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SECOND-ORDER RECONSTRUCTION OF THE INFLATIONARY POTENTIAL

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ABSTRACT

To first order in the deviation from scale invariance the inflationary potential and its first two derivatives can be expressed in terms of the spectral indices of the scalar and tensor perturbations, n and n_T, and their contributions to the variance of the quadrupole CBR temperature anisotropy, S and T. In addition, there is a "consistency relation" between these quantities: $n_T = -\frac{1}{7}\frac{T}{S}$. We derive the second-order expressions for the inflationary potential and its first two derivatives and the first-order expression for its third derivative, in terms of n, n_T, S, T, and $dn/d\ln\lambda$. We also obtain the second-order consistency relation, $n_T = -\frac{1}{7}\frac{T}{S}[1 + 0.11\frac{T}{S} + 0.15(n-1)]$. As an example we consider the exponential potential, the only known case where exact analytic solutions for the perturbation spectra exist. We reconstruct the potential via Taylor expansion (with coefficients calculated at both first and second order), and introduce the Padé approximant as a greatly improved alternative.

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1 Introduction

In inflationary models quantum fluctuations excited on very small length scales ($\sim H_I^{-1} \sim 10^{-23}$cm) are stretched to astrophysical scales ($\sim 10^{25}$cm) by the tremendous growth of the scale factor during inflation (H_I is the value of Hubble parameter during inflation) [1]. This results in almost scale-invariant spectra of scalar (density) [2] and tensor (gravitational-wave) [3] metric perturbations. Together with the prediction of a spatially-flat Universe they provide the means for testing the inflationary paradigm. The tensor fluctuations lead to cosmic background radiation (CBR) anisotropy and a stochastic background of gravitational waves with wavelengths from about 1 km to over 10^4 Mpc. The scalar fluctuations also lead to CBR anisotropy and seed the formation of structure in the Universe.

The amplitudes and spectral indices of the metric fluctuations can be expressed in terms of the inflationary potential and its derivatives, evaluated at the value of the scalar field when astrophysically interesting scales crossed outside the horizon during inflation (from galactic scales to the presently observable Universe, corresponding to the eight e-foldings about 50 e-folds or so before the end of inflation). Techniques have been developed for relating the scalar and tensor spectra to the potential and its derivatives in an expansion whose small parameter is the deviation from scale invariance [4]. In particular, to first order in the deviation from scale invariance the spectral indices and the power spectra of the fluctuations today can be written as [5][1]

$$n = 1 - \frac{x_{50}^2}{8\pi} + \frac{m_{\rm Pl} x'_{50}}{4\pi}, \qquad n_T = -\frac{x_{50}^2}{8\pi}, \tag{1}$$

$$P(k) = A k^n |T(k)|^2, \qquad P_T(k) = A_T k^{n_T - 3} |T_T(k)|^2 \left(\frac{3 j_1(k\tau_0)}{k\tau_0}\right)^2, \tag{2}$$

$$A = \frac{1024\pi^3 k_{50}^{1-n}}{75 H_0^4}\left[1 + \frac{7}{6} n_T + \left[-\frac{7}{3} + \ln 2 + \gamma\right](n-1)\right]\frac{V_{50}}{m_{\rm Pl}^4 x_{50}^2}, \tag{3}$$

$$A_T = \frac{8 k_{50}^{-n_T}}{3\pi}\left[1 + \left(-\frac{7}{6} + \ln 2 + \gamma\right) n_T\right]\frac{V_{50}}{m_{\rm Pl}^4}. \tag{4}$$

[1]Several minor errors in Ref. [5] have been corrected here: the factors of H_0^{3+n} in Eqs. (A5, A7) should be H_0^4; the factor of $1.1(n-1)$ in Eq. (A8) is more precisely $1.3(n-1)$; the factor of $1.2 n_T$ in Eq. (A14) is more precisely $1.4 n_T$.

Here k is the comoving wavenumber, $x = m_{\rm Pl}V'/V$ measures the steepness of the potential, prime denotes derivative with respect to the scalar field that drives inflation, subscript 50 indicates that the quantity is to be evaluated 50 e-folds before the end of inflation,[2] $m_{\rm Pl} = 1.22 \times 10^{19}$ GeV is the Planck mass, H_0 is the present value of the Hubble constant, $\tau_0 \simeq 2H_0^{-1}$ is the present conformal age of the Universe, and $\gamma \simeq 0.577$ is Euler's constant. Scale-invariant metric perturbations correspond to $(n-1) = n_T = 0$. The functions $T(k)$ and $T_T(k)$ are the transfer functions for scalar [7] and tensor [8] metric perturbations respectively; for $k\tau_0 \ll 100$, both $T(k)$ and $T_T(k) \to 1$.

From these expressions the consequences of the scalar and tensor metric fluctuations may be computed. In particular, the contributions to the variance of the angular power spectrum of the CBR anisotropy on large angular scales ($l \ll 200$) that arise predominantly due to the Sachs-Wolfe effect are given by [8]

$$\langle |a^S_{lm}|^2 \rangle = \frac{H_0^4}{2\pi} \int_0^\infty k^{-2} P(k) |j_l(k\tau_0)|^2 dk, \tag{5}$$

$$\langle |a^T_{lm}|^2 \rangle = 36\pi^2 \frac{\Gamma(l+3)}{\Gamma(l-1)} A_T \int_0^\infty k^{n_T+1} |F_l(k)|^2 |T_T(k)|^2 dk, \tag{6}$$

$$F_l(k) = -\int_{\tau_{\rm LSS}}^{\tau_0} \frac{j_2(k\tau_0)}{k\tau_0} \left(\frac{j_l(k\tau_0 - k\tau)}{(k\tau_0 - k\tau)^2} \right) d\tau,$$

where $\tau_{\rm LSS} \simeq \tau_0/(1+z_{\rm LSS})^{1/2} \approx \tau_0/35$ is the conformal age at last scattering ($z_{\rm LSS} \simeq 1100$) and j_l is the spherical Bessel function of order l. (We note in passing that both expressions are based upon the approximation that the Universe is matter-dominated at last-scattering; the small contribution of radiation, about 10%-20%, leads to corrections that are of the same order [8] and would have to be included in a more accurate treatment when the data so demand.)

