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Igor Kaljević
Ohio Aerospace Institute
Cleveland, Ohio

Sunil Saigal
Carnegie Mellon University
Pittsburgh, Pennsylvania

Dale A. Hopkins
Lewis Research Center
Cleveland, Ohio

and

Surya N. Patnaik
Ohio Aerospace Institute
Cleveland, Ohio

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Igor Kaljević
Ohio Aerospace Institute
Cleveland, Ohio 44142

Sunil Saigal
Carnegie Mellon University
Pittsburgh, Pennsylvania 15213

Dale A. Hopkins
National Aeronautics and Space Administration
Lewis Research Center
Cleveland, Ohio 44135

and

Surya N. Patnaik
Ohio Aerospace Institute
Cleveland, Ohio 44142

Summary

A force method formulation, the Completed Beltrami-Michell Formulation (CBMF), has been developed for analyzing boundary value problems in elastic continua. The CBMF is obtained by augmenting the classical Beltrami-Michell Formulation with novel boundary compatibility conditions. It can analyze general elastic continua with stress, displacement, or mixed boundary conditions. The CBMF alleviates the limitations of the classical formulation, which can solve stress boundary value problems only. In this report, the CBMF is specialized for plates and shells. All equations of the CBMF, including the boundary compatibility conditions, are derived from the variational formulation of the Integrated Force Method (IFM). These equations are defined only in terms of stresses. Their solution for kinematically stable elastic continua provides stress fields without any reference to displacements.

In addition, a stress function formulation for plates and shells is developed by augmenting the classical Airy's formulation with boundary compatibility conditions expressed in terms of the stress function. The versatility of the CBMF and the augmented stress function formulation is demonstrated through analytical solutions of several mixed boundary value problems. The example problems include a composite circular plate and a composite circular cylindrical shell under the simultaneous action of mechanical and thermal loads.

Introduction

Boundary value problems (BVP) for analyzing elastic continua have been classified into three categories according to the type of prescribed boundary conditions. These categories are BVP I, where only stresses are prescribed on the boundary; BVP II, where only displacements are prescribed; and BVP III, or mixed BVP, where both stresses and displacements can be prescribed on the boundary.

Two distinct methods (which are classified according to the primary variables used) are available in the literature for solving these problems: (1) the Navier's displacement method (ref. 1) expresses the governing equations in terms of displacement variables, and (2) force methods use stresses in the problem description. The Navier's displacement method can solve all three types of BVP's because the stress boundary conditions may be easily expressed in terms of the kinematic variables. The stresses, however, have to be calculated indirectly when the displacement method is used. This may introduce errors in the stress variables, especially when approximate techniques are followed. Thus, the displacement method may not always be a reliable tool for calculating stresses, and alternative methods are needed for direct and accurate stress analyses.

The force method, known as the Beltrami-Michell's Formulation (BMF) in elasticity, was developed at the turn of the 20th century. It is applicable only to BVP I, where stresses are prescribed on the boundary. Neither BVP II, with prescribed displacement boundary conditions, nor BVP III, with prescribed mixed boundary conditions, can be solved by the BMF because it cannot handle the displacement boundary conditions. Therefore, the BMF had a very limited impact because many realistic engineering problems require the solution of BVP II and III.

The classical BMF was deficient because a set of boundary conditions were unavailable. These boundary conditions, which have recently been derived for both discrete (ref. 2) and continuum analyses (refs. 3 to 5), have been identified as boundary compatibility conditions. Augmenting the BMF with these boundary compatibility conditions resulted in the Completed Beltrami-Michell Formulation (CBMF), which can solve all three types of BVP's. It overcomes the limitation of the classical formulation and is as versatile as the Navier's displacement method. A variational functional for the CBMF, known as the variational formulation of the Integrated Force Method (IFM), also has been developed (ref. 6). The stationary condition of the IFM functional yields both field and boundary equations, including the boundary compatibility conditions that are necessary to solve the problem in terms of stresses. Recently, the IFM functional was used to develop a corresponding finite element formulation (refs. 2 and 7 to 10).

Published CBMF developments include the analysis of flat membrane plates and flat plates in flexure (refs. 3, 4, and 6). For both cases, the response was obtained for mechanical loads only. The CBMF remained to be developed for shell structures wherein the membrane and the flexural responses are coupled and for thermomechanical analysis of structures. In this report, the CBMF is developed for the thermomechanical analysis of composite, circular cylindrical shells in which the membrane and bending responses are coupled. The formulation for circular plates also is extended for the thermomechanical analysis of composite plates.

To introduce the basic concepts of the CBMF, we first consider a two-dimensional plane stress problem. When treated separately, the equilibrium equations cannot be solved for stresses, neither can the compatibility conditions be solved for strains. However, both sets of equations, combined with the traction boundary conditions and the boundary compatibility conditions, provide enough equations to solve all three types of BVP's only in terms of stress variables.

Next, by using the IFM functional, we develop the CBMF for plate and shell bending (ref. 6) problems. The IFM functional is formulated for both problems, and all the necessary field equations and boundary conditions, including the boundary compatibility conditions, are derived from its stationary condition. The boundary conditions are specialized to establish transition (jump) conditions on the interfaces of domains with different material properties. Then, the transition conditions are used to calculate the response of composite plates and shells. The field and the boundary compatibility conditions also are modified to account for initial deformations and are used next for the thermal analysis of circular plates and shells.

Augmented stress function formulations also are developed for both problems. These formulations, which are obtained by adding the boundary compatibility conditions to classical Airy's types of formulations, can solve all three types of BVP. This report demonstrates the capabilities of the CBMF by solving several mixed BVP for plates and shells. The examples include composite domains, mechanical and thermal loadings, and general boundary conditions.

Review of the Completed Beltrami-Michell Formulation

The CBMF represents a method for solving mixed BVP in elasticity solely in terms of stress variables. It concatenates equilibrium equations and St. Venant compatibility conditions both in the field and on the boundary in an attempt to obtain a complete system of equations, all expressed in terms of stresses. To review the basic concepts of the CBMF, we consider a plane stress problem. For simplicity, initial deformations and body forces are neglected in this review, and homogeneous displacement boundary conditions are assumed.

Governing Equations for the Completed Beltrami-Michell Formulation

The CBMF was originally established by augmenting the classical BMF with newly developed boundary compatibility conditions. All equations required for the CBMF have also been derived from the IFM variational functional (ref. 6.). These equations can be divided into five groups: Group Ia, equilibrium equations in the field; Group Ib, equilibrium equations on the boundary; Group IIa, field compatibility conditions; Group IIb, boundary compatibility conditions; and Group III, continuity conditions. For a plane stress problem, these equations are given as follows:

Group Ia: Equilibrium equations in the field.—

$$\left.\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \end{aligned}\right\} \tag{1}$$

Group Ib: Boundary equilibrium equations (or traction conditions).—

$$\left.\begin{aligned} \mathcal{S}_1(\sigma) = \sigma_x n_x + \tau_{xy} n_y - P_x = 0 \\ \mathcal{S}_2(\sigma) = \tau_{xy} n_x + \sigma_y n_y - P_y = 0 \end{aligned}\right\} \qquad (2)$$

where σ_x, σ_y, and τ_{xy} are three components of the stress tensor; n_x and n_y are the direction cosines of the outward normal vector; and P_x and P_y are prescribed boundary tractions. In the field, the equilibrium equations are functionally indeterminate (ref. 11) because three unknown stresses are expressed in terms of two equations of Group Ia.

Group IIa: Field compatibility condition.—The functional indeterminacy in the domain is alleviated with St. Venant's field compatibility condition, which can be written in terms of strain components ε_x, ε_y, and γ_{xy} as

$$\frac{\partial^2 \varepsilon_y}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial y^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = 0 \qquad (3)$$

and in terms of stresses as

$$\nabla^2\left(\sigma_x + \sigma_y\right) = 0 \qquad (4)$$

Equations (1), (2), and (4) represent the stress or the classical BMF in elasticity which was developed in 1900 (ref. 12). This incomplete formulation is not applicable to BVP II and III because of stress indeterminacy on the boundary. This is evidenced by the three stresses on the boundary being expressed in terms of two traction equations (eqs. (2)). Thus, there is one degree of functional indeterminacy. The field compatibility condition given in equation (4) alleviated functional indeterminacy in the field. However, because St. Venant did not formulate the compatibility on the boundary, the stresses remained indeterminate on this type of boundary.

Group IIb: Boundary compatibility condition.—The functional indeterminacy on the boundary that made the BMF incomplete was alleviated when Patnaik (ref. 6) formulated the boundary compatibility condition. This boundary condition, when expressed in terms of stresses, for isotropic material with Poisson's ratio ν has the following form:

$$\mathcal{R}(\sigma) = \frac{\partial}{\partial x}\left(\sigma_y - \nu\sigma_x\right)n_x + \frac{\partial}{\partial y}\left(\sigma_x - \nu\sigma_y\right)n_y - (1+\nu)\left(\frac{\partial \tau_{xy}}{\partial x} n_y + \frac{\partial \tau_{xy}}{\partial y} n_x\right) = 0 \qquad (5)$$

The set of three equations—the traction conditions of equations (2) and the boundary compatibility condition of equation (5)—ensure stress determinacy on the boundary because three unknown stresses are expressed in terms of three equations.

The set of six equations (eqs. (1), (2), (4), and (5)) represents the CBMF, which ensures the determinacy of the stresses in the field and on the boundary of an elastic continuum. The CBMF can solve a general elastic continuum, namely BVP I, II, and III.

