.[1]

For subsequent discussion, it will be useful to refer to a schematic two-photon apparatus (Fig. 1), in which a pair of photons emerges from a source S, one of which propagates in beams A and/or

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A′, and the other in beams B and/or B′, where the locution "and/or" is a brief way of referring to quantum mechanical superposition. For the work on path distinguishability that we shall report, this partial description of Fig. 1 provides the essence. For the work on the complementarity of one-photon and two-photon interference, some further elements are indispensable. There is an ideal symmetric beam splitter H_1 upon which each of the beams A and A′ impinge, from which emerge beams U_2 and L_2. We can speak equivalently of a photon "emerging" in beams U_1, L_1, U_2, L_2 or of its "detection by an ideal photo-detector" in the respective beams. Finally, there are variable phase shifters ϕ_1 and ϕ_2 inserted in beams A and B.

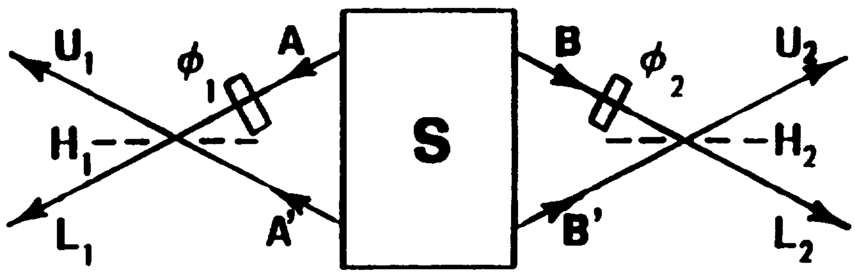

FIG.1. Schematic two-particle four-beam inteferometer.

2 Complementarity.

It was noticed in the past, for instance by Horne and Zeilinger,[2] that when the photon pair is prepared in the entangled state $|\Psi\rangle$,

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\Big[|A\rangle|B\rangle + |A'\rangle|B'\rangle\Big] \ . \tag{1}$$

then probabilities of single detections in the various emerging beams are independent of phase shifts ϕ_1 and ϕ_2, specifically,

$$P(U_1) = P(L_1) = P(U_2) = P(L_2) = \frac{1}{2} \ . \tag{2}$$

whereas the probabilities of joint detection depend on ϕ_1 and ϕ_2, specifically,

$$P(U_1U_2) = P(L_1L_2) = \frac{1}{4}[1 - cos(\phi_1 + \phi_2)]. \tag{3a}$$

$$P(U_1L_2) = P(L_1U_2) = \frac{1}{4}[1 + cos(\phi_1 + \phi_2)]. \tag{3b}$$

Since the probabilities in Eqs.(3a, b) vary from a minimum of zero to a non-zero maximum value, while those of Eq.(2) do not vary at all, it is reasonable to extend standard optical terminology and say that the visibility of one-photon "fringes" is zero and the visibility of two-photon fringes is unity, where "fringe" is a generic way of referring to the dependence of detection probabilities upon variable phase shifts. When the quantum state of the two photons has the product form

$$|\Phi\rangle = \frac{1}{\sqrt{2}}\Big[|A\rangle + |A'\rangle\Big]\frac{1}{\sqrt{2}}\Big[|B\rangle + |B'\rangle\Big], \tag{4}$$

then

$$P(U_i) = \frac{1}{2}(1 - sin\phi_i), \quad i = 1, 2, \tag{5a}$$

$$P(L_i) = \frac{1}{2}(1 + sin\phi_i), \quad i = 1, 2, \tag{5b}$$

and the probabilities of joint detection are the products of respective single detections:

$$P(U_1U_2) = P(U_1)P(U_2) = \frac{1}{4}(1 - sin\phi_1)(1 - sin\phi_2), \ etc. \tag{6}$$

It is reasonable to say in this case that the visibility of one-photon fringes is unity, but the visibility of two-photon fringes is zero (the latter statement in spite of the fact that $P(U_1U_2)$ does vary with ϕ_1 and ϕ_2, because of the consideration that this variation is not a genuine two-photon effect but is derived from the one-photon variation).