[2]The point about which the potential is expanded is in principle arbitrary. However, the spectral indices n and n_T can only plausibly be measured on scales from 1Mpc – 10^4Mpc and S and T depend upon perturbations on these same scales, so it makes sense to choose the expansion point to correspond to when these scales crossed outside the horizon during inflation; in addition, by taking $k_{50}\tau_0 = 1$ several expressions simplify. The precise number of e-folds before the end of inflation when these scales crossed outside the horizon depends logarithmically upon the energy scale of inflation and the reheat temperature, see Refs. [4, 6]; for the sake of definiteness we take this number to be 50, which can easily be changed to the correct value for a given inflationary model.

The contribution of scalar and tensor metric perturbations to the variance of the quadrupole CBR anisotropy can be computed numerically [5]

$$S \equiv \frac{5\langle|a^S_{2m}|^2\rangle}{4\pi} \simeq 2.2\,[1 + 1.2n_T + 0.08(n-1)]\,\frac{V_{50}(k_{50}\tau_0)^{1-n}}{m^4_{\rm Pl}x^2_{50}}, \tag{7}$$

$$T \equiv \frac{5\langle|a^T_{2m}|^2\rangle}{4\pi} \simeq 0.61\,[1 + 1.4n_T]\,\frac{V_{50}(k_{50}\tau_0)^{-n_T}}{m^4_{\rm Pl}}, \tag{8}$$

where the dependence upon $(n-1)$ and n_T is given to first-order. In evaluating these expressions the effect of transfer functions is negligible as the integrals are dominated by $k\tau_0 \sim 2$. For simplicity, following footnote 2 we henceforth omit factors of $(k_{50}\tau_0)^{1-n}$ and $(k_{50}\tau_0)^{-n_T}$; they are easily re-inserted if needed.

1.1 First-order reconstruction

Since S, T, n, and n_T are expressed in terms of the potential and its first two derivatives, one can invert the expressions to solve for the potential and its first two derivatives in terms of S, T, n, and n_T plus a "consistency relation." Those expressions are [9]

$$\begin{aligned} V_{50}/m^4_{\rm Pl} &= 1.65(1-1.4n_T)T, \\ &= 1.65\left(1 + 0.20\frac{T}{S}\right)T, \end{aligned} \tag{9}$$

$$\begin{aligned} V'_{50}/m^3_{\rm Pl} &= \pm 8.3\sqrt{-n_T}T, \\ &= \pm 8.3\sqrt{\frac{1}{7}\frac{T}{S}}T, \end{aligned} \tag{10}$$

$$\begin{aligned} V''_{50}/m^2_{\rm Pl} &= 21[(n-1) - 3n_T]T, \\ &= 21\left[(n-1) + 0.43\frac{T}{S}\right]T, \end{aligned} \tag{11}$$

$$n_T = -\frac{1}{7}\frac{T}{S}. \tag{12}$$

In the second expressions for the potential and its first two derivatives we have used the consistency relation to express n_T in terms of $\frac{T}{S}$, as $\frac{T}{S}$ should be easier to measure [10]. Note that the sign of V' cannot be determined as it can be changed by a field redefinition $\phi \to -\phi$, though a specific choice here determines the signs of various later expressions.

This procedure actually generates the full second-order term for V_{50}, while the other expressions are first-order. Our nomenclature and the overall reconstruction strategy are explained and clarified below.

1.2 The perturbative reconstruction strategy

Before delving into the algebra of the more complicated second-order expressions, let us orient the reader with an overview of the perturbative reconstruction process [6, 9]. The goal of the process is to express the inflationary potential and its derivatives, $V_{50}^{(l)}$ for $l = 0, 1, 2, \cdots$, in an expansion whose terms are powers of observables, T, $\frac{T}{S}$, n, n_T, $dn/d\ln\lambda$, and so on. The choice of the set of observables is of course not unique. From these, the potential can be reconstructed over some finite interval via a series expansion. While one can hope to learn about the potential over the interval that affects astrophysical scales, it is probably not realistic to hope to learn much about the potential globally without some additional a priori knowledge (e.g., the functional form of the potential).[3]

The formal expansion parameter is the deviation from exact scale invariance, which can be expressed as $(n-1)$; the other observables we shall use are: $n_T \sim \mathcal{O}[(n-1)]$, $\frac{T}{S} \sim \mathcal{O}[(n-1)]$, and $dn/d\ln\lambda \sim \mathcal{O}[(n-1)^2]$. (In the scale-invariant limit, $V(\phi) =$const, and $n = 1$, $n_T = 0$, $\frac{T}{S} = 0$, and $dn/d\ln\lambda = 0$.) The expression for $V_{50}^{(l)}$ begins at order $(n-1)^{l/2}T$, with higher-order terms $(n-1)^{k+l/2}T$, $k = 1, 2, 3, \cdots$. The expressions for higher derivatives of the potential are not only of higher order in $(n-1)$, but also involve more observables. For example, to lowest order the expression for V_{50} only involves T; that for V_{50}' involves T and $\frac{T}{S}$; that for V_{50}'' involves T, $\frac{T}{S}$, and $(n-1)$, and, as we shall see, that for V_{50}''' involves T, $\frac{T}{S}$, $(n-1)$, and $dn/d\ln\lambda$. (We have not included n_T in any of the lists, assuming implicitly that it is expressed in terms of the other observables by means of a consistency relation, as we now discuss.) Likewise, the expression for a given derivative involves additional observables as one goes to higher and higher order.

