SMALL-AMPLITUDE DISTURBANCES IN TURBOMACHINE FLOWS WITH SWIRL

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Abstract

A method is proposed for determining the response of the steady swirling potential flow past a blade row in an axial turbomachine to the most general type of 'non-acoustic' incident disturbance. The method is based on Goldstein's (1978) decomposition of the disturbance velocity and only requires solving a linear inhomogeneous wave equation. It is believed that numerical solutions to this wave equation can be obtained more efficiently and will be more accurate than corresponding solutions to the linearized compressible Euler equations.

1. Introduction

The response of the steady compressible flow past a blade row in a turbomachine annulus to both steady and unsteady disturbances is of technological importance because of the effect such disturbances have on the performance of the turbomachine. The disturbances can arise from inlet-flow non-uniformities as well as blade-interaction effects and are often small enough to be described by linear theory. Furthermore, viscous effects are typically confined to thin boundary layers and wakes so the flow can be regarded as inviscid to good approximation.

The behavior of small-amplitude disturbances in uniform flows is well understood

(Kovásznay 1953). The vorticity, entropy and pressure fluctuations associated with such disturbances are decoupled when the flow is unbounded and coupled through the blocking effect caused by the vanishing of the normal velocity on the impermeable surfaces when the flow is bounded. The situation is more complicated when the base flow is not uniform however. Goldstein (1978) presents a general theory for describing the behavior of small-amplitude vortical and entropic disturbances on arbitrary potential flows. This theory shows how the distortion of the vortical part of the disturbance velocity as it is convected by the steady base flow also gives rise to pressure fluctuations. In turbomachine flows, the vorticity and pressure disturbances are coupled through both the blocking and base-flow distortion effects. This is true even between the blade rows since the base-flow velocity there typically has a swirl component that is of the same order as the throughflow component.

The purpose of this paper is to present a relatively simple method for determining the behavior of small-amplitude disturbances on the steady swirling flow past a blade row in an axial turbomachine. To this end, it is assumed that the base flow is irrotational, i.e. that the vortical and entropic parts of the motion are small enough to be completely accounted by the disturbance field. This assumption precludes the possibility of Rayleigh-type instabilities and allows one to use Goldstein's (1978) decomposition of the disturbance velocity. This is done in §2 by writing the disturbance velocity as the sum of a vortical part $\boldsymbol{u}_1^{(I)}$ that is a known function of the imposed disturbance and a potential part $\nabla\phi_1$ that satisfies a linear inhomogeneous wave equation. Since the perturbation potential ϕ_1 determines the pressure fluctuations, it follows from the above discussion that ϕ_1 does not vanish in the upstream region where the affect of the blade row is negligible. In order to complete the formulation, an upstream boundary condition for ϕ_1 must be derived. This is done in §3 where an analytic solution for ϕ_1 is constructed for the case when there is no intervening blade row and the base flow is given by the superposition of a uniform stream and a line vortex. The results of §3 are then used together with the decomposition of the perturbation potential introduced by Atassi & Grzedzinski (1989) to obtain a problem for ϕ_1 that is suitable for numerical integration. Finally, the results are discussed in §5.

2. Formulation

Consider a steady inviscid non-heat-conducting compressible flow past a blade row in an axial turbomachine and suppose that a small-amplitude (steady or unsteady) disturbance is introduced in an upstream region where the effect of the blade row is negligible. It is assumed that the vortical and entropic parts of the motion in this upstream region are small enough to be completely accounted for by the disturbance field and further that any shocks waves that may exist always remain quite weak. The total flow will then be a linear perturbation about a steady irrotational base flow and will have velocity, pressure, density and entropy fields that expand as

$$\boldsymbol{u} = \boldsymbol{u}_0(r, \theta, z) + \boldsymbol{u}_1(r, \theta, z, t) + \cdots, \tag{2.1}$$

$$p = p_0(r, \theta, z) + p_1(r, \theta, z, t) + \cdots, \tag{2.2}$$

$$\rho = \rho_0(r, \theta, z) + \rho_1(r, \theta, z, t) + \cdots, \tag{2.3}$$

$$s = s_1(r, \theta, z, t) + \cdots, \tag{2.4}$$

respectively, where (r, θ, z) are cylindrical coordinates chosen so that the z-axis is coincident with the axis of the annulus and is pointing in the direction of the upstream throughflow, t is time and without loss of generality the constant base-flow entropy has been set equal to zero. The origin of the (r, θ, z) coordinate system is taken to be in the plane of the blade row. The unit vectors in the radial, circumferential and axial directions are denoted $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$ and $\boldsymbol{e}_z$, respectively, while the velocity components in these three directions are denoted u, v and w. Attention is restricted to a calorically perfect ideal gas so the specific heats c_p and c_v are constant and the temperature T is related to the pressure and density by $p = (c_p - c_v)\rho T$.