The two extreme cases of $|\Psi\rangle$ and $|\Phi\rangle$ suggest that there is a complementarity of one-photon and two-photon interference visibility. Jaeger, Horne, and Shimony[3] raised the question of a general complementarity relation, holding for any two-photon state expressible in terms of $|A\rangle$, $|A'\rangle$, $|B\rangle$, $|B'\rangle$. A necessary condition for investigating this question was to define explicitly the "one-photon visibility" v_i ($i = 1, 2$) and the "two-photon visibility" v_{12}. The former is straightforward, simply adapting the standard optical concept introduced by Rayleigh. We state it here only for the beams U_1 and U_2, but parallels hold for L_1 and L_2

$$v_i = \frac{[P(U_i)]_{max} - [P(U_i)]_{min}}{[P(U_i)]_{max} + [P(U_i)]_{min}} \ . \tag{7}$$

For v_{12} Jaeger *et al.* suggested

$$v_{12} = \frac{[\bar{P}(U_1U_2)]_{max} - [\bar{P}(U_1U_2)]_{min}}{[\bar{P}(U_1U_2)]_{max} + [\bar{P}(U_1U_2)]_{min}} \ . \tag{8}$$

The "corrected" joint probability $\bar{P}(U_1U_2)$ is defined as

$$\bar{P}(U_1U_2) = P(U_1U_2) - P(U_1)P(U_2) + \frac{1}{4} \ , \tag{9}$$

where the second term on the right hand side removes the variability that is derived from the single probabilities $P(U_1), P(U_2)$ and the third term is a correction against excessive subtraction in order to agree with intuition in the extreme cases of $|\Psi\rangle$ and $|\Phi\rangle$.

In order to exhibit the desired complementarity relation, it is essential to calculate v_i and v_{12} in the most general two-photon state that can be prepared with $|A\rangle$, $|A'\rangle$ as basis states for photon 1 and $|B\rangle$, $|B'\rangle$ as basis states for photon 2, namely,

$$|\Theta\rangle = cos\alpha[cos\beta|A\rangle|B\rangle + e^{i\lambda}sin\beta|A'\rangle|B'\rangle] \\ + sin\alpha[e^{i\mu}cos\gamma|A\rangle|B'\rangle + e^{i\nu}sin\gamma|A'\rangle|B\rangle] \, . \tag{10}$$

Note that only three phase angles λ, μ, ν are used, because an overall multiplication by a phase factor does not change the quantum state, and this fact can be used to choose the coefficient of $|A\rangle|B\rangle$ to be real. In Ref. 3 it was fallaciously argued that a basis change of

$$|A\rangle = e^{i\rho}|\bar{A}\rangle \, , \quad |A'\rangle = e^{i\rho'}|\bar{A}'\rangle \, , \tag{11}$$

$$|B\rangle = e^{i\sigma}|\bar{B}\rangle \, , \quad |B'\rangle = e^{i\sigma'}|\bar{B}'\rangle \, ,$$

can be used to express $|\Theta\rangle$ in terms of $|A\rangle$, $|A'\rangle$, $|B\rangle$, $|B'\rangle$ with real coefficients. But Prof. Sheldon Goldstein pointed out to us (private communication) that in general only two of the three phase angles in Eq.(10) can be eliminated by a basis change, and therefore the greatest simplification that can be achieved in full generality retains one explicit phase angle, for instance,

$$|\Theta\rangle = cos\alpha[cos\beta|A\rangle|B\rangle + sin\beta|A'\rangle|B'\rangle] \\ + sin\alpha[cos\gamma|A\rangle|B'\rangle + e^{i\tau}sin\gamma|A'\rangle|B\rangle] \, . \tag{12}$$

So far, we have not demonstrated a complementarity relation for the general case of Eq.(12). We therefore report the result in the restricted case of $\tau = 0$, which we have investigated. As stated in Ref. 3, Eqs.(29-32), we obtain

$$v_i^2 = \frac{1}{2}sin^2 2\alpha[1 + sin2\beta \; sin2\gamma + (-1)^i cos2\beta \; cos2\gamma] \, , \tag{13}$$

$$v_{12}^2 = cos^4\alpha \; sin^2 2\beta - 2sin^2\alpha \; cos^2\alpha \; sin2\beta \; sin2\gamma + sin^4\alpha \; sin^2 2\gamma \, , \tag{14}$$

whence

$$v_i^2 + v_{12}^2 \leq 1 \, , \tag{15a}$$

or equivalently,

$$0 \leq v_i v_{12} \leq \frac{1}{2} \, . \tag{15b}$$

Inequalities (15a,b) are our expressions of the complementarity of one-photon and two-photon visibilities. Although we have derived them only for the special case of $\tau = 0$, we are confident that they hold for any τ and hence for the most general $|\Theta\rangle$. Work is in progress on this important question.