[3]The one possible exception involves the accurate measurement of the stochastic background of gravitational waves on scales from 1 km to 3000 Mpc (corresponding to $N \simeq 0 - 50$) in which case the inflationary potential could be mapped out directly since the amplitude of the tensor perturbation on a given scale is related to the value of the potential.

As a matter of nomenclature, we shall refer to the lowest-order term for any $V_{50}^{(l)}$ or for the consistency relation as "first-order," the next as "second-order," and so on. The four expressions from Ref. [9] given above are respectively the second-order expression for V_{50}, the first-order expressions for V'_{50} and V''_{50}, and the first-order consistency relation. In the next Section we shall use second-order results from Ref. [11] to derive the second-order expressions for V'_{50}, V''_{50}, and the consistency relation, as well as the first-order expression for V'''_{50}. Both the second-order expression for V''_{50} and the first-order expression for V'''_{50} involve the derivative of the scalar spectral index, $dn/d\ln\lambda$, which is likely to be very difficult to measure. Higher-order expressions for the potential and its first three derivatives, as well as the fourth derivative, will involve yet another even less accessible observable, and thus there is little motivation at present for proceeding further in the perturbative expansion.

An important feature of reconstruction is that the problem is overdetermined; specifically, a set of $M \geq 3$ observables can be expressed in terms of the potential and its first $M-2$ derivatives. This implies a "consistency relation," which, for increasing M, contains terms of higher and higher order. The lowest-order consistency equation, $n_T = -\frac{1}{7}\frac{T}{S}$, has been much discussed (e.g., in Ref. [4]) and arises through Eqs. (1), (7) and (8) which express n_T, S, and T in terms of V_{50} and V'_{50}.

Calculating higher derivatives alone, while keeping the calculation of each derivative to lowest order, does not lead to the correct second-order term in the consistency equation, and nor does calculating the second-order corrections to the derivatives present. One must systematically do both. The second-order version of the consistency equation is obtained by calculating the potential, its derivative and Eq. (1) to a higher order. Adding an extra order to the calculation of V'_{50} adds a new observable, $(n-1)$, which will appear in the consistency equation at second order. To account for there being still only a single consistency equation, there must be a new equation, and because $(n-1)$ has only entered at second-order in V'_{50}, we only need the first-order equation for V''_{50}. The second-order consistency equation, which we calculate in this paper, therefore relates n_T, $\frac{T}{S}$ and $(n-1)$, with the last only appearing as a second-order correction. Were one to desire a calculation to yet higher order, the same pattern would persist; each existing derivative must be calculated to one extra order and the next derivative to lowest order, introducing a new observable. This will generate next-order terms in the consistency equation with the new observable appearing at that order.

However, this presently cannot be done as third-order expressions for V_{50} and V'_{50} have not been calculated.

2 Second-order Reconstruction Reduced to Practice

The reconstruction equations for the scalar potential and its first two derivatives, evaluated to second-order, are given in Ref. [11], though not in terms of cosmological observables. They are given in terms of the perturbation amplitudes A_G^2 and A_S^2. Very roughly, A_S is the horizon-crossing amplitude of the density perturbation on a given scale and A_G is the horizon-crossing amplitude of the tensor perturbation (in the Appendix we provide some relations between notation used in that paper and this one.) Our purpose here is to express these second-order expressions for the potential and its first two derivatives in terms of the measurable quantities n, $dn/d\ln\lambda$, n_T, S, and T.

The amplitudes A_S^2 and A_G^2 are related to the observables S, T, n_T and n by:

$$A_G^2 = 0.70(1 - 1.3n_T)T, \qquad A_S^2 = 9.6[1 - 1.15(n-1)]S, \tag{13}$$

where the $(n-1)$ and n_T dependencies have been found by evaluating the Sachs-Wolfe integrals numerically. Both expressions are accurate to second-order.

Before deriving second-order expressions for the potential and its derivatives, we calculate the second-order version of the consistency relation. It is obtained from Eq. (2.9) of Ref. [11],

$$-\frac{n_T}{2} = \frac{A_G^2}{A_S^2}\left[1 + 3\epsilon - 2\eta\right], \tag{14}$$

where to the required order the slow-roll parameters ϵ and η (defined in the Appendix) are given by

$$\epsilon = -n_T/2, \qquad \eta = (n-1)/2 - n_T. \tag{15}$$

This gives a simple and very useful relation for A_G^2/A_S^2,

$$\frac{A_G^2}{A_S^2} = -0.5n_T\left[1 - 0.5n_T + 1.0(n-1)\right]. \tag{16}$$

Substituting into Eq. (13), we find the second-order consistency relation

$$n_T = -\frac{1}{7}\frac{T}{S}\left[1 - 0.8n_T + 0.15(n-1)\right], \tag{17}$$

or

$$\frac{T}{S} = -7n_T\left[1 + 0.8n_T - 0.15(n-1)\right]. \tag{18}$$

To the required order we can use the first-order truncation $n_T = -\frac{1}{7}\frac{T}{S}$ inside the brackets, thereby obtaining an alternative form,

$$n_T = -\frac{1}{7}\frac{T}{S}\left[1 + 0.11\frac{T}{S} + 0.15(n-1)\right], \tag{19}$$

where n_T is given in terms of the more accessible quantities $(n-1)$ and $\frac{T}{S}$.

