1. Introduction

The need to solve optimization and control problems arises in many settings. Although in some cases these problems may be easily solved, either analytically or computationally, in many other cases substantial difficulties are encountered. For example, candidate optimal states and controls may belong to infinite dimensional function spaces and one may have to minimize a nonlinear functional of the state and control variables subject to nonlinear constraints that take the form of a system of partial differential equations whose solutions are in general not unique. In this paper, our goal is to construct and analyze a framework for the approximate solution of many such problems. The setting for our framework is a class of nonlinear control or optimization problems which is general enough to be of use in numerous applications. The major steps in the development and analysis of our framework are as follows:

- define an abstract class of nonlinear control or optimization problems;
- show that, under certain assumptions, optimal solutions exist;
- show that, under certain additional assumptions, Lagrange multipliers exist that may be used to enforce the constraints;
- use the Lagrange multiplier technique to derive an optimality system from which optimal states and controls may be deduced;
- define algorithms for the approximation, in finite dimensional spaces, of optimal states and controls; and
- derive estimates for the error in the approximations to the optimal states and controls.

Two of the key ingredients used to carry out the above plan are a theory given in [21] for showing the existence of Lagrange multipliers and a theory first developed in [6] for the approximation of a class of nonlinear problems. In both of these theories, certain properties of compact operators on Banach spaces play a central role. We point out that the nonuniqueness of solutions of the nonlinear constraint equations deems it appropriate to employ Lagrange multiplier principles.

After having developed and analyzed the abstract framework, we will apply it to some specific, concrete problems. In each case, we use the abstract framework to analyze the concrete problems by merely showing that the latter fit into the former. The particular applications we consider are:

- control problems in structural mechanics having geometric nonlinearities that are governed by the von Kármán equations;
- control problems in superconductivity that are governed by the Ginzburg-Landau equations; and
- control problems for incompressible, viscous flows that are governed by the Navier-Stokes equations.

In considering these applications, we will purposely choose different types of controls in order to illustrate how these can be accounted for within the abstract framework. In all three cases, approximation will be effected through the use of finite element methods.

2. The abstract problem and its analysis

In this section we define and analyze an abstract class of constrained nonlinear control problems; an outline of the definitions and results of this section is as follows.

- In §2.1, the abstract class of constrained control problems that we consider is defined.
- In §2.2, a list of assumptions about the class of abstract problems is given.
- In Theorem 2.1 of §2.3, some of the assumptions listed in §2.2 are used to show that optimal solutions of the abstract problem exist.
- In §2.4, some additional assumptions of §2.2 are used to show that Lagrange multipliers exist that may be used to enforce the constraint; also, first-order necessary conditions are given.
- In Theorems 2.5 and 2.6 of §2.4, the first-order necessary conditions for determining optimal states and controls are simplified under additional assumptions about the control set.
- In §2.5, the optimality system from which optimal controls and states can be determined is made more amenable to approximation by simplifying the dependence of the objective functional on the control.

2.1. The abstract setting

We begin with the definition of the abstract class of nonlinear control or optimization problems that we study.

We introduce the spaces and control set as follows. Let G, X, and Y be reflexive Banach spaces whose norms are denoted by $\|\cdot\|_G$, $\|\cdot\|_X$, and $\|\cdot\|_Y$, respectively. Dual spaces will be denoted by $(\cdot)^*$. The duality pairing between X and X^* is denoted by $\langle\cdot,\cdot\rangle_X$; one similarly defines $\langle\cdot,\cdot\rangle_Y$ and $\langle\cdot,\cdot\rangle_G$. The subscripts are often omitted whenever there is no chance for confusion. Let Θ, the control set, be a closed convex subset of G. Let Z be a subspace of Y with a compact imbedding. Note that the compactness of the imbedding $Z \subset Y$ will play an important role.

We assume that the functional to be minimized takes the form

$$\mathcal{J}(v,z) = \lambda\,\mathcal{F}(v) + \lambda\,\mathcal{E}(z) \quad \forall\,(v,z) \in X \times \Theta\,, \tag{2.1}$$

where $\mathcal{F}$ is a functional on X, $\mathcal{E}$ a functional on Θ, and λ is a given parameter which is assumed to belong to a compact interval $\Lambda \subset \mathbb{R}_+$.

The constraint equation $M(v,z) = 0$ relating the state variable v and the control variable z is defined as follow. Let N be a differentiable mapping from X to Y, K a continuous linear operator from Θ to Y, and T a continuous linear operator from Y to X. For any $\lambda \in \Lambda$, we define the mapping M from $X \times \Theta$ to X by

$$M(v,z) = v + \lambda TN(v) + \lambda TK(z) \quad \forall\,(v,z) \in X \times \Theta\,. \tag{2.2}$$

With these definitions we now consider the constrained minimization problem:

$$\min_{(v,z)\in X\times\Theta} \mathcal{J}(v,z) \quad \text{subject to} \quad M(v,z) = 0\,. \tag{2.3}$$

In (2.3), we seek a global minimizer with respect to the set $\{(v,z) \in X \times \Theta : M(v,z) = 0\}$. Although, under suitable hypotheses, we will show that the problem (2.3) has a solution, in practice, one can only characterize local minima, i.e., points $(u,g) \in X \times \Theta$ such that for some $\epsilon > 0$

$$\mathcal{J}(u,g) \leq \mathcal{J}(v,z) \quad \forall\, (v,z) \in X \times \Theta \text{ such that } M(v,z) = 0 \text{ and } \|u - v\|_X \leq \epsilon\,. \tag{2.4}$$

Thus, when we consider algorithms for locating constrained minima of $\mathcal{J}$, we must be content to find local minima in the sense of (2.4).