3 Path Distinguishability.

We return now to Fig. 1 and ask a new question. Suppose that we are allowed to make any observation on photon 1, which is the left-going photon that propagates in A and/or A′; what is the best procedure for predicting which detector will be triggered by photon 2, if ideal detectors are inserted in beams B and B′? This question is related to a question recently raised by Mandel[4] concerning the distinguishability of the path of a photon that propagates in beams B and/or B′. There is, however, an important difference between Mandel's question and ours. He assumes only that one knows the density operator ρ characterizing an ensemble of photons in the beams B and/or B′, and he asks for a measure of distinguishability expressed in terms of ρ. By contrast, we ask for a measure of distinguishability based upon the quantum state $|\Theta\rangle$ of the pair of photons 1 and 2, together with the outcome of an arbitrary measurement upon photon 1. It is possible to compare our result with Mandel's, because when $|\Theta\rangle$ is given a density operator for photon 2 can be calculated[5] by tracing out the appropriate variables of photon 1. But, of course, if only ρ is given, there are many possible preparations of an ensemble of photons propagating in beams B and/or B′ that would yield the same ρ. In other words, the preparation of the ensemble provides additional information that is not included in ρ. Consequently, we anticipate a discrepancy between Mandel's measure of path distinguishability and ours.

As a preliminary to our proposed measure of path distinguishability we suppose that an observable $\mathcal{O}$ is measured on photon 1. Since the space of states that we have allowed for photon 1 is two-dimensional, there is no loss of generality if we restrict the observable $\mathcal{O}$ to the form

$$\mathcal{O} = |\phi_1\rangle\langle\phi_1| - |\phi_2\rangle\langle\phi_2| \,, \tag{16}$$

where $|\phi_1\rangle$ and $|\phi_2\rangle$ are orthonormal kets in the space spanned by $|A\rangle$ and $|A'\rangle$. (We are grateful to Prof. Lev Vaidman for suggesting that we consider any $\mathcal{O}$, rather than just $|A\rangle\langle A| - |A'\rangle\langle A'|$ as in our original preprint.) The eigenvalues of $\mathcal{O}$ are $+1$ and -1. Now formulate a strategy for betting on whether the detector in team B or in beam B′ is triggered, letting the strategy depend upon the quantum state $|\Theta\rangle$ of the photon pair and the outcome $+1$ or -1 of measuring $\mathcal{O}$. If in a single case the correct detector is predicted, the observer wins one unit of utility; if the wrong detector is predicted, the observer loses one unit of utility. Once the strategy is specified, it is straightforward to calculate from $|\Theta\rangle$ the average gain per bet. Let $E_{\mathcal{O}}$ be the largest average gain thus calculated as the strategy is varied but $\mathcal{O}$ is fixed. Finally, our *measure of distinguishability of paths*, which we shall label E, is defined as

$$E = max\ E_{\mathcal{O}}\ \ (over\ the\ set\ of\ allowed\ observables)\,. \tag{17}$$

E is thus the quantum mechanical estimate of the gain per bet when the optimum allowable strategy is followed, the bets being made concerning paths B and B′.

To calculate $E_{\mathcal{O}}$ we first rewrite $|\Theta\rangle$, assumed to be normalized, as

$$|\Theta\rangle = |\chi_1\rangle|B\rangle + |\chi_2\rangle|B'\rangle \,, \tag{18}$$

where, as before, $|B\rangle$ and $|B'\rangle$ are orthonormal, but $|\chi_1\rangle$ and $|\chi_2\rangle$ need not be; however,

$$\langle\chi_1|\chi_1\rangle + \langle\chi_2|\chi_2\rangle = 1 \,. \tag{19}$$

With no loss of generality we can assume that

$$\langle\chi_1|\chi_1\rangle \geq \langle\chi_2|\chi_2\rangle \ , \tag{20}$$

which can be achieved, if necessary, by interchanging the labels B and B′ of the two paths of photon 2. Then we can write

$$|\chi_2\rangle = \lambda|\chi_1\rangle + |\chi_3\rangle \ , \tag{21a}$$

where

$$\lambda = \frac{\langle\chi_1|\chi_2\rangle}{\langle\chi_1|\chi_1\rangle} \ , \tag{21b}$$