Group III: Continuity conditions (or displacement boundary conditions).—The stationary condition of the IFM functional also yields two displacement boundary conditions:

$$u = \bar{u} = 0 \qquad \upsilon = \bar{\upsilon} = 0 \qquad (6)$$

where $\bar{u} = 0$ and $\bar{\upsilon} = 0$ are prescribed boundary displacements. In the CBMF the displacement boundary conditions do not appear explicitly in the stress calculations provided the structure is kinematically stable. Displacements, if required, can be calculated from stresses by integration where the kinematic boundary conditions are required to evaluate the constants of integration (refs. 3 to 5 and 13).

Treatment of Displacement Boundary Conditions in the Force Method of Analysis

Treatment of displacement boundary conditions in force and displacement methods of analyses can differ to some extent. In the displacement method, the two boundary conditions given in equations (6) are used; but in the CBMF for stress calculations, only one boundary compatibility condition, that given in equation (5), is used instead of those in equations (6). The paradox—two conditions given in equations (6) in the displacement method versus one condition given in equation (5) in the force method—is first clarified through the force method analysis of a discrete structure, the two-bay truss shown in figure 1. The concepts are next extended for continuous domains. The truss has 11 members with identical material and geometric properties. Nodes 1 and 3 are restrained in both coordinate directions, resulting in displacement boundary conditions:

$$u_1 = 0 \qquad v_1 = 0 \tag{7}$$

$$u_3 = 0 \qquad v_3 = 0 \tag{8}$$

The truss is subjected to the concentrated force of intensity P as shown in figure 1. The primary set of unknowns consists of 11 member forces, $\{F\}^T = \{F_1, F_2, \cdots, F_{11}\}$, which can be obtained from the following system of equations:

$$\left.\begin{aligned}
F_1 - F_2 + \frac{1}{\sqrt{2}}F_4 - \frac{1}{\sqrt{2}}F_8 &= 0 \\
\frac{1}{\sqrt{2}}F_4 + F_6 + \frac{1}{\sqrt{2}}F_8 &= 0 \\
\frac{1}{\sqrt{2}}F_4 + F_{10} &= 0 \\
F_3 + \frac{1}{\sqrt{2}}F_4 &= 0 \\
\frac{1}{\sqrt{2}}F_5 - \frac{1}{\sqrt{2}}F_7 + F_{10} - F_{11} &= 0 \\
\frac{1}{\sqrt{2}}F_5 + F_6 + \frac{1}{\sqrt{2}}F_7 &= -P \\
\frac{1}{\sqrt{2}}F_8 + F_{11} &= 0 \\
\frac{1}{\sqrt{2}}F_8 + F_9 &= 0
\end{aligned}\right\} \tag{9}$$

$$\left.\begin{aligned}
-\frac{1}{\sqrt{2}}F_1 - \frac{1}{\sqrt{2}}F_3 + \sqrt{2}F_4 + \sqrt{2}F_5 - \frac{1}{\sqrt{2}}F_6 - \frac{1}{\sqrt{2}}F_{10} &= 0 \\
-\frac{1}{\sqrt{2}}F_2 - \frac{1}{\sqrt{2}}F_6 + \sqrt{2}F_7 + \sqrt{2}F_8 - \frac{1}{\sqrt{2}}F_9 - \frac{1}{\sqrt{2}}F_{11} &= 0
\end{aligned}\right\} \tag{10}$$

$$F_1 + F_2 = 0 \tag{11}$$

Equations (9) represent equations of equilibrium (Group Ia); equations (10) represent field compatibility conditions (Group IIa, ref. 14); and equation (11) represents the boundary compatibility condition (Group IIb, ref. 14). Boundary equilibrium equations of Group IIb represent equations for unknown reactions at restrained nodes 1 and 3. These equations are not written here because they are not essential for calculating internal forces.

The displacement boundary conditions for the truss can be divided into two categories: (1) three kinematic boundary conditions ($u_1 = u_3 = v_1 = 0$) and (2) one indeterminate boundary condition ($v_3 = 0$). The three kinematic boundary conditions, which are required to ensure the kinematic stability of the structure, do not appear explicitly in the force method solution of the problem. The indeterminate boundary condition is replaced by the boundary compatibility condition.

The displacement boundary conditions in the CBMF for continuous domains also can be divided into kinematic conditions and boundary compatibility conditions. Kinematic conditions do not enter the CBMF explicitly. These are, however, essential for the overall kinematic stability of the structure, and they are also used to determine the constants of integration when the displacements are calculated from the stresses. Only the boundary compatibility condition is necessary for calculating internal forces.

In summary, the boundary compatibility conditions are sufficient for calculating stresses in kinematically stable domains. However, both kinematic conditions and the boundary compatibility conditions are required to generate the total solution, which consists of stresses and displacements.

Solution for Composite Domains

A brief description of using the CBMF to solve a mixed BVP for a composite elastic medium follows. Figure 2 shows a composite domain Ω that is composed of the two subdomains Ω_1 and Ω_2, which are made of different materials. The subdomains Ω_1 and Ω_2 are bounded by the outer contour Γ and the interface Γ_I. The stresses are assumed to be prescribed on a portion Γ_s of the contour Γ, and the displacements on the portion Γ_u. On both domains, equations (1) and (4) must be satisfied. A solution of these two sets of equations results in a number of integration constants that have to be calculated from conditions on the contours Γ_u, Γ_s, and Γ_I. On the contour Γ_s, equations (2) must be satisfied, and on the contour Γ_u, equation (5) is enforced. Additional equations are provided on the interface Γ_I:

two equilibrium conditions:

$$\mathcal{S}_1^{\mathrm{I}}(\sigma) - \mathcal{S}_2^{\mathrm{I}}(\sigma) = 0 \qquad \mathcal{S}_1^{\mathrm{II}}(\sigma) - \mathcal{S}_2^{\mathrm{II}}(\sigma) = 0 \tag{12}$$

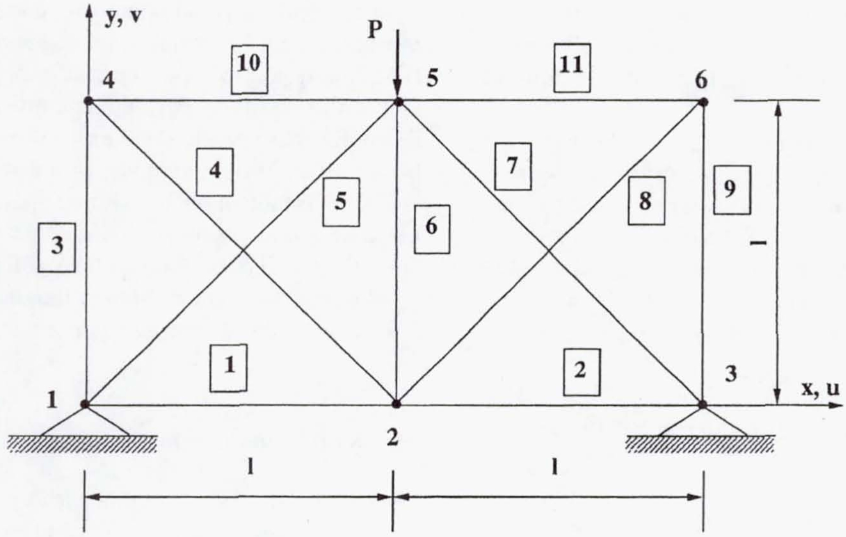

Figure 1.—Two-bay truss.

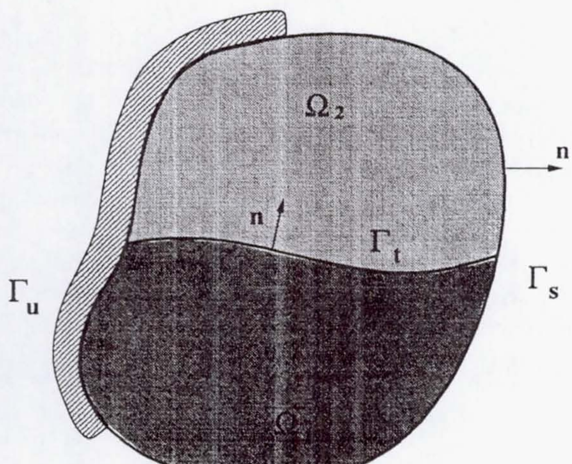

Figure 2.—Composite elastic medium.

and a compatibility condition:

$$\mathcal{R}^{\mathrm{I}}(\sigma) - \mathcal{R}^{\mathrm{II}}(\sigma) = 0 \tag{13}$$

where superscripts I and II denote subdomains Ω_1 and Ω_2, respectively.

Properties of Compatibility Conditions

Some properties of compatibility conditions for a plane stress problem are given in this section.

(1) The field compatibility condition written in terms of strains (such as that given in equation (3)) is an unsymmetrical equation, whereas the condition expressed in terms of the stress function Φ (which is the dual variable of compatibility conditions) becomes a symmetrical equation:

$$\nabla^4 \Phi = 0 \tag{14}$$

(2) When written in terms of strains, the compatibility conditions are independent of the material properties of the object being analyzed.

(3) The compatibility conditions are functionally indeterminate because one condition given by equation (3) contains three unknown strain components. Consequently, strains cannot be determined from compatibility conditions alone.