Independent measurements of n, n_T and $\frac{T}{S}$ provide a powerful test of the inflationary hypothesis; in the space of these parameters inflationary models must lie on the surface defined by Eq. (18). In Figure 1 we illustrate the inflationary surface both without and with second-order corrections. The second-order corrections break the degeneracy in the $(n-1)$ direction, as well as typically reducing $\frac{T}{S}$ viewed as a function of n_T and $(n-1)$. However, the portions of the surface that feature large corrections are not favored by present cosmological data, and further, are susceptible to higher-order corrections. (Indeed, well away from scale invariance the surface would be noticeably different even just using Eq. (19) instead of Eq. (18), which differ by third-order terms.)

Obtaining the reconstruction equations is simply a matter of substituting into Eqs. (3.4), (3.6) and (3.15) of Ref. [11] for V, V' and V'' respectively. We give two alternative forms for each, the first using n_T and the second substituting $\frac{T}{S}$ for n_T using the second-order consistency equation. They are

$$\begin{aligned} V_{50}/m_{\rm Pl}^4 &= 1.65(1 - 1.4n_T)T, \\ &= 1.65\left(1 + 0.20\frac{T}{S}\right)T, \end{aligned} \tag{20}$$

$$\begin{aligned} V'_{50}/m_{\rm Pl}^3 &= \pm 8.3\sqrt{-n_T}\left[1 - 1.1n_T - 0.03(n-1)\right]T, \\ &= \pm 8.3\sqrt{\frac{1}{7}\frac{T}{S}}\left[1 + 0.21\frac{T}{S} - 0.04(n-1)\right]T, \end{aligned} \tag{21}$$

$$V''_{50}/m_{\rm Pl}^2 = 21\Big[(n-1) - 3n_T + 1.4n_T^2$$

$$
\begin{aligned}
& +0.6 n_T(n-1) - 0.2(n-1)^2 - 1.1\frac{dn}{d\ln\lambda}\Bigg] T, \\
= \quad & 21\Bigg[(n-1) + 0.43\frac{T}{S} + 0.073\left(\frac{T}{S}\right)^2 \\
& -0.015\frac{T}{S}(n-1) - 0.2(n-1)^2 - 1.1\frac{dn}{d\ln\lambda}\Bigg] T. \qquad (22)
\end{aligned}
$$

These expressions are accurate to second-order. Naturally, they agree with the first-order expressions given earlier.

Though no expression is given in Ref. [11] for V''', by using the lowest-order expressions for ϵ, η, and a third slow-roll parameter ξ, and Eq. (3.13) which relates the three to $dn/d\ln\lambda$, one can obtain the first-order expression,

$$
\begin{aligned}
V'''_{50}/m_{\rm Pl} &= \pm 104\sqrt{-n_T}\left[-\frac{dn/d\ln\lambda}{n_T} - 6n_T + 4(n-1)\right] T, \\
&= \pm 104\sqrt{\frac{1}{7}\frac{T}{S}}\left[7\frac{dn/d\ln\lambda}{T/S} + 0.9\frac{T}{S} + 4(n-1)\right] T. \qquad (23)
\end{aligned}
$$

where the overall sign is to be the same as that of V'. The second-order term would require yet another observable. As remarked in Ref. [11], even this first-order expression features the rate of change of the scalar spectral index, which is likely to be very difficult to measure. Realistically then, in the near term only the value of the potential and its first two derivatives are likely to be accessible to accurate determination.

The final step in reconstructing the potential is to use $d\phi/dN$, N being the number of e-foldings, to find the range of ϕ that corresponds to the eight or so e-foldings of inflation relevant for astrophysics. In effect, this introduces an additional small parameter, $\nu \equiv (50-N)/50 \leq 8/50$, where $N \simeq 42$ is the number of e-folds before the end of inflation when the smallest scale of interest crossed outside the horizon. The smallness of this parameter is essential if the approximation of the spectra by power-laws is to be valid.

To proceed, we may simply carry out a Taylor expansion of ϕ about ϕ_{50}, to whatever order we believe is appropriate,

$$
\phi_N - \phi_{50} = (N-50)\frac{d\phi}{dN}\bigg|_{\phi_{50}} + \frac{1}{2}(N-50)^2\,\frac{d^2\phi}{dN^2}\bigg|_{\phi_{50}} + \cdots \qquad (24)
$$

We can easily substitute for these derivatives, using as a starting point the *exact* formula

$$\dot{\phi} = -\frac{m_{\rm Pl}^2}{4\pi} H' \,, \tag{25}$$

which, along with $dN/dt = -H$, yields the relation from which the Taylor coefficients may be calculated,

$$\frac{d\phi}{dN} = \frac{m_{\rm Pl}^2}{4\pi} \frac{H'}{H} \,. \tag{26}$$

To get a given coefficient in the Taylor expansion for ϕ_N, one simply calculates $d^i\phi/dN^i$ expanding to the desired order in the deviation from scale invariance.[4] For example, taking only the first term in the ϕ_N expansion and working to first-order one obtains

$$\phi_N - \phi_{50} \simeq \pm \frac{m_{\rm Pl}}{\sqrt{8\pi}} \sqrt{-n_T}\,(N-50), \tag{27}$$

where again the overall sign is the same as that of V'. Next, we give the first coefficient in the ϕ_N expansion to second-order and the second coefficient in the ϕ_N expansion to first-order only,

$$\begin{aligned} \phi_N - \phi_{50} \;&=\; \pm \frac{m_{\rm Pl}}{\sqrt{8\pi}} \sqrt{-n_T}\,\left[1 + 0.1 n_T + 0.1(n-1)\right](N-50) \\ &\quad \pm \frac{m_{\rm Pl}}{4\sqrt{8\pi}} \sqrt{-n_T}\,\left[(n-1) - n_T\right](N-50)^2 + \cdots, \end{aligned} \tag{28}$$

with both signs again agreeing with that of V'.