$$|\lambda| \leq 1 \ , \tag{21c}$$

and

$$\langle\chi_1|\chi_3\rangle = 0 \ . \tag{21d}$$

If we define

$$N_i = \langle\chi_i|\chi_i\rangle \ , \quad i = 1,3 \ , \tag{22a}$$

then the $|\bar{\chi}_i\rangle$, defined by

$$|\bar{\chi}_i\rangle = \frac{|\chi_i\rangle}{\sqrt{N_i}} \ , \quad i = 1,3 \ , \tag{22b}$$

are orthonormal. Furthermore,

$$N_1(1 + |\lambda|^2) + N_3 = 1 \ . \tag{23}$$

Any basis $|\phi_1\rangle, |\phi_2\rangle$ in the space of allowable states of photon 1 can be expressed as

$$|\phi_1\rangle = \mu|\bar{\chi}_1\rangle + \nu|\bar{\chi}_2\rangle \ , \tag{24a}$$

$$|\phi_2\rangle = \nu^*|\bar{\chi}_1\rangle - \mu^*|\bar{\chi}_2\rangle \ , \tag{24b}$$

where

$$|\mu|^2 + |\nu|^2 = 1 \ . \tag{24c}$$

This basis defines the observable $\mathcal{O}$ of Eq.(16). It will also be useful to write

$$\mathcal{B} = |B\rangle\langle B| - |B'\rangle\langle B'| \ , \tag{25}$$

an observable in the allowable space of states of photon 2; clearly $\mathcal{B}$ is observed to have values $+1$ and -1 according as photon 2 is detected in path B or B′.

If $\mathcal{O}$ is the observable chosen to be measured, then there are four pure strategies for bets on the path of photon 2:

(1) If $\mathcal{O} = +1$, predict $\mathcal{B} = +1$; if $\mathcal{O} = -1$, predict $\mathcal{B} = -1$.
(2) If $\mathcal{O} = +1$, predict $\mathcal{B} = -1$; if $\mathcal{O} = -1$, predict $\mathcal{B} = +1$.
(3) Predict $\mathcal{B} = +1$ regardless of the value of $\mathcal{O}$.
(4) Predict $\mathcal{B} = -1$ regardless of the value of $\mathcal{O}$.

In addition to these pure strategies there are mixed strategies, consisting of following (1), (2), (3), (4) with arbitrary probabilities summing to unity. But since the game is not being played against a rational opponent, the average gain in a mixed strategy cannot exceed the maximum of the average gain $E_{\mathcal{O}}^{(i)}$ of the pure strategies,[6] $i = 1, 2, 3, 4$. These are calculated as follows:

$$E_{\mathcal{O}}^{(1)} = P(\mathcal{O} = 1 \ and \ \mathcal{B} = 1) + P(\mathcal{O} = -1 \ and \ \mathcal{B} = -1)$$

$$-P(\mathcal{O} = 1 \ and \ \mathcal{B} = -1) - P(\mathcal{O} = -1 \ and \ \mathcal{B} = 1)$$

$$= |\langle\Theta|\phi_1\rangle|B\rangle|^2 + |\langle\Theta|\phi_2\rangle|B'\rangle|^2 - |\langle\Theta|\phi_1\rangle|B'\rangle|^2 - |\langle\Theta|\phi_2\rangle|B\rangle|^2$$

$$= S(|\mu|^2 - |\nu|^2) - T|\mu||\nu| \ cos(\theta_\lambda + \theta_\nu - \theta_\mu) \ , \tag{26}$$

where

$$S = N_1(1 - |\lambda|^2) + N_3 \ , \tag{27a}$$

$$T = 4N_1^{1/2}N_2^{1/3}|\lambda| \ , \tag{27b}$$

$$\lambda = |\lambda|e^{i\theta_\lambda}, \ \mu = |\mu|e^{i\theta_\mu}, \ \nu = |\nu|e^{i\theta_\nu} \ ; \tag{27c}$$

$$E_{\mathcal{O}}^{(2)} = -E_{\mathcal{O}}^{(1)} \ ; \tag{28}$$

$$\begin{aligned} E_{\mathcal{O}}^{(3)} &= P(\mathcal{B} = +1) - P(\mathcal{B} = -1) = \langle\chi_1|\chi_1\rangle - \langle\chi_2|\chi_2\rangle \\ &= N_1(1 - |\lambda|^2) - N_3 = S - 2N_3 \ ; \end{aligned} \tag{29}$$

$$E_{\mathcal{O}}^{(4)} = P(\mathcal{B} = -1) - P(\mathcal{B} = +1) = -E_{\mathcal{O}}^{(3)} \ . \tag{30}$$