The inner and outer radii of the turbomachine annulus can vary with z, but in the upstream region, it will be assumed that these radii take on the constant values r_i and r_o, respectively. Then the base flow becomes independent of z as $z \to -\infty$ and is given by

$$\left.\begin{aligned} \boldsymbol{u}_0 = \boldsymbol{e}_\theta \Gamma_\infty / r + \boldsymbol{e}_z W_\infty, &\quad p_0/\rho_0^\kappa = \text{constant}, \\ \rho_0^{\kappa-1}/c_0^2 = \text{constant}, &\quad c_0^2 = (\kappa - 1)\,(H - \boldsymbol{u}_0 \cdot \boldsymbol{u}_0/2), \end{aligned}\right\} \tag{2.5}$$

where $2\pi\Gamma_\infty$ and $\boldsymbol{e}_z W_\infty$ are the constant circulation and throughflow velocity of the upstream base flow, $c_0 = (\kappa p_0/\rho_0)^{\frac{1}{2}}$ is the base-flow speed of sound, H is the constant stagnation enthalpy of the base flow and $\kappa = c_p/c_v$.

The perturbation quantities in (2.1)–(2.4) are governed by the linearized compressible Euler equations which Goldstein (1978) shows can be reduced to a single linear inhomogeneous wave equation when the base flow is irrotational. Using this reduction in the present formulation leads to

$$\boldsymbol{u}_1 = \boldsymbol{u}_1^{(I)} + \nabla \phi_1, \tag{2.6}$$

$$p_1 = -\rho_0 \frac{D_0}{Dt} \phi_1, \tag{2.7}$$

$$\rho_1 = \frac{1}{c_0^2} \frac{p_1}{\rho_0} - \frac{s_1}{c_p}, \tag{2.8}$$

$$s_1 = S(\xi_i - \delta_{i3} W_\infty t), \tag{2.9}$$

where $D_0/Dt = \partial/\partial t + \boldsymbol{u}_0\cdot\nabla$ is the convective derivative associated with the steady base flow, δ_{ij} is the Kronecker delta tensor and the $\xi_i - \delta_{i3}W_\infty t = (\xi_1, \xi_2, \xi_3 - W_\infty t)$ denote three functionally independent Lagrangian variables for the base flow.

The vortical part of the velocity perturbation $\boldsymbol{u}_1^{(I)}$ is given by

$$\boldsymbol{u}_1^{(I)} = A_j(\xi_i - \delta_{i3}W_\infty t)\nabla\xi_j + \frac{\boldsymbol{u}_0}{2c_p}S(\xi_i - \delta_{i3}W_\infty t), \tag{2.10}$$

where the repeated index j is summed from 1 to 3. The perturbation potential ϕ_1 satisfies the linear inhomogeneous wave equation,

$$\frac{D_0}{Dt}\left(\frac{1}{c_0^2}\frac{D_0}{Dt}\phi_1\right) - \frac{1}{\rho_0}\nabla\cdot(\rho_0\nabla\phi_1) = \frac{1}{\rho_0}\nabla\cdot(\rho_0\boldsymbol{u}_1^{(I)}), \tag{2.11}$$

within the flow field and the inviscid boundary condition,

$$\boldsymbol{n}\cdot\nabla\phi_1 = -\boldsymbol{n}\cdot\boldsymbol{u}_1^{(I)}, \tag{2.12}$$

on the impermeable boundaries of the flow where $\boldsymbol{n}$ is a unit normal vector.

ξ_1 and ξ_2 are two independent stream functions for the steady base flow and ξ_3/W_∞ is the drift function (Darwin 1954; Lighthill 1956) associated with that flow. Therefore the $\xi_i(r,\theta,z)$ are determined by

$$\boldsymbol{u}_0\cdot\nabla\xi_i = \delta_{i3}W_\infty, \tag{2.13}$$

subject to the upstream boundary conditions,

$$\xi_i = \left(r, \quad \theta - \frac{\Gamma_\infty}{W_\infty}\frac{z}{r^2}, \quad z\right) \tag{2.14}$$

as $z \to -\infty$, where (2.14) is arrived at by using (2.5). These equations can be integrated to give the following analytic expression for ξ_3:

$$\xi_3 = z + \int_{-\infty}^{\phi_0} \frac{W_\infty - w_0\left(r_s(\xi_1,\xi_2,\phi'), \theta_s(\xi_1,\xi_2,\phi'), z_s(\xi_1,\xi_2,\phi')\right)}{|\boldsymbol{u}_0\left(r_s(\xi_1,\xi_2,\phi'), \theta_s(\xi_1,\xi_2,\phi'), z_s(\xi_1,\xi_2,\phi')\right)|^2}\,\mathrm{d}\phi', \tag{2.15}$$

where ϕ_0 is the base-flow velocity potential, i.e. $\boldsymbol{u}_0 = \nabla\phi_0$, and r_s, θ_s and z_s give the variation of the cylindrical coordinates along a base-flow streamline and are determined by solving $\xi_1 = \xi_1(r,\theta,z)$, $\xi_2 = \xi_2(r,\theta,z)$ and $\phi_0 = \phi_0(r,\theta,z)$ for r, θ and z.