(4) When written in terms of displacement variables u and υ, the field compatibility condition becomes a trivial constraint, such as an identity $[f(u,\upsilon) - f(u,\upsilon)] = 0$, where the function f represents the field compatibility condition. However, when written in terms of displacements, the boundary compatibility condition does not become a trivial equation. In terms of displacements, the boundary compatibility condition given by equation (5) becomes

$$\left[\frac{\partial^2 \upsilon}{\partial x \partial y} - \frac{1}{2}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 \upsilon}{\partial x \partial y}\right)\right] n_x + \left[\frac{\partial^2 u}{\partial x \partial y} - \frac{1}{2}\left(\frac{\partial^2 \upsilon}{\partial y^2} + \frac{\partial^2 u}{\partial x \partial y}\right)\right] n_y = 0 \tag{15}$$

The nontrivial property of the boundary compatibility condition contradicts the popular belief that all compatibility conditions are automatically satisfied in the displacement method.

(5) The field compatibility condition can be derived from the strain displacement relation by eliminating the displacement components from the strain displacement relations. This logic, as yet, cannot be extended to the boundary compatibility condition. At present, the boundary compatibility conditions can be generated only from the IFM variational formulation. This is, perhaps, the primary reason why they could not be developed earlier.

Integrated Force Method and the Completed Beltrami-Michell Formulation

The force formulations for stress analysis of structures that incorporate boundary compatibility conditions are, in various contexts, referred to as the CBMF and the IFM. Forces (or stress parameters) are the primary unknowns in both formulations, and these are obtained directly by simultaneously solving both the equations of equilibrium and the compatibility conditions. The CBMF is related to the analytical solutions of the BVP in linear elasticity, whereas the IFM is most often related to the force formulation of the finite element method for the approximate solutions of these problems. The IFM represents the discretized version of the CBMF.

The CBMF described here was developed for analyzing problems in two-dimensional elasticity. When combined with the field compatibility condition written in terms of stresses, the equations of equilibrium represent the complete set of field differential equations. The novel compatibility condition, when combined with the traction boundary conditions, provides a sufficient number of equations to calculate constants resulting from the integration of the field equations. Thus, the solution is obtained solely in terms of stress variables. In subsequent sections, these concepts are applied to circular plates and cylindrical shells. The IFM variational functional is formulated for both problems, and its stationary condition is used to derive all required field differential equations and boundary conditions, including the boundary compatibility conditions. These equations may be solved for stress fields without any reference to displacement variables. The displacements can then be calculated by integrating strain-stress relations and using displacement boundary conditions to calculate resulting integration constants.

Radially Symmetrical Bending of Circular Plates

In this section, the force formulation derived earlier for homogeneous circular plates under mechanical loadings is extended for analyzing composite circular plates under both mechanical and thermal loadings. A modified form of the IFM functional is used, the stationary condition of which yields all previously derived equations, along with an additional boundary condition in terms of bending moments. This boundary condition, together with those derived previously, is used to establish the transition conditions on the interfaces between zones made of different materials. The transition conditions at the interfaces, along with other force boundary conditions, provide a sufficient number of equations for the CBMF solution of composite circular plates.

Stress Function for the Bending of Circular Plates

The stress function must be defined for the IFM functional formulation. For the bending of circular plates, this stress function is derived from the stationary condition of an auxiliary functional Π_s defined by Washizu (ref. 15) as

$$\Pi_s = 2\pi\int_{r_a}^{r_b}\left\{M_r\kappa_r + M_\varphi\kappa_\varphi + \Psi\left[\frac{d}{dr}(r\kappa_\varphi) - \kappa_r\right]\right\}r\,dr \qquad (16)$$

where M_r and M_φ are the radial and tangential moments, respectively; κ_r and κ_φ are the radial and tangential curvatures; Ψ is the stress function; r is the radial coordinate; and the coordinates r_a and r_b represent the inner and outer contours of the plate, respectively, with $r_a = 0$ for the full plate. The stationary condition of the functional Π_s with respect to curvatures κ_r and κ_φ defines the stress function Ψ:

$$M_r = \Psi \qquad (17a)$$

$$M_\varphi = \frac{d}{dr}(r\Psi) \qquad (17b)$$

Variational Formulation of the Integrated Force Method for Circular Plates

The governing equations for the plate bending problem are now derived from the stationary condition of the IFM functional Π_p^c defined as

$$\Pi_p^c = A + B - W \qquad (18)$$

where the strain energy A, the complementary energy B, and the work of external forces W, are given as

$$\left.\begin{aligned} A &= 2\pi\int_{r_a}^{r_b}\left[M_r\left(-\frac{d^2w}{dr^2}\right)+M_\varphi\left(-\frac{1}{r}\frac{dw}{dr}\right)\right]rdr \\ B &= 2\pi\int_{r_a}^{r_b}\left[\Psi\frac{M_r-\nu M_\varphi}{K}+\frac{d}{dr}(r\Psi)\frac{M_\varphi-\nu M_r}{K}\right]rdr \\ W &= 2\pi\int_{r_a}^{r_b} qwrdr \end{aligned}\right\} \tag{19}$$

Here, w is the transverse displacement; ν is the Poisson's ratio of the material; a material constant, K, is defined as $K = Eh^3/12(1-\nu^2)$; h is the thickness of the plate; and E is the modulus of elasticity. The variation of the functional Π_p^c with respect to displacements w and the stress function Ψ can be written as

$$\begin{aligned} \delta\Pi_p^c = 2\pi\Bigg\{ &-\int_{r_a}^{r_b}\left[\frac{d^2}{dr^2}(rM_r)-\frac{dM_\varphi}{dr}+rq\right]dr\delta w \\ &+\int_{r_a}^{r_b}\left[\frac{M_r-\nu M_\varphi}{K}-\frac{d}{dr}\left(r\frac{M_\varphi-\nu M_r}{K}\right)\right]rdr\delta\Psi \\ &+\left[rM_r\left(-\delta\frac{dw}{dr}\right)\right]_{r_a}^{r_b}+\left[\left(\frac{d}{dr}(rM_r)-M_\varphi\right)\delta w\right]_{r_a}^{r_b} \\ &+\left[r^2\frac{M_\varphi-\nu M_r}{K}\delta\Psi\right]_{r_a}^{r_b}\Bigg\} \end{aligned} \tag{20}$$

The stationary condition of the functional given in equation (20) yields all five sets of equations referred to earlier as Groups Ia and Ib, IIa and IIb, and III. The coefficient of δw represents the field equation of equilibrium (Group Ia):

$$\frac{d^2}{dr^2}rM_r-\frac{dM_\varphi}{dr}+rq=0 \tag{21}$$

The coefficient of $\delta\Psi$ represents the field compatibility condition (Group IIa):

$$\frac{d}{dr}\left[r\frac{d}{dr}\left(M_\varphi-\nu M_r\right)\right]-\left(M_r-\nu M_\varphi\right) = 0 \tag{22}$$

Two contour terms in equation (20) that are associated with the variation of displacement w and its derivative provide boundary equilibrium equations (Group Ib), whereas the coefficient of $\delta\Psi$ in the third contour term provides the boundary compatibility condition (Group IIb):

$$\frac{1}{K}\left(M_\varphi-\nu M_r\right)=0 \tag{23}$$

Continuity (displacement) conditions of Group III can also be obtained from the stationary condition. The boundary conditions for various support conditions for a homogeneous circular plate are given by Patnaik and Nagaraj (ref. 4) and are not repeated here. Only transition conditions on composite plate interfaces between zones with different material properties are presented.

Transition Conditions at the Interfaces of Composite Plates

For a plate composed of two regions with different material properties, three constraints, referred to as transition conditions, must be established on the interface between the regions. The transition conditions can be derived from the equilibrium and compatibility conditions at the interface. The first equilibrium condition at the interface is obtained by an appropriate extension of the first contour term of equation (20) as

$$\left[rM_r\left(-\delta\frac{dw}{dx}\right)\right]^{\mathrm{I}}=\left[rM_r\left(-\delta\frac{dw}{dx}\right)\right]^{\mathrm{II}} \tag{24}$$

which results in

$$M_r^{\mathrm{I}}=M_r^{\mathrm{II}} \tag{25a}$$

Similarly, the second equilibrium condition can be obtained as

$$\frac{d}{dr}\left(rM_r^{\mathrm{I}}\right)-M_\varphi^{\mathrm{I}}=\frac{d}{dr}\left(rM_r^{\mathrm{II}}\right)-M_\varphi^{\mathrm{II}} \tag{25b}$$

When similar reasoning is applied to the boundary compatibility condition given in equation (23), the compatibility condition at the interface is obtained as

$$\frac{1}{K^{\mathrm{I}}}\left(M_\varphi^{\mathrm{I}}-\nu^{\mathrm{I}}M_r^{\mathrm{I}}\right)=\frac{1}{K^{\mathrm{II}}}\left(M_\varphi^{\mathrm{II}}-\nu^{\mathrm{II}}M_r^{\mathrm{II}}\right) \tag{25c}$$

The superscripts I and II denote two zones of different material and geometric properties.

With appropriate boundary and transition conditions, equations (21) and (22) represent a sufficient number of equations to solve the radially symmetrical bending problem of composite circular plates in terms of stress variables only. The displacements, if necessary, may be calculated by integrating the moment-curvature relations.