Note that $E_{\mathcal{O}}^{(3)}$ and $E_{\mathcal{O}}^{(4)}$ are independent of $\mathcal{O}$. Then

$$E_{\mathcal{O}} = max\Big\{|S(|\mu|^2 - |\nu|^2) - T|\mu||\nu| \ cos(\theta_\lambda + \theta_\nu - \theta_\mu)| \ , \ |S - 2N_3|\Big\}. \tag{31}$$

In view of Eqs.(17) and (31) one finds the measure E of path distinguishability by investigating $E_{\mathcal{O}}$ as μ and ν are varied, subject to Eq.(24c). We first note that for any $|\Theta\rangle$ there is an $\mathcal{O}$ such that

$$|E_{\mathcal{O}}^{(1)}| \geq |E_{\mathcal{O}}^{(3)}| \ , \tag{32}$$

so that the second option in Eq.(31) can be neglected when we maximize over all possible $\mathcal{O}$. To prove these statements, it suffices in Eqs.(24a,b) to let $\mu = 1$ and $\nu = 0$, determining an $\mathcal{O}'$ such that Eqs.(26), (27), (28) yield

$$|E_{\mathcal{O}'}^{(1)}| = |N_1(1 - |\lambda|^2) + N_3| \ , \tag{33}$$

and

$$|E^{(3)}_{\mathcal{O}'}| = |N_1(1-|\lambda|^2) - N_3| \ . \tag{34}$$

Since N_1 and N_3 are non-negative, and $(1-|\lambda|^2)$ is non-negative by Eq.(21c), we obtain

$$|E^{(1)}_{\mathcal{O}'}| \geq |E^{(3)}_{\mathcal{O}'}| \ , \tag{35}$$

the rhs being the same as $|E^{(3)}_{\mathcal{O}}|$ for all $\mathcal{O}$. E is therefore obtained by maximizing the first option of Eq.(31) for allowable μ and ν, and the result is

$$E = \frac{1}{2}(4S^2 + T^2)^{1/2}. \tag{36}$$

By Eqs. (27a), (27b), and (23) E can be rewritten as

$$E = (1 - 4N_1^2|\lambda|^2)^{1/2}. \tag{37}$$

We can now make a comparison with Mandel's[4] measure of path distinguishability P_D. Mandel notes that in a two-dimensional Hilbert space, any density operator ρ can be expressed uniquely in the form

$$\rho = P_{ID}\ \rho_{ID} + P_D\ \rho_D \ , \tag{38}$$

where ρ_D is diagonal in the $|B\rangle$, $|B'\rangle$ basis, *i.e.*

$$\rho_D = c_{11}|B\rangle\langle B| + c_{22}|B'\rangle\langle B'| \ , \tag{39}$$

(after adaptation to our notation),

$$tr\ \rho_{ID} = tr\ \rho_D = 1 \ , \tag{40}$$

and

$$P_{ID} \geq 0,\ \ P_D \geq 0 \ . \tag{41}$$

Since ρ_D is a diagonal density operator in the specified basis, one can prepare an ensemble with a definite proportion c_{11} in the state $|B\rangle$ and a definite proportion c_{22} in the state $|B'\rangle$ such that this ensemble is represented by ρ_D. It is this consideration that leads Mandel to identify P_D as the degree of path distinguishability when ρ is given. Mandel also shows that

$$P_D = 1 - \frac{|\rho_{12}|}{(\rho_{11}\rho_{22})^{1/2}} \ , \tag{42}$$

where ρ_{ij} is the ij^{th} matrix element of ρ in the $|B\rangle, |B'\rangle$ basis.

Now let us consider the $|\Theta\rangle$ of Eq.(18), which we can rewrite as

$$|\Theta\rangle = N_1^{1/2}|\bar{\chi}_1\rangle|B\rangle + \lambda\ N_1^{1/2}|\bar{\chi}_1\rangle|B'\rangle + N_3^{1/2}|\bar{\chi}_3\rangle|B'\rangle\ . \tag{43}$$

By the standard procedure for writing the density matrix of particle 2 of a two-particle system,[5] we obtain (with the help of Eq.(23)),

$$\begin{gathered} \rho_{11} = N_1\ , \\ \rho_{12} = N_1\lambda\ ,\ \rho_{21} = N_1\lambda^*\ , \\ \rho_{22} = N_1|\lambda|^2 + N_3 = 1 - N_1\ . \end{gathered} \tag{44}$$

Hence, Eq.(37) can be rewritten as

$$E = (1 - 4|\rho_{12}|^2)^{1/2}\ , \tag{45}$$

which can be shown as follows to be greater than or equal to P_D of Eq.(42).