The functions S and A_j of $\xi_i - \delta_{i3}W_\infty t$ are arbitrary apart from the restriction that they be periodic in ξ_2 which follows from (2.14) and the θ-periodicity of the flow field. S is related to the entropy by (2.9) while the A_j can be related to the vorticity which is given as

$$\boldsymbol{\omega}_1 = \nabla\times\boldsymbol{u}_1 = \nabla\xi_k\times\nabla\xi_j\frac{\partial}{\partial\xi_k}\left(A_j + \frac{S}{2c_p}\frac{\partial\phi_0}{\partial\xi_j}\right). \tag{2.16}$$

Before deriving an explicit expression for A_j in terms of $\boldsymbol{\omega}_1$, it is noted that the velocity decomposition (2.6) is not unique since an arbitrary function of $\xi_i - \delta_{i3}W_\infty t$ can be added to ϕ_1 and its gradient subtracted from $\boldsymbol{u}_1^{(I)}$ to produce a new decomposition that also satisfies (2.7)–(2.12). In the present formulation, the velocity decomposition is made unique by requiring that the circumferential average of A_1 and the circumferential derivative of A_2 both equal zero. Forming the scalar product of (2.16) and $\nabla\xi_l$, multiplying the result by the alternating unit tensor ε_{lmn}, and using the identity $\varepsilon_{jkl}\varepsilon_{lmn} = \delta_{jn}\delta_{km} - \delta_{jm}\delta_{kn}$ produces

$$\frac{\partial}{\partial\xi_j}A_k - \frac{\partial}{\partial\xi_k}A_j = \Omega_{jk}, \tag{2.17}$$

where the skew-symmetric tensor Ω_{jk} is given by

$$\Omega_{jk}(\xi_i - \delta_{i3}W_\infty t) = \frac{\varepsilon_{jkl}\boldsymbol{\omega}_1\cdot\nabla\xi_l}{\nabla\xi_1\cdot\nabla\xi_2\times\nabla\xi_3} + \frac{1}{2c_p}\frac{\partial(\phi_0, S)}{\partial(\xi_j,\xi_k)}, \tag{2.18}$$

and $\partial(\cdot,\cdot)/\partial(\cdot,\cdot)$ is the Jacobian derivative. Then, in view of the restrictions on A_1 and A_2 given above, integration of (2.17) yields

$$A_j = \int \bar{\Omega}_{1j}\mathrm{d}\xi_1 + \int\left(\Omega_{2j} - \bar{\Omega}_{2j}\right)\mathrm{d}\xi_2, \tag{2.19}$$

where a bar is used to denote circumferentially averaged quantities. Equations (2.9) and (2.19) show that the functions S and A_j can be completely determined from knowledge of the entropy and vorticity fields on a single surface of constant ξ_3.

The formulation is completed by specifying an upstream boundary condition for (2.11). As indicated above, the coupling of the vorticity and pressure fluctuations through the dipole-type source term $\rho_0^{-1}\nabla\cdot(\rho_0 \boldsymbol{u}_1^{(I)})$ in (2.11) and the $\boldsymbol{n}\cdot\boldsymbol{u}_1^{(I)}$ term in (2.12) continues in the upstream region. Consequently, ϕ_1 does not vanish as $z \to -\infty$. The appropriate upstream limit for ϕ_1 is determined by first constructing a solution to (2.11) and (2.12) that applies when there is no blade row.

3. Solution for ϕ_1 in the absence of the blade row

A formal analytic solution for the perturbation potential in an annulus with constant inner and outer radii where the base flow is given by (2.5) can be derived by considering a single harmonic component of the vorticity and entropy fluctuations, i.e. by letting

$$\begin{pmatrix} A_j \\ S \end{pmatrix} = \begin{pmatrix} \hat{A}_j(\xi_1) \\ \hat{S}(\xi_1) \end{pmatrix} \exp\{\mathrm{i}[k_2\xi_2 + k_3(\xi_3 - W_\infty t)]\} \tag{3.1}$$

where k_2 and k_3 are constant wavenumbers (the former of which is limited to integer values due to the θ-periodicity of the flow field) and

$$\left.\begin{aligned} \hat{A}_1 = 0 \quad \text{when} \quad k_2 = 0 \\ \hat{A}_2 = 0 \quad \text{when} \quad k_2 \neq 0 \end{aligned}\right\} \tag{3.2}$$

in order to satisfy the restrictions on A_1 and A_2. Then since (2.11) and (2.12) are linear the solution for a general vortical/entropic disturbance can be obtained by superposition.

Since interest here is in the long-time response due to a small-amplitude disturbance forced at a fixed axial location say $z = z_\infty$ where $-z_\infty \gg 1$, ϕ_1 and $\partial\phi_1/\partial t$ are taken to be zero at $t = 0$. Substituting (3.1) into (2.10) and the result along with (2.5) into (2.11) and (2.12) and taking the Laplace transform with respect to time then yields

$$\frac{1}{c_0^2}\left(\mathrm{i}\omega + \boldsymbol{u}_0\cdot\nabla\right)^2\breve{\phi}_1 - \frac{1}{\rho_0}\nabla\cdot(\rho_0\nabla\breve{\phi}_1) = \frac{\hat{f}}{\mathrm{i}\nu}\mathrm{e}^{\mathrm{i}(k_2\theta + \alpha z)}, \tag{3.3}$$

for $r_i < r < r_o$ and

$$\frac{\partial}{\partial r}\breve{\phi}_1 = -\frac{\hat{g}}{\mathrm{i}\nu}\mathrm{e}^{\mathrm{i}(k_2\theta + \alpha z)}, \tag{3.4}$$

for $r = r_i, r_o$ where

$$\breve{\phi}_1(r,\theta,z\,|\,\omega) = \int_0^\infty \mathrm{e}^{-\mathrm{i}\omega t}\phi_1(r,\theta,z,t)\mathrm{d}t \tag{3.5}$$

is the temporal Laplace transform of ϕ_1, $\nu = \omega + k_3 W_\infty$ is a shifted transform frequency,

