6p 324762 1995/08120

N95-14534

EQUILIBRIUM FLUID INTERFACE BEHAVIOR UNDER LOW- AND ZERO-GRAVITY CONDITIONS

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INTRODUCTION

We describe here some of our recent mathematical work, which forms a basis for the Interface Configuration Experiment scheduled for USML-2. The work relates to the design of apparatus that exploits microgravity conditions for accurate determination of contact angle. The underlying motivation for the procedures rests on a discontinuous dependence of the capillary free surface interface S on the contact angle γ, in a cylindrical capillary tube whose section (base) Ω contains a protruding corner with opening angle 2α, see Figure 1. Specifically, in a gravity-free environment, Ω can be chosen so that, for all sufficiently large fluid volume, the height of S is uniquely determined as a (single-valued) function $u(x,y)$ entirely covering the base; the height u is bounded over Ω, uniformly in γ throughout the range $\left|\gamma - \frac{\pi}{2}\right| \leq \alpha$, while for $\left|\gamma - \frac{\pi}{2}\right| > \alpha$ the fluid will necessarily move to the corner and uncover the base, rising to infinity (or falling to negative infinity) at the vertex, regardless of volume. Background details and historical discussion are given in [1], [2], [3]. We mention here only that procedures based on the phenomenon promise excellent accuracy when γ is close to $\pi/2$ but may be subject to experimental error when γ is close to zero (or π), as the "singular" part of the domain over which the fluid accumulates (or disappears) when a critical angle γ_0 is crossed then becomes very small and may be difficult to observe. In what follows, we ignore the trivial case $\gamma = \pi/2$ (planar free surface interface), to simplify the discussion.

CANONICAL PROBOSCIS VESSELS

As a way to overcome the experimental difficulty, "canonical proboscis" sections Ω were introduced in [4]. These domains consist of a circular arc attached symmetrically to a (symmetric) pair of curves described by

$$x = \sqrt{{R_0}^2 - y^2} + R_0 \sin\gamma_0 \ln \frac{\sqrt{{R_0}^2 - y^2}\cos\gamma_0 - y\sin\gamma_0}{R_0 + y\cos\gamma_0 + \sqrt{{R_0}^2 - y^2}\,\sin\gamma_0} + C, \qquad (1)$$

and meeting at a point P on the x-axis, see Figure 2. Here R_0, as well as the particular points of attachment, may be chosen arbitrarily. The case $\gamma_0 < \pi/2$ is illustrated; the supplementary one $\gamma_0 > \pi/2$ is reduced to that one on replacing the height u by its negative. We discuss here only the case $\gamma_0 < \pi/2$ in what follows. The (continuum of) circular arcs Γ_0 are all horizontal translates of a given such arc, of radius R_0 and with center on the x-axis, and the curves (1) have the property that they meet all the arcs Γ_0 in the constant angle γ_0. If the radius ρ of the circular boundary arc can be chosen in such a way that

$$|\Sigma| R_0 \cos\gamma_0 = |\Omega|, \qquad (2)$$

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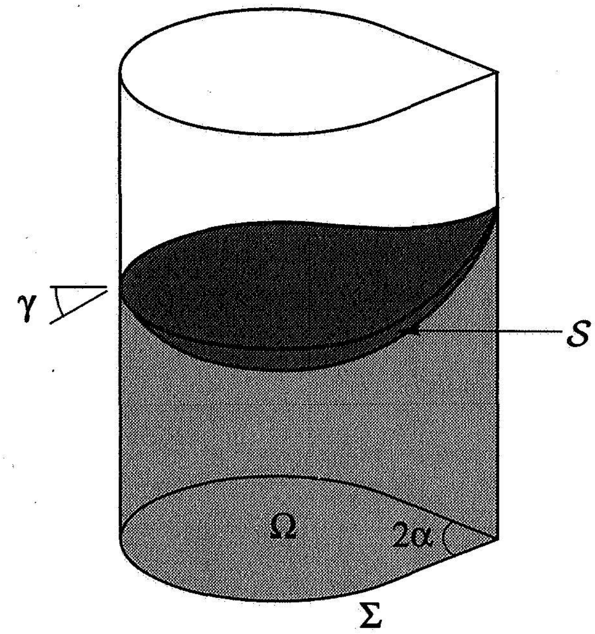