Analysis of Thermal Problems

The treatment of thermal problems for circular plates is now considered. We assume that the upper and lower surfaces of the plate are unevenly heated or cooled, resulting in the initial curvatures

$$\kappa_r^{(t)} = \kappa_\varphi^{(t)} = \alpha_t \frac{\Delta t}{h} \tag{26}$$

where α_t is the thermal coefficient of the plate material; h is the plate thickness; and Δt is the temperature difference between the upper and the lower surface. The total curvatures are

$$\left.\begin{aligned} \kappa_r &= \frac{1}{K}\left(M_r - \nu M_\varphi\right) + \alpha_t \frac{\Delta t}{h} \\ \kappa_\varphi &= \frac{1}{K}\left(M_\varphi - \nu M_r\right) + \alpha_t \frac{\Delta t}{h} \end{aligned}\right\} \tag{27}$$

Equations (27) are introduced into the IFM functional, resulting in the modified expressions for the field and boundary compatibility conditions. The modified field compatibility condition that accounts for initial thermal effects is obtained as an extension of equation (22) and is given as

$$r\frac{d}{dr}\left(M_\varphi - \nu M_r\right) + (1+\nu)\left(M_\varphi - M_r\right) = -Kr\frac{\alpha_t}{h}\frac{d\Delta t}{dr} \tag{28}$$

Likewise, the boundary compatibility condition is

$$\frac{1}{K}\left(M_\varphi - \nu M_r\right) = \alpha_t \frac{\Delta t}{h} \tag{29}$$

Thermal effects do not affect the equations of equilibrium, either in the field or on the boundary.

Augmented Stress Function Formulation for the Plate Bending Problem

For completeness, an augmented stress function formulation of the plate bending problem is now presented. This formulation is obtained from the CBMF by eliminating the moments M_r and M_φ from equations (17) and (22) in favor of the stress function Ψ. The stress function defined by equations (17) identically satisfies the homogeneous form of the equation of equilibrium given in equation (21). In the presence of distributed loads, equation (17b) must be augmented as

$$M_\varphi = \frac{d}{dr}(r\Psi) + \chi(r) \tag{30}$$

where $\chi(r) = \int rq(r)dr$ and $q(r)$ is the distributed load. The governing equation is the field compatibility condition that, when expressed in terms of the stress function, has the following form

$$\frac{d}{dr}\left(r\frac{d\Psi}{dr}\right) + 2\frac{d\Psi}{dr} = -\frac{d\chi}{dr}(r) - \frac{1+\nu}{r}\chi(r) \tag{31}$$

the boundary conditions for this formulation are given as the static (moment) boundary condition,

$$\Psi = 0 \tag{32a}$$

and the boundary compatibility condition,

$$\frac{1}{K}\left[(1-\nu)\Psi + r\frac{d\Psi}{dr} + \chi(r)\right] = 0 \tag{32b}$$

Equations (31) and (32) represent the augmented Airy's stress function formulation of the circular plates, which can solve all three types of BVP.

Example Problem: A Composite Plate Subjected to Mechanical and Thermal Loads

We can illustrate the CBMF by analyzing a composite plate subjected to mechanical and thermal loads (fig. 3). The plate consists of two segments: an inner plate with radius a, material properties E_1 and ν_1, and thickness h_1—hereinafter referred to as region I—and an outer annular plate with inner radius a, outer radius b, material properties E_2 and ν_2, and

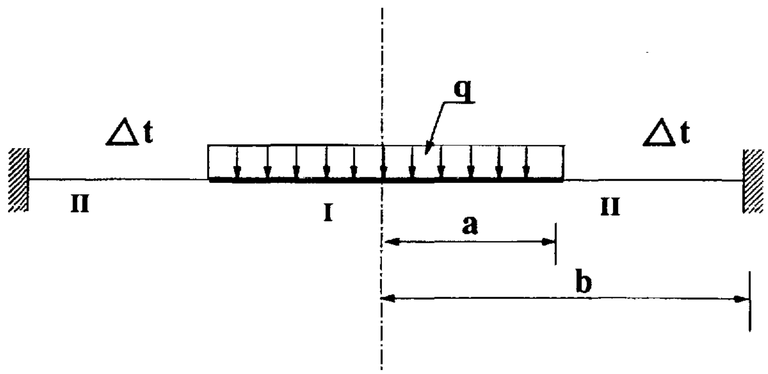

Figure 3.—Composite circular plate subjected to a uniform load, q, and temperature, Δt.

thickness h_2—hereinafter referred to as region II. Region I is subjected to a uniformly distributed mechanical load of intensity q, and region II is exposed to an uneven heating with the temperature difference Δt. The plate is clamped at the outer contour, given by $r = b$. This problem cannot be handled by the classical BMF. The CBMF, however, can readily solve this problem. The solution procedure consists of the following three steps:

Step I: General solution of field equations for region I.—The field equations given in equations (21) and (22) are solved for the moments M_r^{I} and M_φ^{I}. The solution of these equations for region I ($0 \le r \le a$) and for a distributed load of constant intensity q is

$$\left.\begin{aligned} M_r^{\mathrm{I}}(r) &= -\frac{B_1}{r^2} + \frac{1}{2}C_1\left(1+\nu_1\right)\log r \\ &\qquad + \frac{1}{4}C_1\left(1-\nu_1\right) + \frac{1}{2}D_1 - \frac{1}{16}\left(3+\nu_1\right)qr^2 \\ M_\varphi^{\mathrm{I}}(r) &= -\frac{B_1}{r^2} + \frac{1}{2}C_1\left(1+\nu_1\right)\log r \\ &\qquad - \frac{1}{4}C_1\left(1-\nu_1\right) + \frac{1}{2}D_1 - \frac{1}{16}\left(1+3\nu_1\right)qr^2 \end{aligned}\right\} \quad (33)$$

Step II: General solution of field equations for region II.—The system of equations that consists of the homogeneous equation (21) and the modified field compatibility condition given in equation (28) is solved to obtain expressions for the moments in region II ($a \le r \le b$). Because the temperature difference Δt is constant over the region II, the modified compatibility condition also results in a homogeneous equation that is same as equation (22). The solution of this system may be written as

$$\left.\begin{aligned} M_r^{\mathrm{II}}(r) &= -\frac{B_2}{r^2} + \frac{1}{2}C_2\left(1+\nu_2\right)\log r \\ &\qquad + \frac{1}{4}C_2\left(1-\nu_2\right) + \frac{1}{2}D_2 \\ M_\varphi^{\mathrm{II}}(r) &= \frac{B_2}{r^2} + \frac{1}{2}C_2\left(1+\nu_2\right)\log r \\ &\qquad - \frac{1}{4}C_2\left(1-\nu_2\right) + \frac{1}{2}D_2 \end{aligned}\right\} \quad (34)$$

In equations (33) and (34), B_1, C_1, D_1, B_2, C_2, and D_2 are the six constants of integration, and superscripts I and II denote quantities defined in regions I and II, respectively.

Step III: Calculation of constants of integration.—The six constants of integration are calculated by applying boundary conditions at the outer contour of the plate, transition conditions on the interface, and implicit conditions at the center of the plate. The six equations for the six constants of integration consist of one boundary compatibility condition at the outer contour, $r = b$ (eq. (29)); three transition conditions at the interface, $r = a$ (eqs. (25)); and two implicit conditions at the origin, $r = 0$, that ensure the finite values of the moments M_r and M_φ. The implicit conditions result in

$$B_1 = 0 \qquad C_1 = 0 \quad (35)$$

and the transition condition of equation (25b) yields

$$C_2 = -\frac{1}{2}qa^2 \quad (36)$$

After the expressions for moments in regions I and II, and the solutions for constants B_1, C_1, and C_2, are substituted, the remaining three equations have the following forms:

$$\left.\begin{aligned}
&\frac{1}{2}D_1+\frac{B_2}{a^2}-\frac{1}{2}D_2=\frac{1}{16}(3+\nu_1)qa^2-\frac{1}{8}(1-\nu_2)qa^2 \\
&\qquad -\frac{1}{4}(1+\nu_2)qa^2\log a \\
&\frac{1}{2}(1-\nu_1)D_1-\frac{K_1}{K_2}\frac{B_2}{a^2}(1+\nu_2)-\frac{1}{2}\frac{K_1}{K_2}(1-\nu_2)D_2 \\
&\qquad =\frac{K_1}{K_2}\left[-\frac{1}{4}\left(1-\nu_2^2\right)qa^2\log a+\frac{1}{8}\left(1-\nu_2^2\right)qa^2\right] \\
&\qquad +\frac{1}{16}\left(1-\nu_1^2\right)qa^2+K_1\alpha_t\frac{\Delta t}{h_2} \\
&(1+\nu_2)\frac{B_2}{b^2}+\frac{1}{2}D_2(1-\nu_2)=-K_2\alpha_t\frac{\Delta t}{h_2} \\
&\qquad +\frac{1}{4}qa^2\left(1-\nu_2^2\right)\log b-\frac{1}{8}qa^2\left(1-\nu_2^2\right)
\end{aligned}\right\} \qquad (37)$$