After showing that optimal solutions exist and that one is justified in using the Lagrange multiplier rule, we will introduce some simplifications in order to render the abstract problem (2.3), or (2.4), more amenable to approximation. The first is to only consider the control set $\Theta = G$. The second is to only consider Fréchet differentiable functionals $\mathcal{E}(\cdot)$ such that the Fréchet derivative $\mathcal{E}'(g) = E^{-1}g$, where E is an invertible linear operator from G^* to G.

2.2. Hypotheses concerning the abstract problem

The first set of hypotheses will be invoked to prove the existence of optimal solutions. It is given by:

(H1) $\inf_{v\in X} \mathcal{F}(v) > -\infty$;

(H2) *there exist constants* α, $\beta > 0$ *such that* $\mathcal{E}(z) \geq \alpha \|z\|^\beta \ \ \forall\, z \in \Theta$;

(H3) *there exists a* $(v,z) \in X \times \Theta$ *satisfying* $M(v,z) = 0$;

(H4) *if* $u^{(n)} \rightharpoonup u$ *in* X *and* $g^{(n)} \rightharpoonup g$ *in* G *where* $\{(u^{(n)}, g^{(n)})\} \subset X \times \Theta$, *then* $N(u^{(n)}) \rightharpoonup N(u)$ *in* Y *and* $K(g^{(n)}) \rightharpoonup K(g)$ *in* Y;

(H5) $\mathcal{J}(\cdot,\cdot)$ *is weakly lower semicontinuous on* $X \times \Theta$; *and*

(H6) *if* $\{(u^{(n)}, g^{(n)})\} \subset X \times \Theta$ *is such that* $\{\mathcal{F}(u^{(n)})\}$ *is a bounded set in* $\mathbb{R}$ *and* $M(u^{(n)}, g^{(n)}) = 0$, *then* $\{u^{(n)}\}$ *is a bounded set in* X.

The second set of assumptions will be used to justify the use of the Lagrange multiplier rule and to derive an optimality system from which optimal states and controls may be determined. The second set is given by:

(H7) *for each* $z \in \Theta$, $v \mapsto \mathcal{J}(v,z)$ *and* $v \mapsto M(v,z)$ *are Fréchet differentiable*;

(H8) $z \mapsto \mathcal{E}(z)$ *is convex, i.e.,*

$$\mathcal{E}\big(\gamma z_1 + (1-\gamma) z_2\big) \leq \gamma\, \mathcal{E}(z_1) + (1-\gamma)\, \mathcal{E}(z_2) \quad \forall\, z_1, z_2 \in \Theta\,,\ \forall\, \gamma \in [0,1]\,;$$

and

(H9) *for* $v \in X$, $N'(v)$ *maps* X *into* Z.

In (H9), N' denotes the Fréchet derivative of N.

A simplified optimality system may be obtained if one invokes the additional assumption:

(H10) $\Theta = G$, *and the mapping* $z \mapsto \mathcal{E}(z)$ *is Fréchet differentiable on* G.

Hypotheses (H7)-(H10) allow us to obtain a simplified optimality system for almost all values of the parameter $\lambda \in \Lambda$. In many cases, it is possible to show that the same optimality system holds for all values of λ. The following two additional assumptions which will only be invoked in case $(1/\lambda)$ is an eigenvalue of $-TN'(u)$ each provides a setting in which this last result is valid:

(H11) *if* $v^* \in X^*$ *satisfies* $\big(I + \lambda\,[N'(u)]^* T^*\big) v^* = 0$ *and* $K^* T^* v^* = 0$*, then* $v^* = 0$ *; or*

(H11)$'$ *the mapping* $(v, z) \mapsto v + \lambda TN'(u)v + \lambda TKz$ *is onto from* $X \times G$ *to* Y*.*

In order to make the optimality system more amenable to approximation and computation, we will invoke the following additional assumption:

(H12) $\mathcal{E}'(g) = E^{-1} g$ *, where* E *is an invertible linear operator from* G^* *to* G *and* g *is an optimal control for the constrained minimization problem* (2.4)*.*

2.3. Existence of an optimal solution

We first use assumptions (H1)-(H6) to establish that optimal solutions exist.