$$\alpha(r) = -\frac{\Gamma_\infty}{W_\infty}\frac{k_2}{r^2} + k_3 \tag{3.6}$$

is the r-dependent axial wavenumber of the vorticity and entropy fluctuations, and

$$\begin{aligned}\hat{f}(r,z) = \frac{1}{r\rho_0}\left(\frac{\partial}{\partial r} + 2\mathrm{i}k_2\frac{\Gamma_\infty}{W_\infty}\frac{z}{r^3}\right)\left(r\rho_0\hat{A}_1\right) + \frac{\Gamma_\infty}{W_\infty}\frac{1}{r\rho_0}\left(2z\frac{\partial}{\partial r} - \mathrm{i}k_3 r\right)\left(\frac{\rho_0}{r^2}\hat{A}_2\right)\\ + \mathrm{i}\alpha\hat{A}_3 + \mathrm{i}k_3\frac{W_\infty}{2c_p}\hat{S}\end{aligned} \tag{3.7}$$

and

$$\hat{g}(r,z) = \hat{A}_1 + 2\frac{\Gamma_\infty}{W_\infty}\frac{z}{r^3}\hat{A}_2 \tag{3.8}$$

are the harmonic coefficients of the dipole source term $\rho_0^{-1}\nabla\cdot(\rho_0\boldsymbol{u}_1^{(I)})$ and the radial component of the vortical velocity perturbation $u_1^{(I)}$, respectively.

In order to simplify the subsequent analysis, $\breve{\phi}_1$ is decomposed as

$$\breve{\phi}_1 = \frac{1}{\mathrm{i}\nu}\mathrm{e}^{\mathrm{i}k_2\theta}[\breve{q}_1(r,z\,|\,\nu) + \breve{q}_2(r,z\,|\,\nu)] + \breve{\psi}(r,\theta,z\,|\,\omega), \tag{3.9}$$

where the $\check{q}_n$ are determined by the inhomogeneous equations,

$$\left[\frac{W_\infty^2}{c_0^2}\left(\frac{\mathrm{i}\nu}{W_\infty} - \mathrm{i}\alpha + \frac{\partial}{\partial z}\right)^2 - \frac{1}{r\rho_0}\frac{\partial}{\partial r}\left(r\rho_0\frac{\partial}{\partial r}\right) + \frac{k_2^2}{r^2} - \frac{\partial^2}{\partial z^2}\right]\check{q}_n = \delta_{n1}\hat{f}\mathrm{e}^{\mathrm{i}\alpha z}, \tag{3.10}$$

for $r_i < r < r_o$ and

$$\frac{\partial}{\partial r}\check{q}_n = -\delta_{n2}\hat{g}\mathrm{e}^{\mathrm{i}\alpha z}, \tag{3.11}$$

for $r = r_i, r_o$, and $\check{\psi}$ is determined by the homogeneous versions of (3.3) and (3.4). It follows from (2.7) that the $\check{q}_n$ account for the pressure fluctuations associated with the vortical entropic (or 'non-acoustic') part of the disturbance field while $\check{\psi}$ accounts for those fluctuations associated with the irrotational isentropic (or 'acoustic') part of that field.

The solution for $\check{q}_1$ is found by introducing

$$\check{q}_1 = \int_{r_i}^{r_o} \mathrm{e}^{\mathrm{i}\alpha(r')z}\hat{q}_1(r, r', z \,|\, \nu)\mathrm{d}r' \tag{3.12}$$

into (3.10) and (3.11) to obtain

$$\left[\mathrm{L}_{k_2,\alpha',\omega} + 2\mathrm{i}\left(\alpha' - \frac{W_\infty^2}{c_0^2}\lambda_{k_2,\alpha',\omega}\right)\frac{\partial}{\partial z} + \left(1 - \frac{W_\infty^2}{c_0^2}\right)\frac{\partial^2}{\partial z^2}\right]\hat{q}_1 = -\delta(r - r')\hat{f}(r', z), \tag{3.13}$$

and

$$\frac{\partial}{\partial r}\hat{q}_1 = 0, \tag{3.14}$$

respectively, where

$$\mathrm{L}_{m,\sigma,\omega} = \frac{1}{r\rho_0}\frac{\partial}{\partial r}\left(r\rho_0\frac{\partial}{\partial r}\right) - \left(\frac{m^2}{r^2} + \sigma^2 - \frac{W_\infty^2}{c_0^2}\lambda_{m,\sigma,\omega}^2\right), \tag{3.15}$$

$$\lambda_{m,\sigma,\omega}(r) = \frac{\Gamma_\infty}{W_\infty}\frac{m}{r^2} + \sigma + \frac{\omega}{W_\infty}, \tag{3.16}$$

$\alpha' = \alpha(r')$, and $\delta(r)$ is the Dirac delta function. Then, in view of the linear dependence of $\hat{f}$ on z, $\hat{q}_1$ is written as

$$\hat{q}_1 = \left[K_1(r, r' \,|\, \nu) + \mathrm{i}K_2(r, r' \,|\, \nu)\frac{\partial}{\partial z}\right]\hat{f}(r', z). \tag{3.17}$$