The explicit solution of equations (37) for an arbitrary geometry and composite construction of a circular plate is rather cumbersome. Here, the solution is given for a specific plate with region I made of aluminum and region II made of steel. Numerical values for the material parameters in regions I and II, respectively, are taken as $E_1 = 10.6\times10^6$ psi, $\nu_1 = 0.33$, $\alpha_t^{(1)} = 12.6\times10^{-6}$/°F, $E_2 = 30.0\times10^6$ psi, $\nu_2 = 0.30$, and $\alpha_t^{(2)} = 6.3\times10^{-6}$/°F. The radii are $a = 6$ in. and $b = 12$ in.; the thicknesses are $h_1 = 0.2$ in. and $h_2 = 0.15$ in.; the magnitude of the distributed load is $q = 100$ lb/in.^{2}; and the temperature difference is $\Delta t = 50$ °F. The constants of integration for this particular case are

$$\left.\begin{aligned}
B_1 &= 0.0 & B_2 &= -5203.06 \\
C_1 &= 0.0 & C_2 &= -1800.00 \\
D_1 &= 1688.10 & D_2 &= 4723.26
\end{aligned}\right\} \qquad (38)$$

and the solution for the internal forces becomes

$$\left.\begin{aligned}
M_r^{\mathrm{I}}(r) &= 844.05-20.81r^2 \\
M_\varphi^{\mathrm{I}}(r) &= 844.05-12.44r^2
\end{aligned}\right\} \qquad (39)$$

for region I ($0 \le r \le a$)

and

$$\left.\begin{aligned}
M_r^{\mathrm{II}}(r) &= 2046.63-\left(\frac{5203.06}{r^2}\right)-1170\log r \\
M_\varphi^{\mathrm{II}}(r) &= 2676.63-\left(\frac{5203.06}{r^2}\right)-1170\log r
\end{aligned}\right\} \qquad (40)$$

for region II ($a \le r \le b$)

Calculations of Displacements.—The displacements, if required, may be obtained by integrating the moment-curvature relations. These relations for the case of bending of circular plates, are given as

$$-\frac{1}{r}\frac{dw}{dr}=\kappa_\varphi=\frac{1}{K}\left(M_\varphi-\nu M_r\right) \qquad (41)$$

Solving equations (39) to (41) yields the displacement fields w^{I} and w^{II} on regions I and II, respectively, as

$$\left.\begin{aligned}
w^{\mathrm{I}}(r) &= -0.03566r^2+0.1757\times10^{-3}r^4+c_2 \\
w^{\mathrm{II}}(r) &= -0.1344r^2-0.7296\log r+0.04417r^2\log r+c_1
\end{aligned}\right\} \qquad (42)$$

The integration constants c_1 and c_2 are calculated by imposing the kinematic boundary condition on the outer contour, which ensures the kinematic stability of the plate, and by imposing the continuity condition on the interface between regions I and II. These conditions are

$$w^{\mathrm{II}}(b)=0 \qquad w^{\mathrm{I}}(a)=w^{\mathrm{II}}(a) \qquad (43)$$

The constants of integration are determined from equations (43) as $c_1 = 5.3614$ and $c_2 = 3.1209$. Substituting for c_1 and c_2 in equations (42) yields expressions for the displacements w for both regions I and II:

$$\left.\begin{aligned}
w^{\mathrm{I}}(r) &= 3.1209-0.03566r^2+0.1757\times10^{-3}r^4 \\
w^{\mathrm{II}}(r) &= 5.3614-0.1344r^2-0.7296\log r \\
&\qquad +0.04417r^2\log r
\end{aligned}\right\} \qquad (44)$$

The solution for the displacement field given in equations (44) has been verified from the corresponding solution by the displacement method.

Recapitulation

The stationary condition of the IFM variational functional resulted in all previously known equations, along with two new equations: the field compatibility condition given in equation (22) and the boundary compatibility condition given in equation (23). The field compatibility condition complements the equation of equilibrium given in equation (21) to obtain a sufficient number of field differential equations to determine the unknown moments. The traction boundary conditions, when augmented by the boundary compatibility condition, provide enough equations to calculate the integration constants resulting from the solution of the field differential equations. This enables the moments to be calculated without any reference to displacements. Without the boundary compatibility condition, the mixed BVP cannot be solved. Although the field compatibility condition given in equation (22) can be derived by manipulating the strain-displacement relations, the boundary compatibility condition can be deduced only from the stationary condition of the IFM variational functional. The appropriate extension of the boundary compatibility condition resulted in the transition (jump) condition given in equation (25c). This condition enables the solution of composite plates without any references to displacements. Without equation (25c), the solution is not possible in terms of moments, regardless of the type of the BVP being analyzed.

Radially Symmetrical Bending of Circular Cylindrical Shells

In the literature, the bending of circular cylindrical shells has been extensively studied by using the displacement method, and details of the displacement formulations for cylindrical shells can be found in standard textbooks (refs. 16 and 17). The force method, however, has not been developed to analyze the bending of cylindrical shells because the equations of equilibrium are functionally indeterminate. The establishment of boundary compatibility conditions made it possible to solve these problems with the force method. In the following section, the CBMF is extended to analyze the bending of circular cylindrical shells in which the membrane and bending responses are coupled. The stationary condition of the IFM functional yields all the equations required for solving the problem solely in terms of the internal forces and moments.

Stress Function for the Bending of Cylindrical Shells

The IFM functional formulation requires that the stress function be defined. This stress function for cylindrical shells is defined by following a procedure similar to that presented for the bending of circular plates. The corresponding functional Π_s is

$$\Pi_s = \int_\Omega \left[M_x \kappa_x + N_\varphi \varepsilon_\varphi - \Psi \left(\kappa_x - a \frac{d^2 \varepsilon_\varphi}{dx^2} \right) \right] d\Omega \qquad (45)$$

where Ω denotes the domain of the midsurface of the shell, a is the radius of the midsurface, Ψ is the stress function, M_x is the bending moment, N_φ is the tangential force, κ_x is the curvature, and ε_φ is the tangential strain. The stationary condition of the functional Π_s with respect to the strain ε_φ and the curvature κ_x defines the stress function as

$$M_x = \Psi \qquad (46a)$$

$$N_\varphi = -a \frac{d^2 \Psi}{dx^2} \qquad (46b)$$

Variational Formulation of the Integrated Force Method for Cylindrical Shells

The energy terms A, B, and W of the IFM functional for the cylindrical shells are

$$\left.\begin{aligned} A &= \int_\Omega \left[M_x \left(-\frac{d^2 w}{dx^2} \right) + N_\varphi \left(-\frac{w}{a} \right) \right] d\Omega \\ B &= \int_\Omega \left[\Psi \frac{M_x}{K} - a \frac{d^2 \Psi}{dx^2} \frac{N_\varphi}{Eh} \right] d\Omega \\ W &= -\int_\Omega q w \, d\Omega \end{aligned}\right\} \qquad (47)$$

where w is the radial displacement, q is the distributed load, h is the thickness of the shell, and $K = Eh^3/12(1 - \nu^2)$ is the flexural rigidity. The variation of the functional Π_s^c has the following form:

$$\delta \Pi_s^c = -\int_\Omega \left[\frac{d^2 M_x}{dx^2} + \frac{1}{a} N_\varphi + q\right] d\Omega \delta w$$
$$- \int_\Omega \left[\frac{M_x}{K} - \frac{a}{Eh}\frac{d^2 N_\varphi}{dx^2}\right] d\Omega \delta \Psi$$
$$+ 2a\pi \left[M_x \delta\left(-\frac{dw}{dx}\right) + \frac{dM_x}{dx}\delta w\right]_{x_a}^{x_b}$$
$$+ 2a\pi \left[a\frac{N_\varphi}{Eh}\delta\frac{d\Psi}{dx} - \frac{a}{Eh}\frac{dN_\varphi}{dx}\delta\Psi\right]_{x_a}^{x_b} \tag{48}$$

Its stationary condition yields all the equations of the CBMF. The coefficient of δw represents the field equilibrium equation of Group Ia:

$$\frac{1}{a}N_\varphi + \frac{d^2 M_x}{dx^2} + q = 0 \tag{49}$$

The stress function given by equations (46) identically satisfies the homogeneous form of the field equation of equilibrium given in equation (49). If distributed loadings are present, the expression for the force N_φ must be augmented as

$$N_\varphi = -a\left(\frac{d^2\Psi}{dx^2} + q\right) \tag{50}$$

The coefficient of $\delta\Psi$ represents the field compatibility condition of Group IIa:

$$M_x - \frac{Ka}{Eh}\frac{d^2 N_\varphi}{dx^2} = 0 \tag{51}$$

The contour terms associated with the variations of the displacements w represent the static boundary conditions:

$$M_x \delta\left(-\frac{dw}{dx}\right)\Bigg|_{x_a}^{x_b} = 0 \qquad \frac{dM_x}{dx}\delta w\Bigg|_{x_a}^{x_b} = 0 \tag{52}$$

and the terms associated with the variations of the stress function Ψ represent novel boundary compatibility conditions:

$$N_\varphi \delta \frac{d\Psi}{dx}\Bigg|_{x_a}^{x_b} = 0 \tag{53a}$$

$$\frac{dN_\varphi}{dx}\delta\Psi\Bigg|_{x_a}^{x_b} = 0 \tag{53b}$$

The boundary conditions given in equations (52) and (53) are specialized for various support conditions.

Free contour.—On a free contour, both moments and their derivatives must vanish because $w \neq 0$ and $dw/dx \neq 0$. This case results in the static boundary conditions:

$$M_x = 0 \qquad \frac{d}{dx}(M_x) = 0 \tag{54}$$

Simply supported contour.—For a simply supported contour, the transverse displacement w is equal to zero. The derivative dM_x/dx, therefore, is not zero on such a contour. The satisfaction of the boundary compatibility condition of equation (53a) for this case results in

$$N_\varphi = 0 \tag{55a}$$

The rotation of the cross section is not prevented on a simply supported boundary. The condition $\delta dw/dr \neq 0$ results in

$$M_x = 0 \tag{55b}$$

Clamped contour.—For the clamped contour, both displacements and rotations are zero, resulting in a nonvanishing moment and its derivative. For this case, both compatibility conditions must be satisfied, which results in

$$N_\varphi = 0 \qquad \frac{dN_\varphi}{dx} = 0 \tag{56}$$

The field equations given in equations (49) and (51), with appropriate boundary conditions, represent a sufficient number of equations to solve the shell bending problem in terms of stress variables. Equations (54) represent the boundary equilibrium equations of Group Ib. Though these equations are written here for a case with no prescribed contour loads, they can be easily extended to incorporate prescribed contour tractions and moments. The variation of contour terms also yields the displacement boundary conditions of Group III. Because the displacement boundary conditions are substituted by the boundary compatibility conditions of equations (56), these derivations are not elaborated here.