THEOREM 2.1. *Assume that the functional* $\mathcal{J}$ *and mapping* M *defined by* (2.1) *and* (2.2)*, respectively, satisfy the hypotheses* (H1)-(H6)*. Then, there exists a solution to the minimization problem* (2.3)*.*

Proof: Assumption (H3) simply asserts that there is at least one element of $X \times \Theta$ that satisfies the constraint. Thus, we may choose a minimizing sequence $\{(u^{(n)}, g^{(n)})\} \subset X \times \Theta$ such that

$$\lim_{n\to\infty} \mathcal{J}(u^{(n)}, g^{(n)}) = \inf_{(v,z)\in X\times\Theta} \mathcal{J}(v, z)$$

and

$$M(u^{(n)}, g^{(n)}) = 0\,.$$

By (H1) and (H2), the boundedness of $\{\mathcal{J}(u^{(n)}, g^{(n)})\}$ implies the boundedness of the sequences $\{\|g^{(n)}\|_G\}$ and $\{\mathcal{F}(u^{(n)})\}$. Then, by (H6), we deduce that $\{\|u^{(n)}\|_X\}$ is bounded. Thus, we may extract a subsequence $\{(u^{(n)}, g^{(n)})\}$ such that $u^{(n)} \rightharpoonup u$ in X and $g^{(n)} \rightharpoonup g$ in G. Since Θ is closed and convex, we have $g \in \Theta$. Of course, $u \in X$. We next show that (u, g) satisfies the constraint equation. Using (H4), we have that

$$\lim_{n\to\infty} \langle TN(u^{(n)}), f\rangle = \lim_{n\to\infty} \langle N(u^{(n)}), T^* f\rangle = \langle N(u), T^* f\rangle = \langle TN(u), f\rangle \quad \forall\, f \in X^*$$

and

$$\lim_{n\to\infty} \langle TK(g^{(n)}), f\rangle = \lim_{n\to\infty} \langle K(g^{(n)}), T^* f\rangle = \langle K(g), T^* f\rangle = \langle TK(g), f\rangle \quad \forall\, f \in X^*$$

so that

$$0 = \lim_{n\to\infty} \langle M(u^{(n)}, g^{(n)}), f\rangle = \langle u + \lambda TN(u) + \lambda TK(g), f\rangle \quad \forall\, f \in X^*\,,$$

i.e., $M(u, g) = 0$. Finally, we use (H5), the weak lower semicontinuity of $\mathcal{J}(\cdot,\cdot)$, to conclude that (u, g) is indeed a minimizer in $X \times \Theta$ satisfying the constraint $M(u, g) = 0$. □

2.4. Existence of Lagrange multipliers

We now wish to use the additional assumptions (H7)-(H9) to show that the Lagrange multiplier rule may be used to turn the constrained minimization problem

(2.3) into an unconstrained one. Actually, the Lagrange multiplier rule will only enable us to find local minima in the sense of (2.4). We first quote the following abstract Lagrange multiplier rule whose proof can be found in [21].

THEOREM 2.2. *Let X_1 and X_2 be two Banach spaces and Θ an arbitrary set. Suppose $\mathcal{J}$ is a functional on $X_1 \times \Theta$ and M a mapping from $X_1 \times \Theta$ to X_2. Assume that $(u,g) \in X_1 \times \Theta$ is a solution to the following constrained minimization problem:*

$$\begin{aligned} &M(u,g) = 0 \text{ and there exists an } \epsilon > 0 \text{ such that } \mathcal{J}(u,g) \le \mathcal{J}(v,z) \\ &\text{for all } (v,z) \in X_1 \times \Theta \text{ such that } \|u - v\|_{X_1} \le \epsilon \text{ and } M(v,z) = 0\,. \end{aligned} \tag{2.5}$$

Let U be an open neighborhood of u in X_1. Assume further that the following conditions are satisfied:

(2.6) *for each $z \in \Theta$, $v \mapsto \mathcal{J}(v,z)$ and $v \mapsto M(v,z)$ are Fréchet-differentiable at $v = u$;*

(2.7) *for any $v \in U$, $z_1, z_2 \in \Theta$, and $\gamma \in [0,1]$, there exists a $z_\gamma = z_\gamma(v, z_1, z_2)$ such that*

$$M(v, z_\gamma) = \gamma M(v, z_1) + (1-\gamma) M(v, z_2)$$

and

$$\mathcal{J}(v, z_\gamma) \le \gamma \mathcal{J}(v, z_1) + (1-\gamma)\mathcal{J}(v, z_2)\,;$$

and

(2.8) $\text{Range}\big(M_u(u,g)\big)$ *is closed with a finite codimension,*

where $M_u(u,g)$ denotes the Fréchet derivative of M with respect to u. Then, there exists a $k \in \mathbb{R}$ and a $\mu \in X_2^*$ that are not both equal to zero such that

$$k\,\langle \mathcal{J}_u(u,g), v\rangle - \langle \mu, M_u(u,g)v\rangle = 0 \quad \forall\, v \in X_1$$

and

$$\min_{z \in \Theta} \mathcal{L}(u,z,\mu,k) = \mathcal{L}(u,g,\mu,k)\,,$$

where $\mathcal{L}(u,g,\mu,k) = k\,\mathcal{J}(u,g) - \langle \mu, M(u,g)\rangle$ is the Lagrangian for the constrained minimization problem (2.5) and where $\mathcal{J}_u(u,g)$ denotes the Fréchet derivative of $\mathcal{J}$ with respect to u. Moreover, if