Figure 1. *Capillary surface in cylindrical tube with protruding corner in section.*

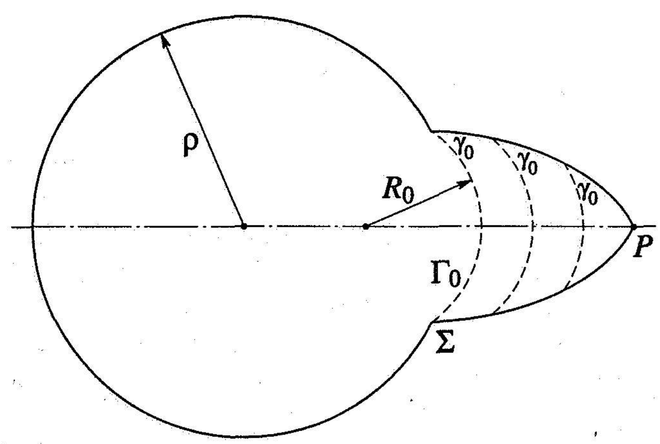

Figure 2. *Proboscis domain showing three members of the continuum of extremal arcs.*

then the arcs Γ_0 become extremals for a "subsidiary" variational problem ([5], see also [2, Chap. 6], [3]) determined by the functional

$$\Phi \equiv |\Gamma| - |\Sigma^*| \cos\gamma + \frac{|\Sigma|}{|\Omega|} |\Omega^*| \cos\gamma \tag{3}$$

defined over piecewise smooth arcs Γ, see Figure 3. ($|\Sigma|$ and $|\Omega|$ denote respectively the length of Σ and area of Ω.) It can be shown [5], [2] that every extremal for Φ is a subarc of a semicircle of radius R_0, with center on the side of Γ exterior to Ω^*, and that it meets Σ in angles $\geq \gamma_0$ on the side of Γ within Ω^*, and $\geq \pi - \gamma_0$ on the other side of Γ (and thus in angle γ_0 within Ω^* whenever the intersection point is a smooth point of Σ). It is remarkable that whenever (2) holds, $\Phi = 0$ for every Ω^* that is cut off in the proboscis by one of the arcs Γ_0; see [4] and the references cited there.

In [4], a value for ρ was obtained empirically from (2) in a range of configurations, and it was conjectured that the angle γ_0 on which the construction is based would be critical for the geometry.

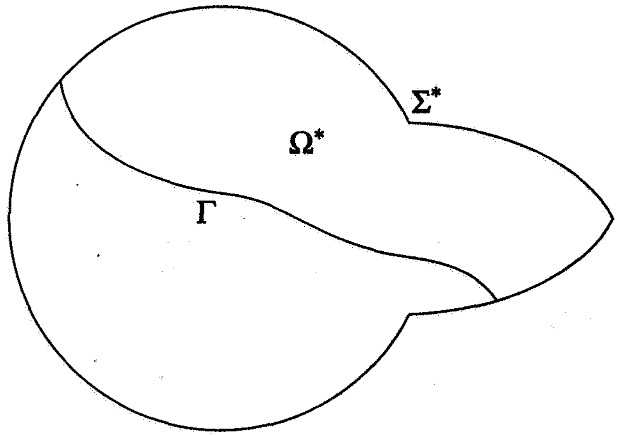

Figure 3. *Admissible configuration for variational problem.*

That is, a solution of the Young-Laplace capillary free-surface equation [2, Chap. 1]

$$\operatorname{div} Tu = \frac{|\Sigma|}{|\Omega|}\cos\gamma \quad \text{in } \Omega,$$