Augmented Stress Function Formulation for Cylindrical Shells

The CBMF for the bending of cylindrical shells is formulated in terms of two unknown internal forces. The problem also can be formulated in terms of the stress function, which results in a single differential equation. When augmented with the boundary compatibility condition, the classical stress function formulation can solve shell bending problems with general boundary conditions. The CBMF derived from the stationary condition of the IFM functional is cast as an augmented stress function formulation with the elimination of the internal forces M_x and N_φ in favor of the stress function Ψ. Written in terms of the stress function, the field compatibility condition becomes a fourth-order differential equation,

$$\frac{d^4\Psi}{dx^4} + 4\beta^4\Psi = -\frac{d^2q}{dx^2} \tag{57}$$

where $\beta = 3(1 - \nu^2)/(ah)^2$. The constants of integration are determined from various support conditions as

Free contour.—

$$\Psi = 0 \qquad \frac{d\Psi}{dx} = 0 \tag{58}$$

Simply supported contour.—

$$\Psi = 0 \qquad \frac{d^2\Psi}{dx^2} + q = 0 \tag{59}$$

Clamped contour.—

$$\frac{d^2\Psi}{dx^2} + q = 0 \qquad \frac{d^3\Psi}{dx^3} + \frac{dq}{dx} = 0 \tag{60}$$

Two appropriate sets of boundary equations out of the three sets given in equations (58) to (60) are sufficient to uniquely determine four integration constants in the solution of equation (57).

Analysis of Thermal Effects

Next, the augmented stress function formulation is extended to analyze thermal effects. We assume that the inner and the outer surface of the shell are unevenly heated, resulting in a temperature difference Δt. The initial curvature $\kappa_x^{(t)}$ due to the temperature difference Δt is calculated as $\kappa_x^{(t)} = (1 + \nu)\alpha_t\Delta t/h$, and the initial strain $\varepsilon_\varphi^{(+)}$ is equal to zero.

The total curvature is now written as

$$\kappa_x = \frac{M_x}{K} + (1+\nu)\alpha_t \frac{\Delta t}{h} \tag{61}$$

Equation (61) is introduced into the expression for the IFM functional given in equation (48) to derive the modified field compatibility condition as

$$\frac{M_x}{K} + (1+\nu)\alpha_t \frac{\Delta t}{h} = \frac{a^2}{Eh}\left(\frac{d^2N_\varphi}{dx^2}\right) \tag{62}$$

The boundary compatibility conditions are not changed because the thermal strain $\varepsilon_\varphi^{(t)} = 0$. Combining equations (46a) and (50) with the modified boundary compatibility condition given in equation (62) yields the differential equation for the bending of the cylindrical shell due to the thermal effects:

$$\frac{d^4\Psi}{dx^4} + 4\beta^4\Psi = -\frac{Eh}{a^2}(1+\nu)\alpha_t \frac{\Delta t}{h} \tag{63}$$

Substituting $q = 0$ in equations (58) to (60) yields the boundary conditions for thermal effects.

Completed Beltrami-Michell Formulation Versus Augmented Stress Function Formulation

As seen from equations (49), (51), and (57), the CBMF is represented by two coupled, second-order differential equations in terms of the internal forces M_x and N_φ. The augmented stress function formulation, however, is given as a single differential equation of the fourth order. A sufficient number of boundary conditions are available to calculate integration constants for both formulations. As seen from equations (54) to (56), the boundary conditions for the CBMF are given in terms of internal forces and their first-order derivatives. However, for the augmented stress function formulations, the boundary conditions given in equations (58) to (60) are expressed in terms of the stress function and its derivatives up to the third order. The augmented stress function formulation may be regarded as a solution technique for the CBMF. The two procedures are equivalent because they are derived with the same set of assumptions. The augmented stress function formulation, at first glance, appears to be more applicable for an analytical solution of the problem because it involves only one variable. However, reducing the number of variables involved results in a higher order governing differential equation. Thus, the solution method should be chosen according to the problem under consideration.

The CBMF, however, is more appropriate for an approximate solution of the problem using numerical techniques, particularly the finite element method. Because lower order governing equations are used, fewer constraints are imposed on the interpolation functions used to approximate the response variables. Moreover, in the CBMF, the internal forces are obtained as the primary result of the analysis, whereas in the augmented stress function formulation, they have to be calculated indirectly from the stress function values. Because the internal forces are expressed in terms of derivatives of stress functions, these calculations may introduce additional errors in the results.

Examples

Two examples are presented to illustrate CBMF analysis of circular cylindrical shells. A simply supported short, cylindrical shell and a composite shell composed of two long, cylindrical shells subjected to combined thermal and mechanical loadings are analyzed. Because the CBMF was used to analyze the composite circular plate, the shell examples are solved with the augmented stress function formulation.

Analysis of a Short Circular Cylindrical Shell.—Figure 4 shows a simply supported cylindrical shell of length L, radius a, and thickness h. The shell is made of a homogeneous and isotropic material with elastic modulus E and Poisson's ratio ν. The analysis is performed for two cases: (1) a uniformly distributed load of intensity q and (2) uneven heating with a temperature difference of Δt. The origin of the coordinate system is located at the centroid of the shell (fig. 4). The material and the geometric parameters of the shell are such that the product $\beta L < 5$; hence, it must be analyzed as a short shell. Equations (57) and (63), which describe the response of the shell for the mechanical and thermal loadings, respectively, have a similar form. The solution for both cases can be written as

$$\Psi(x) = C_1 \cosh \beta x \cos \beta x + C_2 \cosh \beta x \sin \beta x$$

$$+ C_3 \sinh \beta x \cos \beta x + C_4 \sinh \beta x \sin \beta x + \Psi^{(p)} \qquad (64)$$

where C_1, C_2, C_3, and C_4 are the integration constants and $\Psi^{(p)}$ is a particular integral of equation (57) or (63).

Solution for a uniformly distributed load: For a uniform load, equation (57) is homogeneous, and particular integral $\Psi^{(p)} = 0$. Because the shell is simply supported, the boundary conditions given in equations (59) are used for both contours. The expressions for the stress function Ψ and its second derivative are introduced into expressions for boundary conditions to obtain the system of four equations in terms of four constants $\{C\}^T = \{C_1\ C_2\ C_3\ C_4\}$:

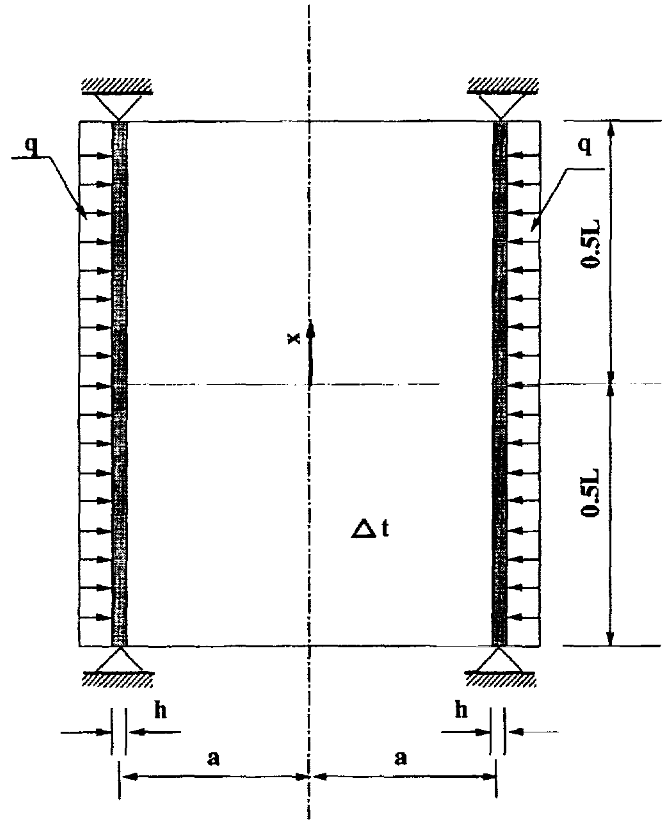

Figure 4.—Short, circular cylindrical shell.

$$[G]\{C\} = \{Q\} \qquad (65)$$

where $\{Q\}^T = -(q/2\beta^2)\{0\ 0\ 1\ 1\}$ is the right side vector and $[G]$ is the system matrix defined as

$$[G] = \begin{bmatrix} g_1 & -g_2 & -g_3 & g_4 \\ g_1 & g_2 & g_3 & g_4 \\ g_4 & g_3 & -g_2 & -g_1 \\ g_4 & -g_3 & g_2 & -g_1 \end{bmatrix} \qquad (66)$$

with $g_1 = \cosh \lambda \cos \lambda$, $g_2 = \cosh \lambda \sin \lambda$, $g_3 = \sinh \lambda \cos \lambda$, $g_4 = \sinh \lambda \sin \lambda$, and $\lambda = \beta L/2$. The solution of the system given in equation (65) yields the constants of integration as

$$\{C\}^T = \frac{q}{2\beta^3 D}\{g_4\ 0\ 0\ -g_1\} \qquad (67)$$

where $D = g_1^2 + g_4^2$. The expressions for the moment and normal force are

$$M_x(x) = \frac{q}{2\beta^2 D}(g_4 \cosh \beta x \cos \beta x - g_1 \sinh \beta x \sin \beta x)$$
$$N_\varphi(x) = \frac{qa}{D}\left[g_4(g_4 - \sinh \beta x \sin \beta x) + g_1(g_1 - \cosh \beta x \cos \beta x)\right] \quad (68)$$