(2.9) *the algebraic sum $M_u(u,g)X_1 + M(u,\Theta)$ contains $0 \in X_2$ as an interior point,*

then we may choose $k = 1$, i.e., there exists a $\mu \in X_2^*$ such that

$$\langle \mathcal{J}_u(u,g), v\rangle - \langle \mu, M_u(u,g)v\rangle = 0 \quad \forall\, v \in X_1$$

and

$$\min_{z \in \Theta} \mathcal{L}(u,z,\mu,1) = \mathcal{L}(u,g,\mu,1)\,.$$

Proof: See [21]. □

Next, we apply Theorem 2.2 to the optimization problem (2.4). In doing so, we will need the following result.

LEMMA 2.3. *Let the spaces X, Y, and Z and operators T and N be defined as in §2.1. For $v \in X$, assume that $N'(v)$ maps X into Z. Then, $TN'(v)$ is a compact operator* *from X to X, and therefore $\sigma\big(-TN'(v)\big)$, the spectrum of the operator $\big(-TN'(v)\big)$, is at most countable with zero being the only possible limit point.*

Proof: Since $Z \hookrightarrow\hookrightarrow Y$, we see that $N'(v)$ is a compact linear operator from X to Y. Also, T is a bounded linear operator from Y to X so that $TN'(v)$ is a compact operator from X to X. Hence, $\sigma\big(-TN'(v)\big)$ is at most countable and consists only of 0 and the eigenvalues of $\big(-TN'(v)\big)$. $\square$

Note that in the following result, the existence of at least one pair (u, g) satisfying (2.4) is guaranteed by Theorem 2.1.

THEOREM 2.4. *Let $\lambda \in \Lambda$ be given. Assume that assumptions* (H1)-(H9) *hold. Let $(u, g) \in X \times \Theta$ be an optimal solution satisfying* (2.4). *Then, there exists a $k \in \mathbb{R}$ and a $\mu \in X^*$ that are not both equal to zero such that*

$$k\,\langle \mathcal{J}_u(u,g), w\rangle - \langle \mu, M_u(u,g)\cdot w\rangle = 0 \quad \forall\, w \in X \tag{2.10}$$

and

$$\min_{z\in\Theta} \mathcal{L}(u,z,\mu,k) = \mathcal{L}(u,g,\mu,k)\,. \tag{2.11}$$

Furthermore, if $(1/\lambda) \notin \sigma\big(-TN'(u)\big)$, we may choose $k = 1$, i.e., for almost all λ, there exists a $\mu \in X^*$ such that

$$\langle \mathcal{J}_u(u,g), w\rangle - \langle \mu, M_u(\lambda,u,g)\cdot w\rangle = 0 \quad \forall\, w \in X \tag{2.12}$$

and

$$\min_{z\in\Theta} \mathcal{L}(u,z,\mu,1) = \mathcal{L}(u,g,\mu,1)\,. \tag{2.13}$$

Proof: Let $\lambda \in \Lambda$ be given. To show the existence of k and μ such that (2.10) and (2.11) are valid, we only need to verify that the hypotheses (2.6)-(2.8) of Theorem 2.2 hold with $X_1 = X_2 = X$, since in this case (2.5) reduces to (2.4). Obviously, (2.6) is merely a restatement of (H7). Since Θ is convex and since the mappings T and K are linear, we have that if $z_\gamma = \gamma z_1 + (1-\gamma)z_2$, then

$$\begin{aligned} M(v,z_\gamma) &= v + \lambda TN(v) + \lambda TKz_\gamma \\ &= \gamma\big(v + \lambda TN(v)\big) + (1-\gamma)\big(v+\lambda TN(v)\big) + \gamma\lambda\big(TKz_1 + (1-\gamma)TKz_2\big)\big) \\ &= \gamma M(v,z_1) + (1-\gamma)M(v,z_2)\,. \end{aligned}$$

Moreover, (H8) implies that

$$\begin{aligned} \mathcal{J}(v,z_\gamma) &= \lambda\,\mathcal{F}(v) + \lambda\,\mathcal{E}(z_\gamma) = \lambda\,\mathcal{F}(v) + \lambda\,\mathcal{E}\big(\gamma z_1 + (1-\gamma)z_2\big) \\ &\le \lambda\,\mathcal{F}(v) + \lambda\left(\gamma\mathcal{E}(z_1) + (1-\gamma)\,\mathcal{E}(z_2)\right) = \gamma\,\mathcal{J}(v,z_1) + (1-\gamma)\,\mathcal{J}(v,z_2)\,. \end{aligned}$$

Thus, (2.7) holds. The operator $M_u(u,g)$ from X to X is defined by

$$M_u(u,g)\cdot w = w + \lambda TN'(u)\cdot w \quad \forall\, w \in X$$

or simply,

$$M_u(u,g) = I + \lambda TN'(u)\,.$$

From (H9) and Lemma 2.3, we have that $TN'(u)$ is a compact operator from X to X. As a result, $M_u(u,g) = I + \lambda TN'(u)$ is a Fredholm operator so that it has a closed range with a finite codimension, i.e., (2.8) holds. Thus, by Theorem 2.2, there exists a $k \in \mathbb{R}$ and a $\mu \in X^*$ which are not both equal to zero such that (2.10) and (2.11) hold.