By substituting this expression into (3.13) and (3.14) and using the results of appendix A, one can show that

$$K_1(r, r' \,|\, \nu) = G(r, r' \,|\, k_2, \alpha', \nu - k_3 W_\infty), \tag{3.18}$$

$$K_2(r, r' \,|\, \nu) = -\frac{\partial}{\partial \alpha'} G(r, r' \,|\, k_2, \alpha', \nu - k_3 W_\infty), \tag{3.19}$$

where the Green's function $G(r, r' \,|\, m, \sigma, \omega)$ associated with the operator $\mathrm{L}_{m,\sigma,\omega}$ is given by (A 3). It is now a relatively simple matter to show, by using (A 3) and (A 4), that

$$\check{q}_2 = -\mathrm{e}^{\mathrm{i}\alpha' z} \left[K_1(r, r' \,|\, \nu) + \mathrm{i} K_2(r, r' \,|\, \nu) \frac{\partial}{\partial z} \right] \hat{g}(r', z) \Big|_{r'=r_i}^{r_o}, \tag{3.20}$$

since $\hat{g}$ is also linearly dependent on z.

As indicated above, $\check{\psi}$ arises from the 'acoustic' part of the disturbance field imposed at $z = z_\infty$. The portion of the 'acoustic' field propagating inwards (in the direction of the throughflow) at $z = z_\infty$ can be specified arbitrarily, but the outward-propagating portion must be determined by the solution for $\check{\psi}$. No outward-propagating 'acoustic' waves arise in the absence of the blade row because the base flow is independent of z. Therefore $\check{\psi} = 0$ for $z < z_\infty$ and the solution for $\check{\psi}$ can be found by taking the Fourier transform with respect to θ and the Laplace transform with respect to z of the homogeneous versions of (3.3) and (3.4) to get

$$\left.\begin{aligned} \mathrm{L}_{m,\sigma,\omega} \hat{\psi} &= -\hat{h} \quad && \text{for} \quad r_i < r < r_o \\ \frac{\partial}{\partial r} \hat{\psi} &= 0 && \text{for} \quad r = r_i, r_o \end{aligned}\right\}, \tag{3.21}$$

where

$$\hat{\psi}(r \,|\, m, \sigma, \omega) = \int_0^{2\pi} \int_{z_\infty}^{\infty} \mathrm{e}^{-\mathrm{i}[m\theta + \sigma(z - z_\infty)]} \check{\psi}(r, \theta, z \,|\, \omega) \mathrm{d}z \mathrm{d}\theta, \tag{3.22}$$

and

$$\hat{h}(r \,|\, m, \sigma, \omega) =$$

$$\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}m\theta} \left[2\mathrm{i} \frac{W_\infty^2}{c_0^2} (\lambda_{m,\sigma,\omega} - \sigma) \left(\frac{W_\infty^2}{c_0^2} - 1 \right) \left(\frac{\partial}{\partial z} + \mathrm{i}\sigma \right) \right] \hat{\psi}(r, \theta, z \,|\, \omega) \mathrm{d}\theta \Big|_{z=z_\infty} \quad (3.23)$$

is determined by the inward-propagating 'acoustic' field at $z = z_\infty$. The solution for $\hat{\psi}$ is then just

$$\hat{\psi} = \int_{r_i}^{r_o} G(r, r' \,|\, m, \sigma, \omega) \hat{h}(r' \,|\, m, \sigma, \omega) \mathrm{d}r' \quad (3.24)$$

which when combined with (3.22) leads to

$$\check{\psi} = \frac{1}{(2\pi)^2} \int_{r_i}^{r_o} \sum_{m=-\infty}^{+\infty} \int_{C_\sigma} \mathrm{e}^{\mathrm{i}[m\theta + \sigma(z - z_\infty)]} G(r, r' \,|\, m, \sigma, \omega) \hat{h}(r' \,|\, m, \sigma, \omega) \mathrm{d}\sigma \mathrm{d}r' \quad (3.25)$$

where the integration contour C_σ must lie below the singularities of the integrand so that $\check{\psi} = 0$ for $z < z_\infty$. The singularities correspond to singularities in G which, in view of (A 3), arise whenever

$$D(m, \sigma, \omega) = 0 \quad (3.26)$$

where D is given by (A 4). Therefore the singular points of G result from the discrete spectrum of the wave equation.

It now follows from (3.5), (3.9), (3.17), and (3.20) that the solution for the perturbation potential is

$$\phi_1 = \frac{1}{2\pi} \int_{C_\nu} \mathrm{e}^{\mathrm{i}[k_2\theta + (\nu - k_3 W_\infty)t]} \frac{1}{\mathrm{i}\nu} \check{q}(r, z \,|\, \nu) \mathrm{d}\nu + \frac{1}{2\pi} \int_{C_\omega} \mathrm{e}^{\mathrm{i}\omega t} \check{\psi}(r, \theta, z \,|\, \omega) \mathrm{d}\omega \quad (3.27)$$

where

$$\begin{aligned} \check{q} = \check{q}_1 + \check{q}_2 = \int_{r_i}^{r_o} & \mathrm{e}^{\mathrm{i}\alpha' z} \left[K_1(r, r' \,|\, \nu) + \mathrm{i} K_2(r, r' \,|\, \nu) \frac{\partial}{\partial z} \right] \hat{f}(r', z) \mathrm{d}r' \\ & - \mathrm{e}^{\mathrm{i}\alpha' z} \left[K_1(r, r' \,|\, \nu) + \mathrm{i} K_2(r, r' \,|\, \nu) \frac{\partial}{\partial z} \right] \hat{g}(r', z) \Big|_{r'=r_i}^{r_o}, \end{aligned} \quad (3.28)$$

$\check{\psi}$ is given by (3.25), and in order for the solution to be causal, i.e. $\phi_1 = 0$ for $t < 0$, the integration contours C_ν and C_ω must lie below the singularities of $\check{q}/\nu$ and $\check{\psi}$, respectively.