Proof: First note that if x and y are real numbers in the interval $[0,1]$ which sum to unity, then

$$xy \leq \frac{1}{4}\ , \tag{46}$$

from which it follows that

$$(\rho_{11})^{1/2}(\rho_{22})^{1/2} \leq \frac{1}{2}\ . \tag{47}$$

Furthermore, since, by Eq.(23)

$$N_1^2\ |\lambda|^2 \leq N_1(1 - N_1 - N_3) \leq N_1(1 - N_1)\ ,$$

we have

$$|\rho_{12}| = N_1|\lambda| \leq \frac{1}{2}\ . \tag{48}$$

From Eqs.(47) and (48) we obtain

$$1 - 4|\rho_{12}|^2 \geq 1 - 2|\rho_{12}| \geq 1 - \frac{|\rho_{12}|}{(\rho_{11}\rho_{22})^{1/2}}\ , \tag{49}$$

where the lhs of this inequality is E^2 and the rhs is P_D^2. Since both E and P_D are non-negative, it follows that

$$E \geq P_D\ . \tag{50}$$

We note that when E is unity, so is P_D: that is, perfect distinguishability (in our sense) on the basis of the two-photon state $|\Theta\rangle$ implies perfect distinguishability (in Mandel's sense) on the basis of the density operator. There is an intuitive reason for this agreement: $E = 1$ implies that there is perfect correlation between the behavior of photon 1 and the entrance of photon 2 into $|B\rangle$ or $|B'\rangle$, but perfect correlation requires the orthogonality of $|\chi_1\rangle$ and $|\chi_2\rangle$ in Eq.(18). This orthogonality, in turn, guarantees that the density operator of photon 2 is diagonal in the $|B\rangle, |B'\rangle$ basis.

If we look at the other extreme, however, we find that $P_D = 0$ does not imply that $E = 0$. Again there is an intuitive reason. When $P_D = 0$, then ρ is a pure case, derived from a quantum state of the form

$$|\psi\rangle = c|B\rangle + c'|B'\rangle \ , \tag{51}$$

so that

$$\begin{aligned} \rho_{11} &= |c|^2 \ , \\ \rho_{12} = cc'^*, \ \rho_{21} &= c^*c' \ , \\ \rho_{22} &= |c'|^2 \ , \end{aligned} \tag{52}$$

Then

$$\begin{aligned} E - P_D &= -4|\rho_{12}|^2 + \frac{|\rho_{12}|}{(\rho_{11}\rho_{22})^{1/2}} \ , \\ &= -4|c|^2|c'|^2 + 1 \ , \end{aligned} \tag{53}$$

and this vanishes if and only if $|c|^2 = |c'|^2 = \frac{1}{2}$. But when the amplitudes of $|B\rangle$ and $|B'\rangle$ in the pure state $|\psi\rangle$ are equal, there is no strategy for betting on the path that will yield a net gain on the average. On the other hand, when $|c|^2$ and $|c'|^2$ are unequal, the strategy of betting on the path associated with the larger coefficient will yield a net gain on the average. The advantage of our E over P_D is the ability of the former to take advantage of inequalities in the amplitudes associated with the two paths.

Mandel also relates path distinguishability to the visibility v_2 of the interference pattern, where

$$v_2 = 2|\rho_{12}| \ . \tag{54}$$

He obtains the inequality

$$v_2 \leq P_{ID} = 1 - P_D \ , \tag{55}$$

with equality holding only when $\rho_{11} = \rho_{22}$. We obtain from the expressions for E and v_2 in Eqs.(45) and (54) the equation

$$v_2 = (1 - E^2)^{1/2} \ , \tag{56}$$

which holds for any preparation of an ensemble of photons in states $|B\rangle$ and $|B'\rangle$ derived from a two-photon state of the form $|\Theta\rangle$. Hence, for the preparation of photon 2 that we have been studying, the visibility v_2 is a natural measure of path indistinguishibility.

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