To show the existence of a μ such that (2.12) and (2.13) are valid, we only need to verify that the additional hypothesis (2.9) of Theorem 2.2 holds. In fact, if, in addition $(1/\lambda) \notin \sigma\big(-TN'(u)\big)$, then it follows that $X = \text{Range}\big(I + \lambda TN'(u)\big) = \text{Range}\big(M_u(u,g)\big)$ so that $\text{Range}\big(M_u(u,g)\big)$ contains $0 \in X$ as an interior point, i.e., (2.9) holds. Hence, by Theorem 2.2 and Lemma 2.3, we conclude that for almost all λ, there exists a $\mu \in X^*$ such that (2.12) and (2.13) hold. □

So far Θ has only been assumed to be a closed and convex subset of G. No smoothness condition on the control variable g has been assumed in the functional or in the constraint. Thus, the necessary condition of optimality with respect to variations in the control variable is expressed in the cumbersome relation (2.11). We now turn to the case where Θ contains a neighborhood of g, where (u,g) is an optimal solution. In particular, we assume that $\Theta = G$. In this case, (2.11) can be given a more concrete structure.

THEOREM 2.5. *Let $\lambda \in \Lambda$ be given. Assume that assumptions* (H1)-(H10) *hold. Let $(u,g) \in X \times G$ be a solution of the problem* (2.4). *Then, there exists a $k \in \mathbb{R}$ and a $\mu \in X^*$ that are not both equal to zero such that*

$$k\,\langle \mathcal{J}_u(u,g), w\rangle - \langle \mu, \big(I + \lambda TN'(u)\big)w\rangle = 0 \quad \forall\, w \in X \tag{2.14}$$

and

$$k\langle \mathcal{E}'(g), z\rangle - \langle \mu, TKz\rangle = 0 \quad \forall\, z \in G\,. \tag{2.15}$$

Furthermore, if $(1/\lambda) \notin \sigma\big(-TN'(u)\big)$, we may choose $k = 1$, i.e., there exists a $\mu \in X^*$ such that

$$\langle \mathcal{J}_u(u,g), w\rangle - \langle \mu, \big(I + \lambda TN'(u)\big)w\rangle = 0 \quad \forall\, w \in X \tag{2.16}$$

and

$$\langle \mathcal{E}'(g), z\rangle - \langle \mu, TKz\rangle = 0 \quad \forall\, z \in G\,. \tag{2.17}$$

hold.

Proof: Since the hypotheses imply that $\mathcal{J}(v,z)$ is Fréchet differentiable with respect to z, (2.14)-(2.17) follow easily from Theorem 2.4. □

Remark. If $k = 0$, then there exists a $\mu \neq 0$ such that

$$-\langle \mu, M_u(u,g)w\rangle = 0 \quad \forall\, w \in X$$

so that the optimality system necessarily has infinitely many solutions. In fact, for any $C \in \mathbb{R}$, $(C\mu)$ is a solution whenever μ is a solution. This creates both theoretical and numerical difficulties. Thus, it is of great interest to try to eliminate this situation. Fortunately, Lemma 2.3 and Theorem 2.4 tell us that we may set $k = 1 \neq 0$ for almost all values of $(1/\lambda)$, i.e., except for the at most countable set of values in $\sigma\big(-TN'(u)\big)$. □

If the control g enters the constraint in a favorable manner, then we may take $k = 1$ even when $(1/\lambda) \in \sigma\big(-TN'(u)\big)$. Specifically, we invoke one of the assumptions (H11) and (H11)$'$. We then have the following result.

THEOREM 2.6. *Assume that the hypotheses of Theorem 2.5 hold. Assume that if $(1/\lambda) \in \sigma\big(-TN'(u)\big)$, then either (H11) or (H11)$'$ holds. Then, for all $\lambda \in \Lambda$, there exists a $\mu \in X^*$ such that (2.16) and (2.17) hold.*

Proof: Because of Theorem 2.5, we only need to examine the case $(1/\lambda) \in \sigma\big(-TN'(u)\big)$ and show that the algebraic sum $M_u(u,g)X + M(u,G) = X$. If (H11)$'$ holds, the result is a direct application of Theorem 2.2.