The long-time behavior of the 'acoustic' part of the perturbation potential is determined by those singularities associated with the discrete spectrum that lie on or below the real-ω axis. The discrete spectrum associated with the base-flow solution (2.5) is considered in Kerrebrock (1977) and the interested reader is referred to that paper for a detailed discussion. However it should be noted that the 'unstable' modes found in Kerrebrock (1977) are excluded from (3.27) by causality. For the remainder of the present study, attention will be restricted to 'non-acoustic' disturbances so $\breve{\psi}$ will be set equal to zero. In addition to contributions from the discrete-spectrum singularities, the long-time behavior of the 'non-acoustic' part of the perturbation potential includes a contribution from the singularity at $\nu = 0$ which determines the response at the forcing frequency of the imposed vortical/entropic disturbance.

4. Solution for ϕ_1 in presence of the blade row

The solution for the perturbation potential derived in the preceding section provides the upstream boundary condition needed to complete the formulation begun in §2. Once the base-flow solution $(\boldsymbol{u}_0, p_0, \rho_0)$ is known and the ξ_i have been determined from (2.13)–(2.15), the response of an actual swirling turbomachine flow to the most general type of 'non-acoustic' incident disturbance can be found from (2.6)–(2.12). However difficulties arise when one tries to solve (2.11) and (2.12) numerically subject to (3.27) as $z \to -\infty$.

It is clear from (2.15) that ξ_3 becomes singular at the base-flow stagnation points, i.e. the points where $\boldsymbol{u}_0 = 0$. The zeros of the Jacobian determinant $\partial(\xi_1, \xi_2, \phi_0)/\partial(r, \theta, z)$ also give rise to singularities in ξ_3 but these points can be shown to coincide with the zeros of $\boldsymbol{u}_0$. The singular behavior of ξ_3 is, in general, transferred through $\boldsymbol{u}_1^{(I)}$ to the perturbation potential ϕ_1 thereby making (2.11), (2.12) and (3.27) unsuitable for numerical integration. Atassi & Grzedzinski (1989) overcome this difficulty by expressing the solution for ϕ_1 as

$$\phi_1 = \varphi_1(\xi_i - \delta_{i3}W_\infty t) + \phi_1^*(r,\theta,z,t), \tag{4.1}$$

where φ_1 is a known function that is singular at the base-flow stagnation points and ϕ_1^* is a non-singular function that satisfies a linear inhomogeneous wave equation.

For simplicity, it is assumed that the base flow has only one stagnation point which is denoted P. The relevant results for flows with multiple stagnation points and/or stagnation lines can be obtained by superposition. The surface formed by the base-flow streamlines issuing from P is denoted Σ and, in general, consists of a portion Σ_b that lies on an impermeable boundary of the flow and a portion Σ_w that lies within the flow itself. By assuming Σ is smooth near P, Atassi & Grzedzinski (1989) show that φ_1 will contain all the singular behavior in ϕ_1 if

$$\boldsymbol{n}\cdot(\boldsymbol{u}_1^{(I)} + \nabla\varphi_1) \to 0 \quad \text{and} \quad \boldsymbol{u}_0\cdot(\boldsymbol{u}_1^{(I)} + \nabla\varphi_1) \to \frac{1}{2c_p}|\boldsymbol{u}_0|^2 S, \tag{4.2}$$

as Σ is approached and furthermore, that it is sufficient to take

$$\varphi_1 = -\int^{\xi_3 - W_\infty t} \left\{A_3(\xi_1^{(0)}, \xi_2^{(0)}, \xi') + \exp\left[a(\xi_3 - \xi' - W_\infty t)\right] B(\xi_1, \xi_2, \xi')\right\} \mathrm{d}\xi', \tag{4.3}$$

where $a = W_\infty^{-1}(\partial|\boldsymbol{u}_0|/\partial n)_P$,

$$\begin{aligned} B(\xi_i - \delta_{i3}W_\infty t) = &- a\left[(\xi_1 - \xi_1^{(0)})A_1(\xi_i - \delta_{i3}W_\infty t) + (\xi_2 - \xi_2^{(0)})A_2(\xi_i - \delta_{i3}W_\infty t)\right] \\ &+ A_3(\xi_i - \delta_{i3}W_\infty t) - A_3(\xi_1^{(0)}, \xi_2^{(0)}, \xi_3 - W_\infty t), \end{aligned} \tag{4.4}$$

and a superscript (0) indicates evaluation on Σ.

It now follows from (2.6), (2.7) and (4.1) that the velocity and pressure perturbations produced by the imposed disturbance can be expressed as

$$\boldsymbol{u}_1 = \boldsymbol{u}_1^{(R)} + \nabla\phi_1^*, \tag{4.5}$$

and

$$p_1 = -\rho_0 \frac{D_0}{Dt}\phi_1^*, \tag{4.6}$$

where $D_0/Dt = \partial/\partial t + \boldsymbol{u}_0 \cdot \nabla$ is the convective derivative associated with the steady base flow. The known solution for $\boldsymbol{u}_1^{(R)}$ is

$$\boldsymbol{u}_1^{(R)} = A_j(\xi_i - \delta_{i3}W_\infty t)\nabla\xi_j + \frac{\boldsymbol{u}_0}{2c_p}S(\xi_i - \delta_{i3}W_\infty t) + \nabla\varphi_1, \tag{4.7}$$

where (2.5), (2.9), (2.14) and (2.19) can be used to relate the functions S and A_j to the vorticity and entropy fluctuations of the incident disturbance at $z = z_\infty$.