3 Techniques and an Example

Before going on to specifics, let us again consider perturbative reconstruction from the larger perspective. In constructing the Taylor series for the value of

[4]Note this procedure differs slightly from that in Ref. [9], where $d\phi/dN$ was expanded linearly about ϕ_{50} and ϕ_N was solved for exactly, cf. Eq. (8). This results in an exponential, whose expansion picks up the $(N-50)$ and $(N-50)^2$ terms correctly to lowest order in the deviation from scale invariance, though not the higher-order terms in the $(N-50)$ term which would require higher-order terms in the expansion of $d\phi/dN$. There is an overall sign error in Eq. (8) of Ref. [9].

the potential at a given point the l-th term is given by $V_{50}^{(l)}(\phi-\phi_{50})^l/l!$. As mentioned earlier $V_{50}^{(l)}$ starts at order $(n-1)^{l/2}T$; from Eq. (28) we see that $(\phi-\phi_{50})^l$ is of order $(n-1)^{l/2}\Delta N^l$, where $\Delta N = N-50$ is around eight for the region of the potential corresponding to astrophysically accessible scales. Thus, the l-th term in the Taylor expansion to lowest order is

$$V_{50}^{(l)}(\phi-\phi_{50})^l/l! \sim \mathcal{O}[(n-1)^l\,\Delta N^l\,T] + \cdots. \tag{29}$$

In order that the series be clearly convergent, the deviation from scale invariant $(n-1)$ should be less than about $\Delta N^{-1} \sim 0.1$. In the case that $(n-1)\Delta N \sim \mathcal{O}(1)$, as in our example below, Eq. (29) suggests that the expression for each derivative be expanded to the same order in the deviation from scale invariance. On the other hand, if $(n-1) \ll 0.1$, then clearly the higher-derivative terms in the Taylor expansion are less important, and a sound case could be made for expanding the lower derivatives of the potential to higher order in $(n-1)$. However, because of the existence of two expansion parameters, ΔN and $(n-1)$, that are a priori unrelated, in the general case there is no clear prescription. In our example below, where $(n-1)\Delta N \sim \mathcal{O}(1)$, we will explore several possibilities.

3.1 Expansion techniques

Given the value of the potential and its first two or three derivatives at a point and the ϕ_N relation just obtained, one can reconstruct the potential on the observationally relevant scales (i.e., $N \simeq 42-50$). The standard technique used previously is the Taylor expansion

$$V(\phi) = V_{50} + V'_{50}(\phi-\phi_{50}) + \frac{1}{2}V''_{50}(\phi-\phi_{50})^2 + \cdots \tag{30}$$

For many situations this is perfectly fine (e.g., when n_T and $n-1$ are small, see Ref. [9]). However, if the range of eight or so e-foldings corresponds to a large range in ϕ the convergence may not be very good because of the abrupt truncation of the Taylor series. Specifically, for large $(\phi-\phi_{50})$ the shape of the reconstructed potential is dictated, rightly or wrongly, by the last term in the expansion (quadratic or cubic).

An often used alternative is the Padé approximant [12], which can be generated directly from a truncated power series. For a power series that

extends to order N, the Padé approximants are quotients of two polynomials of order L (numerator) and M (denominator) denoted by $[L, M]$, where $L + M = N$. By construction, the expansion of $[L, M]$ matches that of the power series to order N, but of course is not truncated. Very often, Padé approximants provide a very good approximation over a wider range of values than the Taylor series from which they are derived; they in some way encode better estimates of the higher-order terms than does truncation. If we truncate the Taylor series at the second derivative, then the associated diagonal Padé approximant $[1, 1]$ is a ratio of two first-order polynomials given by[5]

$$R(\phi) = \frac{a_0 + a_1\phi}{1 + b_1\phi}, \tag{31}$$

with

$$a_0 = V_{50}; \quad b_1 = -V''_{50}/2V'_{50}; \quad a_1 = V'_{50} - V_{50}V''_{50}/2V'_{50}. \tag{32}$$

As we shall now illustrate by specific example, Padé approximants have a lot to offer when the Taylor series proves a poor approximation.

3.2 Reconstructing an exponential potential

A useful testing ground for reconstruction is the exponential potential, the only known case where the perturbation spectra can be derived exactly analytically [13]. For the potential

$$V(\phi) = V_0 \exp\left(-\sqrt{\frac{16\pi}{p}}\,\frac{\phi}{m_{Pl}}\right), \tag{33}$$

the scale factor grows exactly as t^p. Compared with the lowest order expressions, the amplitudes A and A_T, or A_S^2 and A_G^2, are both multiplied by the same p-dependent factor $R^2(p)$, where

$$R(p) = 2^{1/(p-1)}\frac{\Gamma\left[3/2 + 1/(p-1)\right]}{\Gamma[3/2]}\left(1 - 1/p\right)^{p/(p-1)}, \tag{34}$$

[5]The $[2, 0]$ approximant is just the truncated Taylor series; in addition to simplicity, there is some motivation for using the diagonal approximant rather than the $[0, 2]$ approximant as it is asymptotically constant, consistent with the flatness of inflationary potentials.