$$\nu \cdot Tu = \cos\gamma \quad \text{on } \Sigma,$$

where

$$Tu \equiv \frac{\nabla u}{\sqrt{1+|\nabla u|^2}}$$

and ν is the exterior unit normal on Σ, should exist in Ω if and only if $\left|\gamma - \frac{\pi}{2}\right| < \left|\gamma_0 - \frac{\pi}{2}\right|$. Additionally, the surface height should approach infinity as $\left|\gamma - \frac{\pi}{2}\right| \nearrow \left|\gamma_0 - \frac{\pi}{2}\right|$, exactly in the region swept out by the arcs Γ_0. And if upper and lower bounds for ρ can be found, independent of γ and of the points of attachment of the circular arc portion of the boundary, then the section Ω can be designed so as to be contained in a fixed rectangle and so that the "singular" ${\Omega_0}^*$ contains another fixed rectangle, for all choices of γ bounded away from $\pi/2$. If γ is close to $\pi/2$ then previously applied techniques for determining contact angle in corner domains with planar walls are effective, see [6, p.136]. Thus, the experimental apparatus can be built so as to remain bounded in size, and so that the observed singular behavior occurs over a fixed rectangle, for all eventual measurements in the range of γ for which the previous techniques do not apply well.

For the above conjectures, which form the basis of our proposed procedure and for which the mathematical underpinnings were proved only partially in [4], complete mathematical proofs have been obtained. Specifically, it is shown in [7]:

Theorem 1. *For any γ_0 in the range $0 \le \gamma_0 < \pi/2$ and for any point of attachment, there exists a unique solution ρ of* (2) *as an unbranched continuation of the unique value $\rho = 2R_0\cos\gamma_0$ obtained when the point of attachment is at P, and there holds $R_0\cos\gamma_0 \le \rho \le 2R_0$.*

Theorem 2. *Let ρ be determined as in Theorem* 1 *for a proboscis domain Ω, and let Γ be a subarc in Ω of a semicircle of radius R_0. Let Ω_0^* be the portion of Ω cut out by Γ on the side exterior to its center. Then $\Phi[\Omega_0^*;\gamma_0] \ge 0$, equality holding if and only if Γ is one of the extremal arcs indicated in Figure* 2.

The stated geometrical properties of Ω follow from Theorem 1, from the relation $2\alpha = \pi - 2\gamma_0$ for the angle 2α formed at P, and from the convexity properties of the curves (1) that are considered. Using general results established in [5] and in [2, Chap. 6], Theorem 2 establishes that γ_0 is the