Substituting (4.1) into (2.11) and (2.12) and using (4.2) one finds that

$$\frac{D_0}{Dt}\left(\frac{1}{c_0^2}\frac{D_0}{Dt}\phi_1^*\right) - \frac{1}{\rho_0}\nabla\cdot(\rho_0\nabla\phi_1^*) = \frac{1}{\rho_0}\nabla\cdot(\rho_0\boldsymbol{u}_1^{(R)}), \tag{4.8}$$

which must be solved subject to

$$\left.\begin{aligned} \boldsymbol{n}\cdot\nabla\phi_1^* &= 0 \qquad \text{on} \quad \Sigma_b, \\ \Delta(\boldsymbol{n}\cdot\nabla\phi_1^*) &= 0 \qquad \text{on} \quad \Sigma_w, \\ \boldsymbol{n}\cdot\nabla\phi_1^* &= -\boldsymbol{n}\cdot\boldsymbol{u}_1^{(R)} \quad \text{on} \quad \Pi, \end{aligned}\right\} \tag{4.9}$$

where Δ denotes the jump across Σ_w and Π denotes the parts of the impermeable boundaries not included in Σ_b. The solution for ϕ_1^* must also satisfy the upstream condition

$$\phi_1^* \to -\varphi_1 + \phi_\infty \tag{4.10}$$

as $z \to -\infty$, where ϕ_∞ is given by the long-time behavior of the right-hand side of (3.27) but with $\check{\psi}$ set equal to zero.

5. Discussion

The determination of the response of a steady swirling potential flow past a turbomachine blade row to the most general type of 'non-acoustic' incident disturbance has been reduced to solving the linear inhomogeneous wave equation (4.8) subject to (4.9) and (4.10). This reduction required modifying the upstream boundary conditions given in Goldstein (1978) because the vorticity induced pressure fluctuations do not vanish in the upstream region of the present problem. These fluctuations result from the coupling of the vorticity and pressure fields through the blocking and base-flow distortion effects and persist in the upstream region due to the vanishing of the radial velocity on $r = r_i, r_o$ as well as the swirling nature of the base flow.

A numerical solution to (4.8)–(4.10) for the flow past a blade row is beyond the scope of the present study but some of the results obtained from the solution that applies in the absence of the blade row will be discussed. The results of §3 imply that the long-time behavior of the 'acoustic' part of the disturbance field is given by a sum over the discrete spectrum associated with the wave equation but that the long-time behavior of the 'non-acoustic' disturbance field can not be given by such a sum.

Tan & Greitzer (1986) considered the behavior of steady non-axisymmetric disturbances on compressible swirling flow in turbomachine annuli. The authors found that the amplitudes of these disturbances decrease as the axial distance from the point of excitation increases. The large-z behavior of $\check{q}$ is given by (B 6) of appendix B for the case when $k_2 \neq 0$ and this result together with (3.27) shows that the amplitudes of the non-axisymmetric 'non-acoustic' disturbances decrease like z^{-1} as $z \to \infty$. In contrast, (3.7), (3.8) and (3.28) imply that the amplitudes of the axisymmetric 'non-acoustic' disturbances grow linearly with z in this limit.

The difference between the large-z behaviors of axisymmetric and non-axisymmetric parts of the 'non-acoustic' disturbance disturbance is a consequence of the way in which the relative motion between particles frozen in the base flow (i.e. purely convected with it) affects the solutions to (2.11) and (2.12). Equation (2.14) shows that the base-flow particle trajectories in the upstream region are helical and that the particle velocity has an axial component equal to W_∞ and a circumferential component inversely proportional to the (constant) radius of the trajectory. Therefore the relative motion only occurs in the circumferential direction and then only between particles of differing radial positions. This means that the terms in (2.11) and (2.12) that involve derivatives with respect to r become dominant as z becomes large but only when the 'non-acoustic' part of ϕ_1 varies with θ.

Appendix A. The Green's function G

In this appendix, a Green's function G associated with the operator $\mathrm{L}_{m,\sigma,\omega}$ is constructed. The governing equations for $G(r, r' \,|\, m, \sigma, \omega)$ are taken to be

$$\mathrm{L}_{m,\sigma,\omega} G = -\delta(r - r') \tag{A 1}$$

for $r_i < r < r_o$ and

$$\frac{\partial}{\partial r} G = 0 \tag{A 2}$$

for $r = r_i, r_o$. Let $G_1(r \,|\, m, \sigma, \omega)$ and $G_2(r \,|\, m, \sigma, \omega)$ be two linearly independent homogeneous solutions to (A 1) that satisfy the initial conditions $G_1 = 1/r\rho_0$, $\partial G_1/\partial r = 0$ at $r = r_i$ and $G_2 = 1/r\rho_0$, $\partial G_2/\partial r = 0$ at $r = r_o$. Then the solution to (A 1) and (A 2) can be written as

$$G(r, r' \mid m, \sigma, \omega) = -\frac{r'\rho_0(r')}{D(m, \sigma, \omega)} G_1(r_< \mid m, \sigma, \omega) G_2(r_> \mid m, \sigma, \omega) \tag{A 3}$$

where

$$D = -\frac{\partial}{\partial r} G_1(r_o \mid m, \sigma, \omega) = \frac{\partial}{\partial r} G_2(r_i \mid m, \sigma, \omega) \tag{A 4}$$

is the product of $r\rho_0$ and the Wronskian of G_1 and G_2 and $r_<$ ($r_>$) is the lesser (greater) of r and r'.