where $\Gamma(\cdots)$ is the usual gamma function. Both scalar and tensor spectra are exact power laws with spectral indices $(n-1) = n_T = -2/(p-1)$. The scalar-field solution is characterized by

$$\dot{\phi} = \sqrt{\frac{p}{4\pi}}\frac{m_{\rm Pl}}{t}; \quad \frac{d\phi}{dN} = -\frac{m_{\rm Pl}}{\sqrt{4\pi p}}; \quad V(\phi_N) = V_{50}^{\rm true}\exp\left[2(N-50)/p\right]. \quad (35)$$

The expressions for T and S can be obtained exactly by integrating Eqs. (5,6),

$$S = 2.2 f(n) R^2(p) \frac{V_{50}^{\rm true}}{m_{\rm Pl}^4 x_{50}^2}, \quad (36)$$

$$T = 0.61 g(n_T) R^2(p) \frac{V_{50}^{\rm true}}{m_{\rm Pl}^4}, \quad (37)$$

where the numerical factors $f(n) = 1 + 1.15(n-1) + \cdots$ and $g(n_T) = 1 + 1.3 n_T + \cdots$ arise from the n, n_T dependence of the Sachs-Wolfe integrals, cf. Eqs. (5, 6).

We are now ready to carry out an array of reconstruction methods. Because we are using exact expressions to generate the spectra, this procedure is more ambitious, and more realistic, than those attempted thus far [6, 9], where the trial spectra were produced using the slow-roll approximation. For the general inflationary potential, exact results are not known, and so this procedure is not possible. However, our method here should give a more realistic estimate of inherent errors even in the general case.

There are two distinct types of error. The first is error in the value of the potential at ϕ_{50}, due to third-order and higher terms. By substituting the expression for T in Eq. (37) into Eq. (9) or (20) for V_{50} we can compute that error:

$$V_{50}/V_{50}^{\rm true} = g(n_T)(1 - 1.4 n_T) R(p)^2. \quad (38)$$

The second error involves the shape of the potential, which depends on the ability of the chosen expansion to match the potential over the eight interesting e-foldings.

We have chosen as a specific example an exponential potential with $p = 43/3$. We did so because this leads to about the largest departure from scale invariance that can still be regarded as observationally viable, $(n-1) = n_T = -0.15$ and $\frac{T}{S} \simeq 1$, and thus realistically represents the most challenging example of reconstruction. The exact potential is shown in Fig. 2 along with the results of five different reconstructions.

To begin, consider the error in estimating $V_{50}^{\rm true}$; $g(n_T = -0.15) = 0.824$ and so $V_{50}/V_{50}^{\rm true} \simeq 0.95$, a modest 5% due to the neglected higher-order terms. As we always include the second-order term in V_{50}, the error is the same in every method we look at. Had the first-order expression for V_{50} been used instead, corresponding to the neglect of the factor of $(1 - 1.4n_T)$ in Eq. (9), then the underestimation would have been about 20%.

Let us now consider the shape, which we note depends on T and S only through their ratio. The important distinction between different methods is the difference in required input data; methods needing only n and $\frac{T}{S}$ have the advantage of depending only on the information that is easiest to obtain. Requiring $dn/d\ln\lambda$ in addition, while offering more accuracy, is setting a much trickier observational task.

As a starting point, let us take the equations derived in Ref. [9], which are primarily first-order though they include the second-order correction to V_{50}, cf. Eqs. (9–11). In this extreme example, the quadratic Taylor series based upon this does a bad job of approximating the shape of the potential, as it turns upward for large $(\phi - \phi_{50})$ due to the truncation at the $(\phi - \phi_{50})^2$ term (see Fig. 2).

If we now require knowledge of $dn/d\ln\lambda$, the Taylor series approach can be improved in two ways. We can now take V_{50}, V'_{50}, and V''_{50} to second-order; however, the improvement is rather minimal. Alternatively, we can stick to first-order expressions, but include the V'''_{50} cubic term. Again the improvement is modest, though at least the unwanted minimum has been eliminated. One could go further and take V_{50}, V'_{50}, and V''_{50} to second-order and V'''_{50} to lowest order, which we haven't illustrated, again seeing only modest gains for the increased observational requirement.

The Taylor series having been unimpressive, let us progress in a different direction. With only n and $\frac{T}{S}$, as an alternative to the Taylor series one can construct the Padé approximant based upon it, taking V_{50} to second-order and V'_{50} and V''_{50} to first-order. This represents a substantial gain on the Taylor series to that order *without requiring any additional input information.* With this minimal information, it is a much better method. Reintroducing $dn/d\ln\lambda$ allows this method to be extended to second-order, where the reproduction of the shape of the potential is excellent. To include the third derivative term would necessitate a more complicated (non-diagonal) Padé approximant, which doesn't seem warranted at the moment.

What is the upshot of this comparison? Recalling that we have chosen

an example with extreme deviation from scale-invariance, the second-order corrections are reassuringly small and only improve the shape of the reconstructed potential slightly. The addition of the third derivative term in the Taylor series gives a slightly more significant improvement, but at the price of its dependence upon $dn/d\ln\lambda$ even at lowest order. The most remarkable improvement involves the use of Padé approximants. Even without knowledge of $dn/d\ln\lambda$ the shape of the potential is reproduced far better than with the higher-order Taylor series which does require that knowledge. As noted previously, the improvement results from the fact that the Padé approximant is not truncated; further, even in situations where truncation of the Taylor series does not lead to problems, the Padé approximant still proves valuable as its Taylor expansion coincides with that of the original expansion. We therefore conclude that Padé approximants provide a significant improvement in the perturbative reconstruction of the inflationary potential.

4 Discussion

By presenting the second-order reconstruction equations directly in terms of observables, we have been able to assemble and to compare an array of different perturbative reconstruction techniques based upon cosmological observables. Our work extends previous work in several important ways.