If (H11) holds, let $(1/\lambda)$ be a nonzero eigenvalue of $\big(-TN'(u)\big)$. Then, λ is also an eigenvalue of $\big(-N'(u)^*T^*\big)$ with a finite dimensional eigenspace having the corresponding eigenfunctions $\{v_i^*\}_{i=1}^m \subset X^*$ as a basis. We claim that $\{K^*T^*v_i^*\}_{i=1}^m \subset G^*$ is a linearly independent set. To see this, we assume $\sum_{i=1}^m \alpha_i\, K^*T^*v_i^* = 0$ with $\alpha_i \in \mathbb{R}$; this expression can be rewritten as $K^*T^*\big(\sum_{i=1}^m \alpha_i\, v_i^*\big) = 0$. Because each v_i^* is an eigenvector, we have $\big(I + \lambda\, N'(u)^*T^*\big)\sum_{i=1}^m \alpha_i\, v_i^* = 0$. Thus, the assumption (H11) implies that $\sum_{i=1}^m \alpha_i\, v_i^* = 0$. Since $\{v_i^*\}_{i=1}^m$ is an eigenbasis, and is therefore a linearly independent set, we have each $\alpha_i = 0$. This shows that $\{K^*T^*v_i^*\}_{i=1}^m$ is linearly independent set in G^*. Hence, we may choose an orthonormal dual basis $\{z_i\}_{i=1}^m \subset G$ such that $\langle z_i, K^*T^*v_j^*\rangle = \delta_{ij}$.

Now, let $w \in X$ be given. We choose $z = \frac{1}{\lambda}\sum_{i=1}^m \langle w, v_i^*\rangle z_i$. Then $\langle w, v_j^*\rangle - \lambda\langle TKz, v_j^*\rangle = \langle w, v_j^*\rangle - \lambda\langle z, K^*T^*v_j^*\rangle = \langle w, v_j^*\rangle - \sum_{i=1}^m \langle w, v_i^*\rangle\delta_{ij} = 0$ for $j = 1,\ldots,m$. Thus, by Fredholm alternatives, there exists a unique $v \in X$ that satisfies $\big(I + \lambda TN'(u)\big)v = w - \lambda TKz$, or, $\big(I + \lambda TN'(u)\big)v + \lambda TKz = w$; thus, we have shown that $M_u(u,g)X + M(u,G) = X$. Hence, by Theorem 2.2, there exists a $\mu \in X^*$ such that (2.16) and (2.17) hold. □

2.5. The optimality system

Under the assumptions of Theorem 2.6, an optimal state $u \in X$, an optimal control $g \in G$, and the corresponding Lagrange multiplier $\mu \in X^*$ satisfy the optimality system of equations formed by (2.2), (2.16), and (2.17). From (2.1) we have that $\mathcal{J}_u = \lambda\mathcal{F}'$ and $\mathcal{J}_g = \lambda\mathcal{E}'$, where $\mathcal{F}'$ denotes the obvious Fréchet derivative. Then, (2.16)-(2.17) may be rewritten in the form

$$\mu + \lambda\,[N'(u)]^*T^*\mu - \lambda\,\mathcal{F}'(u) = 0 \quad \text{in } X^* \tag{2.18}$$

and

$$\mathcal{E}'(g) - K^*T^*\mu = 0 \quad \text{in } G^*\,. \tag{2.19}$$

For purposes of numerical approximations, it turns out to be convenient to make the change of variable $\xi = T^*\mu$. Then, the optimality system (2.2), (2.18), and (2.19) for $u \in X$, $g \in G$, and $\xi \in Y^*$ takes the form

$$u + \lambda TN(u) + \lambda TKg = 0 \quad \text{in } X\,, \tag{2.20}$$

$$\xi + \lambda T^*[N'(u)]^*\xi - \lambda T^*\mathcal{F}'(u) = 0 \quad \text{in } Y^*\,, \tag{2.21}$$

and

$$\mathcal{E}'(g) - K^*\xi = 0 \quad \text{in } G^*\,. \tag{2.22}$$

It will also be convenient to invoke an additional simplifying assumption concerning the dependence of the objective functional on the control. Specifically, we assume that (H12) holds. Then, (2.20)-(2.22) can be rewritten as

$$u + \lambda T N(u) + \lambda T K g = 0 \quad \text{in } X\,, \tag{2.23}$$

$$\xi + \lambda T^*[N'(u)]^*\xi - \lambda T^* \mathcal{F}'(u) = 0 \quad \text{in } Y^*\,, \tag{2.24}$$

and

$$g - E K^* \xi = 0 \quad \text{in } G\,. \tag{2.25}$$

Remark. Note that the optimality systems, e.g., (2.23)-(2.25), are linear in the adjoint variable ξ. Also, note that the control g may be eliminated from the optimality system (2.23)-(2.25). Indeed, the substitution of (2.25) into (2.23) yields

$$u + \lambda T N(u) + \lambda T K E K^* \xi = 0 \quad \text{in } X\,. \tag{2.26}$$

Thus, (2.24) and (2.26) determine the optimal state u and adjoint state ξ; subsequently, (2.25) may be used to determine the optimal control g from ξ. This observation serves to emphasize the important, direct role that the adjoint state plays in the determination of the optimal control. □