Appendix B. The large-z behavior of $\check{q}$

In this appendix, the behavior of $\check{q}$ in the limit of $z \to \infty$ is determined for the case when $k_2 \neq 0$. Considering $\check{q}_1$ first, (3.12) is rewritten as

$$\check{q}_1 = \int_{r_i}^{r-} \mathrm{e}^{\mathrm{i}\alpha' z} \hat{q}_1 \mathrm{d}r' + \int_{r+}^{r_o} \mathrm{e}^{\mathrm{i}\alpha' z} \hat{q}_1 \mathrm{d}r' \tag{B 1}$$

so that $\hat{q}_1(r, r', z \mid \nu)$ can be continuously differentiated with respect to r'. Integrating by parts and using (3.2), (3.7), and (3.17) then shows that

$$\begin{aligned} \int \mathrm{e}^{\mathrm{i}\alpha' z} \hat{q}_1 \mathrm{d}r' = \frac{1}{\mathrm{i}\beta' z} \mathrm{e}^{\mathrm{i}\alpha' z} \Big\{ & \hat{f}(r', z) K_1(r, r' \mid \nu) - \hat{A}_1(r') K_2(r, r' \mid \nu) \beta' \\ & - \frac{\partial}{\partial r'} \left[\hat{A}_1(r') K_1(r, r' \mid \nu) \right] \Big\} + O(z^{-2}) \end{aligned} \tag{B 2}$$

as $z \to \infty$ where

$$\beta(r) = \frac{\mathrm{d}\alpha}{\mathrm{d}r} = 2 \frac{\Gamma_\infty}{W_\infty} \frac{k_2}{r^3}, \tag{B 3}$$

and $\beta' = \beta(r')$. Equations (3.18), (3.19) and (A 3) imply

$$K_1(r, r' \mid \nu) \Big|_{r'=r+}^{r-} = K_2(r, r' \mid \nu) \Big|_{r'=r+}^{r-} = 0, \quad \frac{\partial}{\partial r'} K_1(r, r' \mid \nu) \Big|_{r'=r+}^{r-} = 1,$$

and

$$\frac{\partial}{\partial r'}K_1(r,r'\,|\,\nu)\Big|_{r'=r_i}^{r_o} = \left\{\frac{1}{r'\rho_0(r')}\frac{\mathrm{d}}{\mathrm{d}r'}\left[r'\rho_0(r')\right]K_1(r,r'\,|\,\nu) - \beta' K_2(r,r'\,|\,\nu)\right\}\Big|_{r'=r_i}^{r_o},$$

so substituting (B 2) into (B 1) yields

$$\begin{aligned}\check{q}_1 = -\frac{1}{\mathrm{i}\beta z}\mathrm{e}^{\mathrm{i}\alpha z}\hat{A}_1 + \frac{1}{\mathrm{i}\beta' z}\mathrm{e}^{\mathrm{i}\alpha' z}\Big[\mathrm{i}\beta' z\hat{A}_1(r') + \mathrm{i}\alpha'\hat{A}_3(r') \\ + \mathrm{i}k_3\frac{W_\infty}{2c_p}\hat{S}(r')\Big]K_1(r,r'\,|\,\nu)\Big|_{r'=r_i}^{r_o} + O(z^{-2})\end{aligned} \quad \text{(B 4)}$$

as $z \to \infty$. Then, since (3.20) reduces to

$$\check{q}_2 = -\mathrm{e}^{\mathrm{i}\alpha' z}\hat{A}_1(r')K_1(r,r'\,|\,\nu)\Big|_{r'=r_i}^{r_o} \quad \text{(B 5)}$$

when $k_2 \neq 0$,

$$\check{q} = -\frac{1}{\mathrm{i}\beta z}\mathrm{e}^{\mathrm{i}\alpha z}\hat{A}_1 + \frac{1}{\mathrm{i}\beta' z}\mathrm{e}^{\mathrm{i}\alpha' z}\left[\mathrm{i}\alpha'\hat{A}_3(r') + \mathrm{i}k_3\frac{W_\infty}{2c_p}\hat{S}(r')\right]K_1(r,r'\,|\,\nu)\Big|_{r'=r_i}^{r_o} + O(z^{-2}) \quad \text{(B 6)}$$

as $z \to \infty$.

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A method is proposed for determining the response of the steady swirling potential flow past a blade row in an axial turbomachine to the most general type of 'non-acoustic' incident disturbance. The method is based on Goldstein's (Unsteady vortical and entropic distortions of potential flows round arbitrary obstacles. *J. Fluid Mech.* **89**, 433–468, 1978) decomposition of the disturbance velocity and only requires solving a linear inhomogeneous wave equation. It is believed that numerical solutions to this wave equation can be obtained more efficiently and will be more accurate than corresponding solutions to the linearized compressible Euler equations.

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