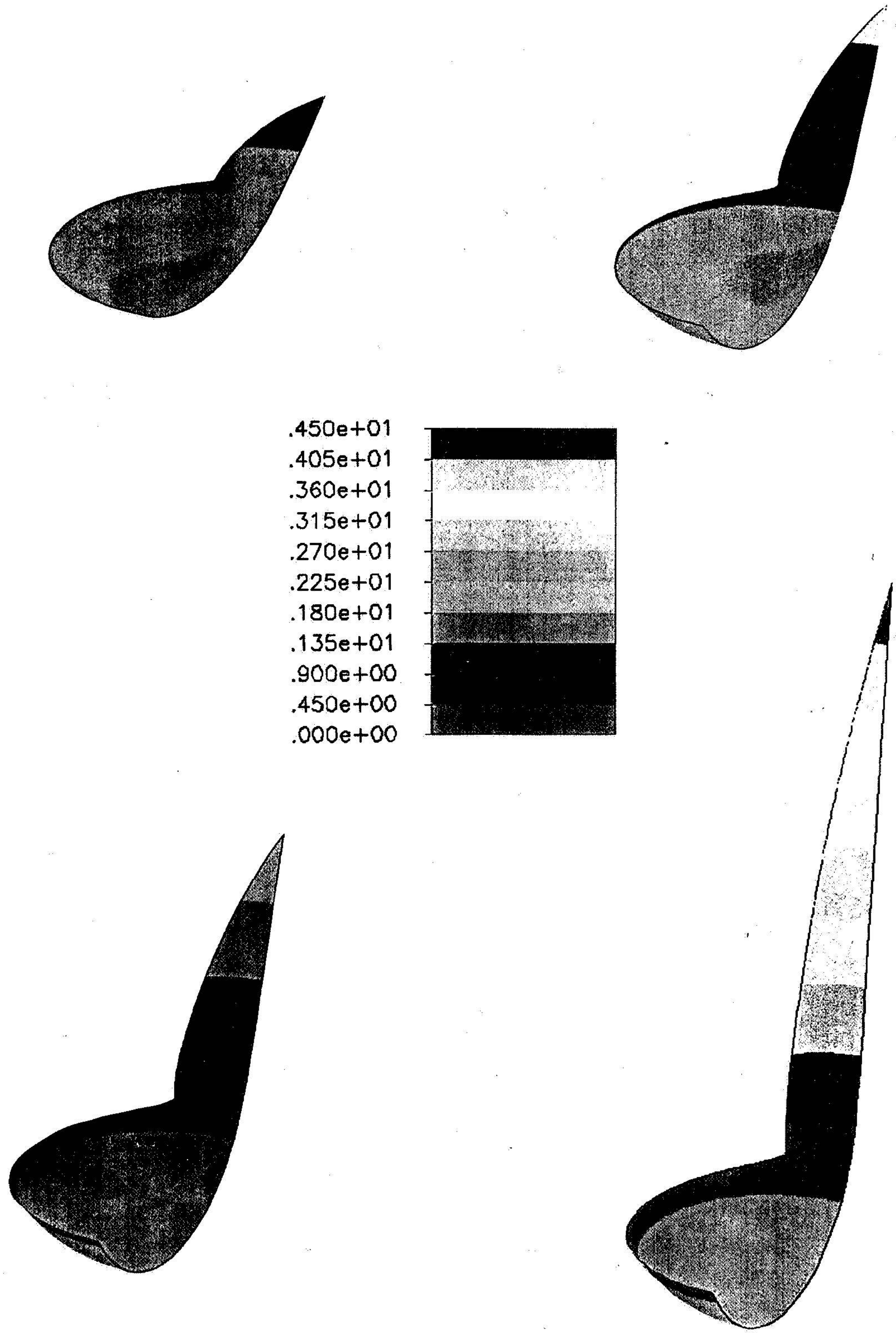

Figure 4. *Equilibrium interfaces, proboscis domain;* $\gamma = 60°, 40°, 35°, 31°$; $\gamma_0 = 30°$.

critical angle and also that there are no "extraneous" extremals, that is, the proboscis domain swept out by the extremal arcs Γ indicated in Figure 2 is exactly the domain in which the fluid height will rise unboundedly as $\left|\gamma - \frac{\pi}{2}\right| \nearrow \left|\gamma_0 - \frac{\pi}{2}\right|$.

Figure 4 is taken from [8] and shows results of computer calculations for a particular proboscis domain with $\rho = 1$ and proboscis length approximately 2/3, for four contact angles γ decreasing to the critical $\gamma_0 = 30°$. The individual figures show projections of the three dimensional surfaces, cut along the plane of symmetry, with height above the center of the circular arc portion indicated according to the scale of shadings shown. It is seen that although the fluid rise in the corner is not discontinuous as occurs for a planar wedge, the rise height in the proboscis is relatively modest until γ becomes very close to γ_0, and then becomes extremely rapid (at $\gamma = \gamma_0$ the rise height would be infinite). The proximity of γ to γ_0 could thus be readily evidenced by sensors close to the vertex, near the top of a container of carefully selected height. The method appears to open a prospect for contact angle measurements more accurate than can be obtained with presently available methods. The experiment for testing the proposed procedure on board the forthcoming USML-2 Space Shuttle flight is being designed and fabricated in collaboration with M. Weislogel of NASA Lewis Research Center. It is planned to use video cameras for recording the fluid behavior. The fabricated test containers will have two diametrically opposed canonical proboscis extensions to a circular cylinder, instead of just the one shown in Figure 2. These extensions will correspond to differing critical contact angles, to allow bracketing of the determined contact angle value.

CONCLUSION

The mathematical underpinnings have been completed for design of the canonical proboscis vessels for the USML-2 Interface Configuration Experiment. Testing of the mathematical results, in the presence of physical effects not included in the idealized Young-Laplace mathematical theory, such as those associated with contact-line resistance and inaccuracies in fabricating the vessels, awaits completion of the design and fabrication of the vessels and commencement of the experiment.

ACKNOWLEDGMENTS

This work was supported in part by the Applied Mathematical Sciences Subprogram of the Office of Energy Research, Department of Energy, under Contract Number DE-AC03-76SF00098, by the National Aeronautics and Space Administration under Grant NCC3-329, and by the National Science Foundation under Grant DMS89-02831. Part of the work was facilitated by the Undergraduate Research Opportunities Program at Stanford University.

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