The displacement w, if required, may be calculated from the stress-strain relations as

$$w(x) = -\frac{qa^2}{DEh}\left[g_4(g_4 - \sinh \beta x \sin \beta x) + g_1(g_1 - \cosh \beta x \cos \beta x)\right] \quad (69)$$

Solution for a thermal load: Next, the shell is analyzed for unevenly heated inner and the outer surfaces with a temperature difference of Δt. The particular integral for this case is given as

$$\psi^{(p)} = -K(1+\nu)\alpha_t \Delta t / h \quad (70)$$

the constants of integration of thermal loads are obtained as

$$\{C\}^T = -\frac{\psi^{(p)}}{D}\{g_1 \;\; 0 \;\; 0 \;\; g_4\} \quad (71)$$

and the expressions for the internal forces due to thermal loading have the following form:

$$M_x(x) = -\frac{\psi^{(p)}}{D}\left[g_1(g_1 - \cosh \beta x \cos \beta x) + g_4(g_4 - \sinh \beta x \cos \beta x)\right]$$
$$N_\varphi(x) = 2\beta^2 \frac{\psi^{(p)}}{D}(g_4 \cosh \beta x \cos \beta x - g_1 \sinh \beta x \sin \beta x) \quad (72)$$

The displacement w, if required, can be calculated as

$$w(x) = -2\frac{a\beta^2\psi^{(p)}}{EDh}(g_4 \cosh \beta x \cos \beta x - g_1 \sinh \beta x \sin \beta x) \quad (73)$$

Analysis of a Long Composite Shell.—In this section, we apply the augmented stress function formulation to a composite cylindrical shell subjected to both mechanical and thermal loadings. This example is presented to (1) demonstrate the CBMF procedure for different types of boundary conditions, (2) develop the solution for long cylindrical shells, and (3) demonstrate the ability of CBMF to solve problems that involve composite cylindrical shells. Figure 5 shows a cylindrical shell of radius a and length $2L$. It is composed of two regions with different material and geometrical properties. Region I, bounded by contours 1–1 and 2–2, has material parameters E_1 and ν_1 and thickness h_1; and region II, bounded by contours 2–2 and 3–3, has material and geometric properties E_2, ν_2, and h_2. The shell is clamped along contour 1–1 and simply supported along the contour 3–3. Both regions are subjected to a uniformly distributed load of intensity q. Region I also is subjected to the temperature change Δt. The material and geometric parameters for both regions are such that the products $\beta_1 L \geq 5$ and $\beta_2 L \geq 5$. This allows both regions to be treated as long shells, where the edge

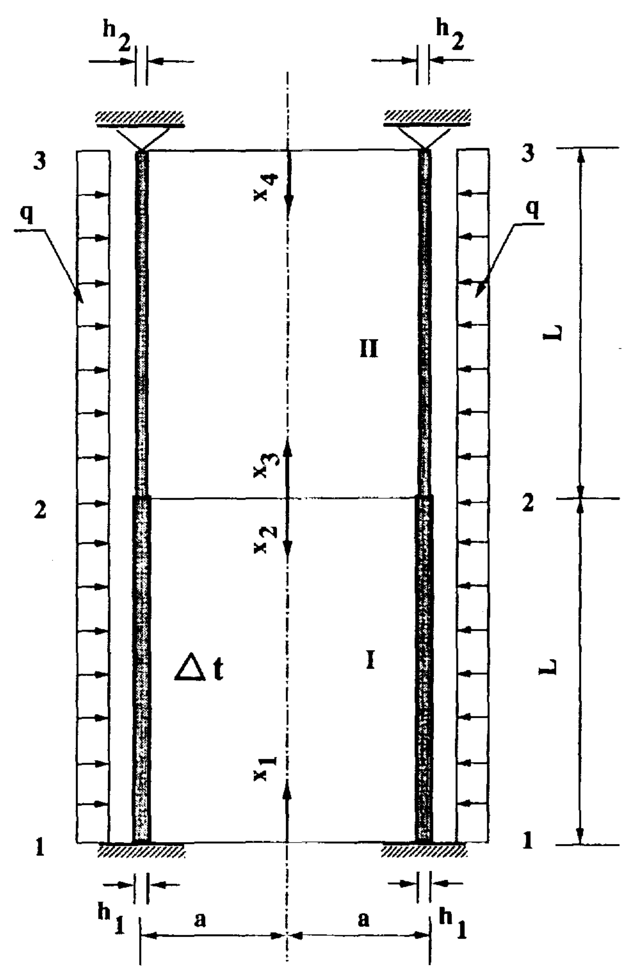

Figure 5.—Long, composite circular cylindrical shell.

effects introduced by the discontinuities in the material properties and the loading conditions along contour 2–2 have negligible effects at lines 1–1 and 3–3. This assumption also makes the effects of the supports negligible along contours 1–1 and 3–3 at the line of contact 2–2.

The expression for the stress function $\Psi(x)$ for the case of a long cylindrical shell is deduced from equation (64) by setting integration constants corresponding to the term $e^{\beta x}$ to be equal to zero. This stress function is written as

$$\Psi(x) = e^{-\beta x}\left(C_1 \cos \beta x + C_2 \sin \beta x\right) + \Psi_p \tag{74}$$

where C_1 and C_2 are integration constants and Ψ_p is the particular integral. Introducing equation (74) into equations (46a) and (50) yields the internal forces:

$$\left.\begin{aligned} M_x(x) &= e^{-\beta x}\left(C_1 \cos \beta x + C_2 \sin \beta x\right) + \Psi_p \\ N_\varphi(x) &= -2a\beta^2 e^{-\beta x}\left(-C_2 \cos \beta x + C_1 \sin \beta x\right) - a\frac{d^2\Psi_p}{dx^2} - aq \end{aligned}\right\} \tag{75}$$

The response of the shell consists of three parts: (1) the response due to constraints imposed on contour 1–1, (2) the response due to the discontinuity of the material properties and loadings along contour 2–2, and (3) the response due to constraints along contour 3–3. Because a long shell is assumed, these responses can be treated separately. The total response at any point of the shell can be obtained by superposing the individual responses.

Response of the shell due to the edge effects along contour 1–1: First, the expressions for the moment M_x and the force N_φ are derived for the edge effects on contour 1–1. The coordinate system is defined such that the axis x_1 is placed along the axis of the shell, with the origin in the plane defined by contour 1–1. The stress function Ψ is given as

$$\Psi(x_1) = e^{-\beta_1 x_1}\left(C_1 \cos \beta_1 x_1 + C_2 \sin \beta_1 x_1\right) + \Psi_p^{(\Delta t)} \tag{76}$$

where $\Psi_p^{\Delta t}$, the particular integral due to thermal effects, is calculated as $\Psi_p^{(\Delta t)} = K_1(1+\nu_1)\alpha_1 \Delta t / h_1$. The integration constants C_1 and C_2 are calculated by imposing the appropriate boundary conditions. For the clamped contour 1–1, both of the boundary compatibility conditions given in equations (60) need to be satisfied, which gives

$$C_1 = -C_2 = -\frac{q}{2\beta_1^2} \tag{77}$$

The solution for the internal forces due to the edge effects along contour 1–1 may be written as

$$M_x(x_1) = -\frac{q}{2\beta_1^2} e^{-\beta_1 x_1}\left(\cos \beta_1 x_1 - \sin \beta_1 x_1\right) + \Psi_p^{(\Delta t)} \tag{78a}$$

$$N_\varphi(x_1) = aq\left[e^{-\beta_1 x_1}\left(\cos \beta_1 x_1 - \sin \beta_1 x_1\right) - 1\right] \tag{78b}$$

Using equation (78b) yields the following expression for the displacement w:

$$w(x_1) = -\frac{qa^2}{Eh_1}\left[e^{-\beta_1 x_1}\left(\cos \beta_1 x_1 - \sin \beta_1 x_1\right) - 1\right] \tag{79}$$

Response of the shell due to the discontinuities along contour 2–2: The expressions for the stress functions Ψ_1 and Ψ_2, defined for regions I and II, respectively, may be written as

$$\left.\begin{aligned} \Psi_1(x_2) &= e^{-\beta_1 x_2}\left(A_1 \cos \beta_1 x_2 + B_1 \sin \beta_1 x_2\right) + \Psi_p^{(\Delta t)} \\ \Psi_2(x_3) &= e^{-\beta_2 x_3}\left(A_2 \cos \beta_2 x_3 + B_2 \sin \beta_2 x_3\right) \end{aligned}\right\} \tag{80}$$

where the coordinate axes x_2 and x_3 are defined separately for each region (fig. 5). Four constants of integration are calculated by imposing transition conditions along contour 2–2. They consist of two force equilibrium conditions,

$$\Psi_2(0) - \Psi_2(0) - \Psi_p^{(\Delta t)} = 0 \tag{81a}$$

$$\frac{d\Psi_1}{dx_2}(0) + \frac{d\Psi_2}{dx_3}(0) = 0 \tag{81b}$$

and two boundary compatibility conditions,

$$\frac{1}{E_1 h_1}\left[\frac{d^2\Psi_1}{dx_2^2}(0) + q\right] = \frac{1}{E_2 h_2}\left[\frac{d^2\Psi_2}{dx_3^2}(0) + q\right] \tag{81c}$$

$$\frac{1}{E_1 h_1}\frac{d^2\Psi_1}{dx_2^3}(0) + \frac{1}{E_2 h_2}\frac{d^3\Psi_2}{dx_3^3}(0) = 0 \tag{81d}$$