Remark. Given a $\xi \in Y^*$, it is not always possible to evaluate g exactly from (2.25). For example, the application of the operator E may involve the solution of a partial differential equation. Thus, although it is often convenient to devise algorithms for the approximation of optimal control and states based on the simplified optimality system (2.24) and (2.26), in some other cases it is best to deal with the full form (2.23)-(2.25). Thus, when we consider approximations of optimal controls and states, we will deal with the latter. □

Remark. In many applications we have that $X^* = Y$. Since these spaces are assumed to be reflexive, we also have that $Y^* = X$. In this case, we have that both u and ξ belong to X. □

3. Finite dimensional approximations of the abstract problem

In this section we define and analyze algorithms for the finite dimensional approximation of solutions of the optimality system (2.23)-(2.25); an outline of the definitions and results of this section is as follows.

- In §3.1, we define the finite dimensional approximate problems that we consider.
- In §3.2, a list of assumptions about the approximate problems is given.
- In §3.3, we quote a result of [6] that we will use to analyze approximations obtained as solutions of the approximate problems defined in §3.1-3.2.
- In §3.4, we provide error estimates for the approximation of solutions of the optimality system (2.23)-(2.25)

3.1. Formulation of finite dimensional approximate problems

A finite dimensional discretization of the optimality system (2.23)-(2.25) is defined as follows. First, one chooses families of finite dimensional subspaces $X^h \subset X$, $(Y^*)^h \subset Y^*$, and $G^h \subset G$. These families are parameterized by a parameter h that tends to zero. (For example, this parameter can be chosen to be some measure of the grid size in a subdivision of Ω into finite elements.) Next, we define approximate operators $T^h : Y \to X^h$, $E^h : G^* \to G^h$, and $(T^*)^h : X^* \to (Y^*)^h$. Of course, one views T^h, E^h, and $(T^*)^h$ as approximations to the operators T, E, and T^*, respectively. Note that $(T^*)^h$ is not necessarily the same as $(T^h)^*$. The former is a discretization of an adjoint operator while the later is the adjoint of a discrete operator.

Once the approximating subspaces and operators have been chosen, an approximate problem is defined as follows. We seek $u^h \in X^h$, $g^h \in G^h$, and $\xi^h \in (Y^*)^h$ such that

$$u^h + \lambda T^h N(u^h) + \lambda T^h K g^h = 0 \quad \text{in } X^h\,, \tag{3.1}$$

$$\xi^h + \lambda (T^*)^h [N'(u^h)]^* \xi^h - \lambda (T^*)^h \mathcal{F}'(u^h) = 0 \quad \text{in } (Y^*)^h\,, \tag{3.2}$$

and

$$g^h - E^h K^* \xi^h = 0 \quad \text{in } G^h\,. \tag{3.3}$$

3.2. Hypotheses concerning the abstract problem and the approximate problem

We make the following hypotheses concerning the approximate operators T^h, $(T^*)^h$, and E^h:

(H13) $$\lim_{h\to 0} \|(T - T^h)y\|_X = 0 \quad \forall\, y \in Y\,,$$

(H14) $$\lim_{h\to 0} \|(T^* - (T^*)^h)v\|_{Y^*} = 0 \quad \forall\, v \in X^*\,,$$

and

(H15) $$\lim_{h\to 0} \|(E - E^h)s\|_G = 0 \quad \forall\, s \in G^*\,.$$

We also need the following additional hypotheses on the operators appearing in the definition of the abstract problem (2.4):

(H16) *$N \in C^3(X;Y)$ and $\mathcal{F} \in C^3(X;\mathbb{R})$;*

(H17) *N'', N''', $\mathcal{F}''$, and $\mathcal{F}'''$ are locally bounded, i.e., they map bounded sets to bounded sets;*

(H18) *for $v \in X$, in addition to* (H9), *i.e., $N'(v) \in \mathcal{L}(X;Z)$ where $Z \hookrightarrow\hookrightarrow Y$, we have that $[N'(v)]^* \in \mathcal{L}(Y^*;\hat{Z})$ where $\hat{Z} \hookrightarrow\hookrightarrow X^*$, that for $\eta \in Y^*$, $[N''(v)]^* \cdot \eta \in \mathcal{L}(Y^*;\hat{Z})$, and that for $w \in X$, $\mathcal{F}''(v) \cdot w \in \mathcal{L}(X;\hat{Z})$; and*

(H19) *K maps G into Z.*

Here, $(\cdot)''$ and $(\cdot)'''$ denote second and third Fréchet derivatives, respectively.

3.3. Quotation of results concerning the approximation of a class of nonlinear problems

The error estimate to be derived in Section 3.4 makes use of results of [6] and [10] (see also [13]) concerning the approximation of a class of nonlinear problems. These results imply that, under certain hypotheses, the error of approximation of solutions of certain nonlinear problems is basically the same as the error of approximation of solutions of related linear problems. Here, for the sake of completeness, we will state the relevant results, specialized to our needs.