The most interesting result is the introduction of the Padé approximant as an alternative to the Taylor series in perturbative reconstruction. It can be obtained from a Taylor series regardless of the order (in the deviation from scale invariance) to which the coefficients of the Taylor series has been obtained. In our worked example, the improvement in reproducing the shape of the potential as compared to the Taylor series is striking, especially considering that no extra observables are required.

We have shown that the second-order corrections to the Taylor series coefficients are generally small, and that those for V_{50} and V'_{50} only depend upon the same quantities as the first-order expressions (S, T, and n). The corrections to V''_{50} however require a new observable such as $dn/d\ln\lambda$, and by deriving for the first time an explicit expression we have confirmed that even the lowest-order term in V'''_{50} requires this challenging observable.

Finally, one of the most important aspects of reconstruction is that it is overdetermined: Any set of cosmological observables supplies degenerate

information regarding the potential and its derivatives, thereby providing an important consistency check. In particular, the tensor spectral index can be expressed to second-order in terms of S, T, and n by the relation: $n_T = -\frac{1}{7}\frac{T}{S}[1 + 0.11\frac{T}{S} + 0.15(n-1)]$. In cases that are observationally viable, the second-order corrections are small.

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Appendix: Some Relations between Notation

For the convenience of the reader, we summarise here some relations between the notation used here and that in Ref. [11], from which several important results were taken. In that paper, the spectra A_S and A_G were defined so as to include any scale-dependence within them, i.e., they are functions of k. In the perturbative reconstruction regime, where for most results the spectra can be approximated by power laws, these are related at lowest-order to the amplitudes A and A_T in this paper, which are just numbers, by

$$A(k/k_{50})^{n-1} = \frac{2\pi^2}{H_0^4} A_S^2(k), \tag{39}$$

$$A_T(k/k_{50})^{n_T} = 2A_G^2(k). \tag{40}$$

In making the connection, note that $8\pi \simeq 25$. For the higher-order terms, the scalar field kinetic energy must be accounted for in translating between H and V, which means these relations break down at higher order.

In Ref. [11], slow-roll parameters ϵ and η are introduced,

$$\epsilon = \frac{m_{Pl}^2}{4\pi}\left(\frac{H'}{H}\right)^2, \qquad \eta = \frac{m_{Pl}^2}{4\pi}\frac{H''}{H}, \tag{41}$$

which are again in general k-dependent. As indicated in Section 2 of the present paper, they can be related to the spectral indices to various orders, ϵ and η being of the same order in perturbation theory as $(n-1)$ and n_T. To lowest-order they are constant, corresponding to power-law spectra. At lowest-order $\epsilon = 16\pi x^2$, but once more higher order corrections break this relation.