The transition conditions given in equations (81) yield

$$\left.\begin{aligned}
&A_1 - A_2 = -\Psi_p^{(\Delta t)} \\
&\beta_1\left(B_1 - A_1\right) + \beta_2\left(B_2 - A_2\right) = 0 \\
&-2\frac{\beta_1^2}{E_1 h_1} B_1 + 2\frac{\beta_2^2}{E_2 h_2} B_2 = \frac{q}{E_2 h_2} - \frac{q}{E_1 h_1} \\
&\frac{\beta_1^3}{E_1 h_1}\left(A_1 + B_1\right) + \frac{\beta_2^3}{E_2 h_2}\left(A_2 + B_2\right) = 0
\end{aligned}\right\} \tag{82}$$

The solution of equations (82) yields the four integration constants as

$$\left.\begin{aligned}
A_1 &= \frac{1}{\overline{D}}\left[\Psi_p^{(\Delta t)}\beta_2^2\left(-2k\beta_1\beta_2 - k\beta_1^2 - k^2\beta_2^2\right) + \frac{q}{2}(1-k)\left(k\beta_2^2 - \beta_1^2\right)\right] \\
A_2 &= \frac{1}{\overline{D}}\left[\Psi_p^{(\Delta t)}\beta_1^2\left(\beta_1^2 + k\beta_2^2 + 2k\beta_1\beta_2\right) + \frac{q}{2}(1-k)\left(k\beta_2^2 - \beta_1^2\right)\right] \\
B_1 &= \frac{1}{\overline{D}}\left[\Psi_p^{(\Delta t)}k\beta_2^2\left(\beta_1^2 - k\beta_2^2\right) + \frac{q}{2\beta_1}(1-k)\left(2k\beta_2^3 + k\beta_1\beta_2^2 + \beta_1^3\right)\right] \\
B_2 &= \frac{1}{\overline{D}}\left[\Psi_p^{(\Delta t)}\beta_1^2\left(\beta_1^2 - k\beta_2^2\right) - \frac{q}{2\beta_2}(1-k)\left(2\beta_1^3 + \beta_1^2\beta_2 + k\beta_2^3\right)\right]
\end{aligned}\right\} \tag{83}$$

where $\overline{D} = 2k\beta_1\beta_2(\beta_1^2 + \beta_2^2) + (\beta_1^2 + k\beta_2^2)^2$ and $k = (E_1 h_1)/(E_2 h_2)$. The integration constants given in equations (83) are introduced into equations (80) to obtain expressions for stress function Ψ for both regions. The expressions for the moment M_x and the force N_φ are obtained next by substituting equations (80) into equations (46a) and (50). The displacement w can then be calculated similarly as in the previous cases.

Response of the shell due to the support conditions on contour 3–3: For this case, a procedure similar to that presented for the edge effects along contour 1–1 is followed. The coordinate axis x_4 is defined as shown in figure 5, contour 3–3 is simply supported, and the conditions given in equation (60) are applied to obtain the constants of integration:

$$C_1 = 0 \qquad C_2 = \frac{q}{2\beta_2^2} \tag{84}$$

The expressions for the internal forces are written as

$$\left.\begin{aligned}
M_x(x_4) &= \frac{q}{2\beta_2^2} e^{-\beta_2 x_4} \sin \beta_2 x_4 \\
N_\varphi(x_4) &= -qa\left(-e^{-\beta_2 x_4} \cos \beta_2 x_4 + 1\right)
\end{aligned}\right\} \tag{85}$$

and the displacement w is calculated as

$$w(x_4) = \frac{qa^2}{Eh_2}\left(-e^{-\beta_2 x_4} \cos \beta_2 x_4 + 1\right) \tag{86}$$

Recapitulation

Similar to that for circular plates, the stationary condition of the IFM variational functional yields all the equations required to solve the shell bending problem in terms of stress variables. These equations include the field compatibility condition given in equation (51) and the boundary compatibility conditions given in equations (56), which were derived for the first time here. Without the boundary compatibility conditions, it is not possible to solve the shell bending problem in terms of stress parameters only for the mixed BVP. The transition conditions given in equations (81c) and (81d) make it possible to solve composite shells without any reference to displacements. Again, without equations (81c) and (81d), such a solution is not possible regardless of the type of boundary conditions.

Concluding Remarks

The Completed Beltrami-Michell Formulation (CBMF) has been established for analyzing circular plates and cylindrical shells subjected to thermal and mechanical loads and described by boundary value problems with stress, displacement, and mixed boundary conditions. The CBMF alleviates the limitations of the classical Beltrami-Michell's Formulation, which could analyze first or stress boundary value problems only. It can be considered as an alternative to Navier's formulation of elasticity. All the equations of the Beltrami-Michell Formulation were derived from the stationary condition of the IFM variational functional that was originally developed for problems in two-dimensional elasticity. The definitions of stress functions, along with the boundary compatibility conditions, also have been established. Transition conditions on the contours between the zones made of different materials have been established for composite plates and shells. For thermal analysis, the original IFM functional has been expanded to include the effects of initial strains. Its stationary condition modified the field and boundary equations. Finally, several mixed boundary value problems have been solved with these formulations to demonstrate the capability of the CBMF to solve mixed boundary value problems solely in terms of stress variables.

Lewis Research Center
National Aeronautics and Space Administration
Cleveland, Ohio, August 26, 1994

Appendix—Symbols

A	potential energy in the IFM functional
a	radius of the shell
B	complementary energy in the IFM functional
E	modulus of elasticity
$F_1, F_2, \ldots$	truss member forces
f	field compatibility condition
$[G]$	material matrix
h	plate or shell thickness
K	plate or shell flexural rigidity
L	length of shell
M_r, M_φ	plate bending moments
M_x	shell bending moment
N_φ	shell tangential force
n_x, n_y	directional cosines
P_x, P_y	components of surface tractions
q	intensity of the distributed load
r, r_a, r_b	radial coordinates
$\mathcal{R}(\sigma)$	boundary compatibility condition in terms of stresses
$\mathcal{S}_1(\sigma), \mathcal{S}_2(\sigma)$	surface tractions in terms of stresses
u, υ, w	displacement components
$\bar{u}, \bar{\upsilon}$	prescribed displacements
W	potential of external loads in the IFM functional
$w^{\mathrm{I}}, w^{\mathrm{II}}$	displacement fields on regions I and II, respectively
x, y	Cartesian coordinates
x_a, x_b	coordinates of shell contours
α_t	coefficient of thermal expansion
β	cylindrical shell parameter
Γ	boundary of an elastic domain
Δt	temperature difference between inner and outer surfaces
$\varepsilon_x, \varepsilon_y, \gamma_{xy}$	strain tensor components for plane stress
ε_φ	shell tangential strain
κ_r, κ_φ	plate curvatures
κ_x	shell curvature
ν	Poisson's ratio
Π_p^c, Π_s^c	IFM variational functional for plane stress, plates and shells, respectively
$\sigma_x, \sigma_y, \tau_{xy}$	components of the stress tensor

Φ	Airy's stress function for plane stress
χ	particular integral for external loading
Ψ	shell stress function
$\Psi^{(p)}$	particular integral for stress function
Ω	domain of the midsurface of the shell

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Igor Kaljević, Sunil Saigal, Dale A. Hopkins, and Surya N. Patnaik

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Igor Kaljević, Ohio Aerospace Institute, 22800 Cedar Point Road, Cleveland, Ohio 44142; Sunil Saigal, Department of Civil Engineering, Carnegie Mellon University, Pittsburgh, Pennsylvania 15213; Dale A. Hopkins, NASA Lewis Research Center; and Surya N. Patnaik, Ohio Aerospace Institute, 22800 Cedar Point Road, Cleveland, Ohio 44142, and NASA Resident Research Associate at Lewis Research Center. Responsible person, Dale A. Hopkins, organization code 5210, (216) 433-3260.

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13. ABSTRACT ***(Maximum 200 words)***

A force method formulation, the Completed Beltrami-Michell Formulation (CBMF), has been developed for analyzing boundary value problems in elastic continua. The CBMF is obtained by augmenting the classical Beltrami-Michell Formulation with novel boundary compatibility conditions. It can analyze general elastic continua with stress, displacement, or mixed boundary conditions. The CBMF alleviates the limitations of the classical formulation, which can solve stress boundary value problems only. In this report, the CBMF is specialized for plates and shells. All equations of the CBMF, including the boundary compatibility conditions, are derived from the variational formulation of the Integrated Force Method (IFM). These equations are defined only in terms of stresses. Their solution for kinematically stable elastic continua provides stress fields without any reference to displacements. In addition, a stress function formulation for plates and shells is developed by augmenting the classical Airy's formulation with boundary compatibility conditions expressed in terms of the stress function. The versatility of the CBMF and the augmented stress function formulation is demonstrated through analytical solutions of several mixed boundary value problems. The example problems include a composite circular plate and a composite circular cylindrical shell under the simultaneous actions of mechanical and thermal loads.

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