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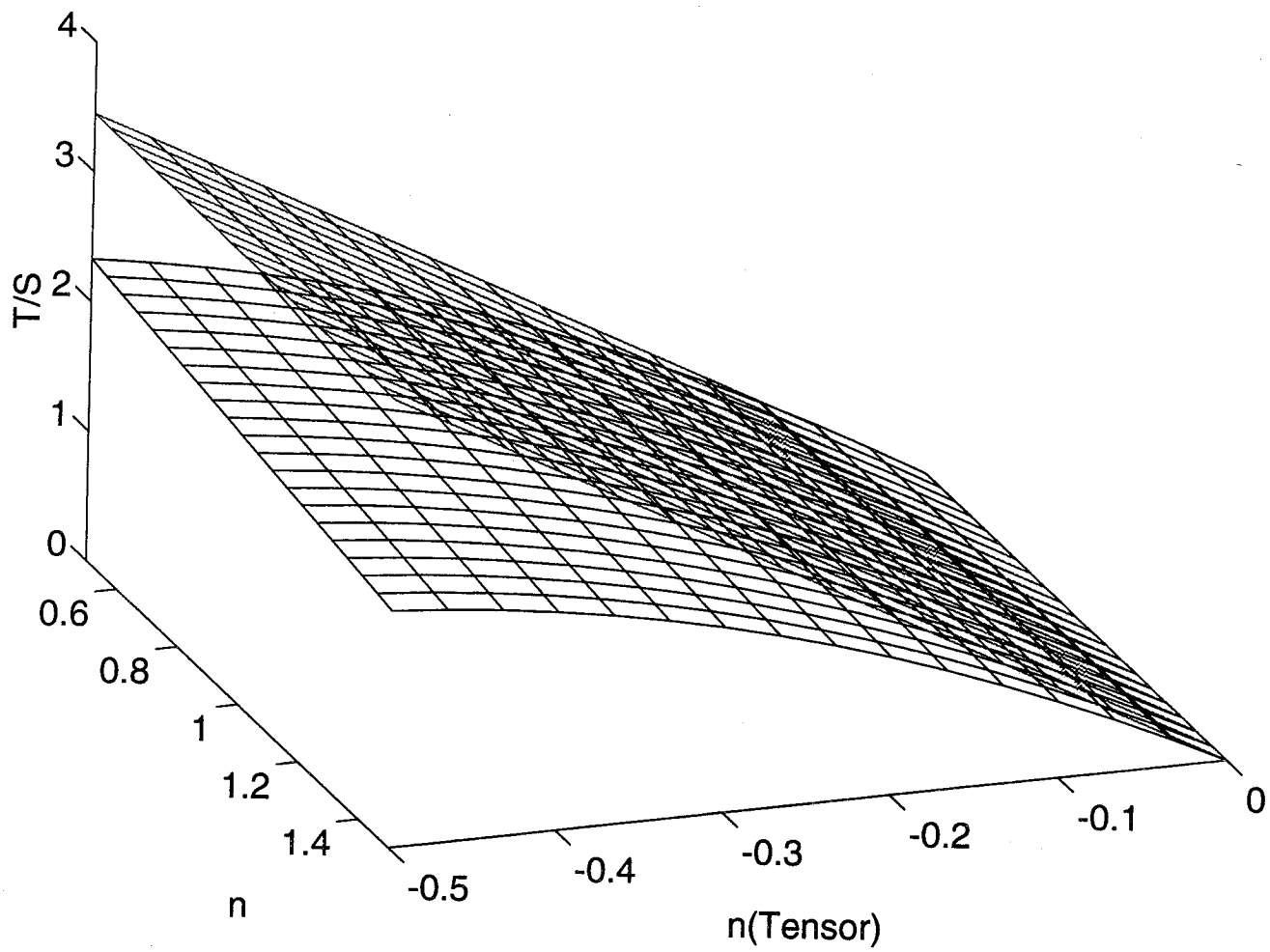
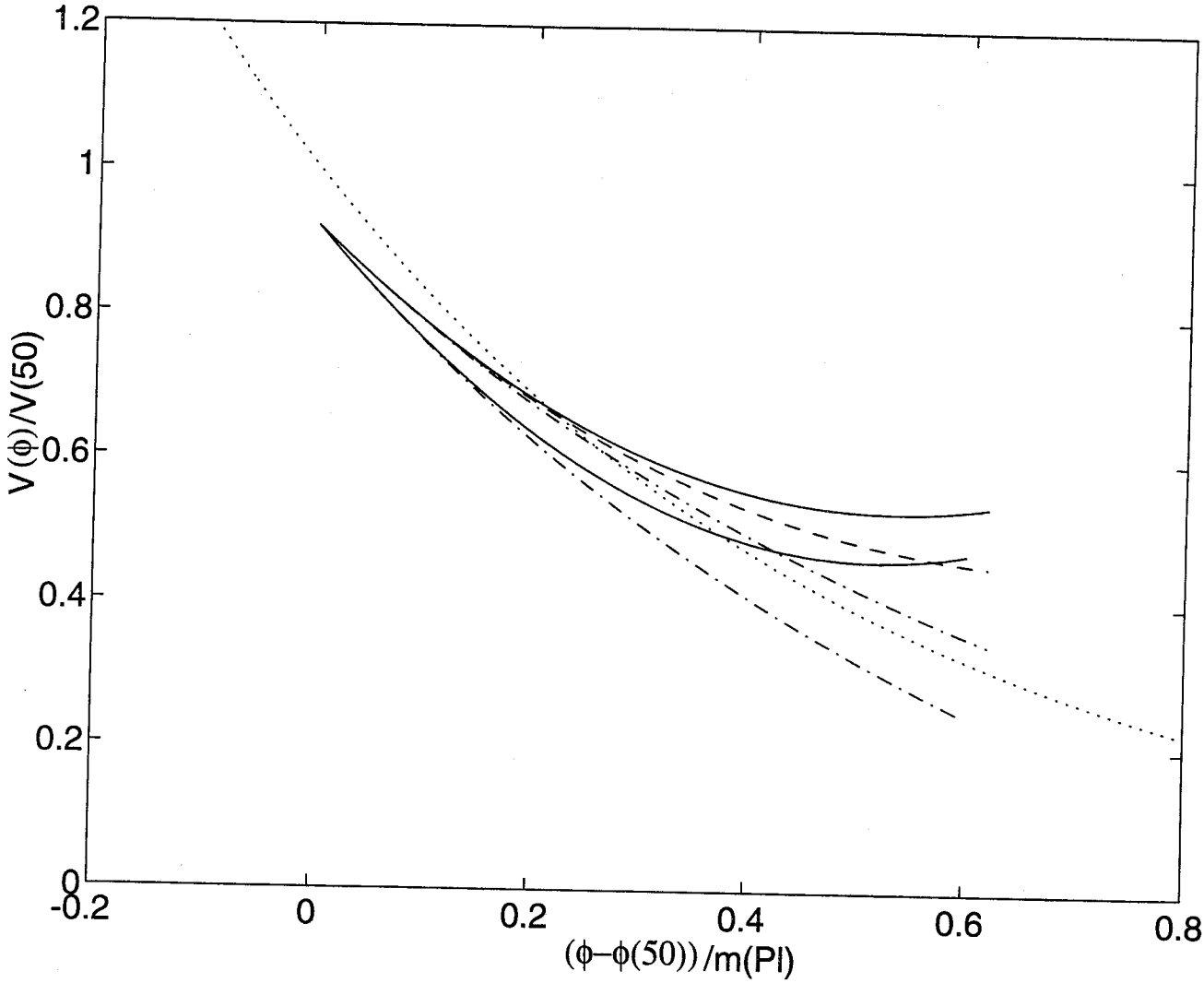
Figure Captions

Figure 1: The consistency plane for inflation in $n - n_T - \frac{T}{S}$ space, the flat surface being the lowest-order result and the curved one incorporating the second-order corrections, given by Eq. (18).

Figure 2: An array of different reconstructions of an exponential potential with $(n-1) = n_T = -0.15$ ($p = 43/3$). The longer dotted line indicates the exact potential. The three different line styles correspond to three different reconstruction strategies; solid is Taylor series truncated at $(\phi - \phi_{50})^2$, dashed is Taylor series truncated at $(\phi - \phi_{50})^3$ and dash-dotted is the Padé approximant based on the former of these. The upper line of a given style uses coefficients to first-order in the deviation from scale invariance (save V_{50}, which is always second-order), while the lower, where plotted, is second-order in all coefficients. The length of the curves corresponds to eight e-foldings.



- FIG 1 -



- FIG 2 -