The nonlinear problems considered in [6], [10], and [13] are of the following type. For given $\lambda \in \Lambda$, we seek $\psi \in \mathcal{X}$ such that

$$\mathcal{H}(\lambda, \psi) \equiv \psi + \mathcal{T}\mathcal{G}(\lambda, \psi) = 0\,, \tag{3.4}$$

where $\mathcal{T} \in \mathcal{L}(\mathcal{Y};\mathcal{X})$, $\mathcal{G}$ is a C^2 mapping from $\Lambda \times \mathcal{X}$ into $\mathcal{Y}$, $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces, and Λ is a compact interval of $\mathbb{R}$. We say that $\{(\lambda, \psi(\lambda)) : \lambda \in \Lambda\}$ is a *branch of solutions* of (3.4) if $\lambda \to \psi(\lambda)$ is a continuous function from Λ into $\mathcal{X}$ such that $\mathcal{H}(\lambda, \psi(\lambda)) = 0$. The branch is called a *regular branch* if we also have that $\mathcal{H}_\psi(\lambda, \psi(\lambda))$ is an isomorphism from $\mathcal{X}$ into $\mathcal{X}$ for all $\lambda \in \Lambda$. Here, $\mathcal{H}_\psi(\cdot,\cdot)$ denotes the Fréchet derivative of $\mathcal{H}(\cdot,\cdot)$ with respect to the second argument. We assume that there exists another Banach space $\mathcal{Z}$, contained in $\mathcal{Y}$, with continuous imbedding, such that

$$\mathcal{G}_\psi(\lambda, \psi) \in \mathcal{L}(\mathcal{X};\mathcal{Z}) \quad \forall\, \lambda \in \Lambda \text{ and } \psi \in \mathcal{X}\,, \tag{3.5}$$

where $\mathcal{G}_\psi(\cdot,\cdot)$ denotes the Fréchet derivative of $\mathcal{G}(\cdot,\cdot)$ with respect to the second argument.

Approximations are defined by introducing a subspace $\mathcal{X}^h \subset \mathcal{X}$ and an approximating operator $\mathcal{T}^h \in \mathcal{L}(\mathcal{Y};\mathcal{X}^h)$. Then, given $\lambda \in \Lambda$, we seek $\psi^h \in \mathcal{X}^h$ such that

$$\mathcal{H}^h(\lambda, \psi^h) \equiv \psi^h + \mathcal{T}^h\mathcal{G}(\lambda, \psi^h) = 0\,. \tag{3.6}$$

Concerning the operator $\mathcal{T}^h$, we assume the approximation properties

$$\lim_{h\to 0} \|(\mathcal{T}^h - \mathcal{T})\omega\|_{\mathcal{X}} = 0 \quad \forall\, \omega \in \mathcal{Y} \tag{3.7}$$

and

$$\lim_{h\to 0} \|(\mathcal{T}^h - \mathcal{T})\|_{\mathcal{L}(\mathcal{Z};\mathcal{X})} = 0\,. \tag{3.8}$$

Note that whenever the imbedding $\mathcal{Z} \subset \mathcal{Y}$ is compact, (3.8) follows from (3.7) and, moreover, (3.5) implies that the operator $\mathcal{T}\mathcal{G}_\psi(\lambda, \psi) \in \mathcal{L}(\mathcal{X};\mathcal{X})$ is compact.

We can now state the result of [6] or [10] that will be used in the sequel. In the statement of the theorem, $D^2\mathcal{G}$ represents any and all second Fréchet derivatives of $\mathcal{G}$.

THEOREM 3.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and Λ a compact subset of* $\mathbb{R}$. Assume that $\mathcal{G}$ is a C^2 mapping from $\Lambda \times \mathcal{X}$ into $\mathcal{Y}$ and that $D^2\mathcal{G}$ is bounded on all bounded sets of $\Lambda \times \mathcal{X}$. Assume that (3.5), (3.7), and (3.8) hold and that $\{(\lambda, \psi(\lambda)); \lambda \in \Lambda\}$ is a branch of regular solutions of (3.4). Then, there exists a neighborhood $\mathcal{O}$ of the origin in $\mathcal{X}$ and, for $h \le h_0$ small enough, a unique C^2 function $\lambda \to \psi^h(\lambda) \in \mathcal{X}^h$ such that $\{(\lambda, \psi^h(\lambda)); \lambda \in \Lambda\}$ is a branch of regular solutions of (3.6) and $\psi^h(\lambda) - \psi(\lambda) \in \mathcal{O}$ for all $\lambda \in \Lambda$. Moreover, there exists a constant $C > 0$, independent of h and λ, such that

$$\|\psi^h(\lambda) - \psi(\lambda)\|_{\mathcal{X}} \le C\|(\mathcal{T}^h - \mathcal{T})\mathcal{G}(\lambda, \psi(\lambda))\|_{\mathcal{X}} \quad \forall\, \lambda \in \Lambda\,. \qquad (3.9)$$ □

3.4. Error estimates for the approximation of solutions of the optimality system

We now apply the result of Theorem 3.1 to study the approximation of solutions of the optimality system. Set $\mathcal{X} = X \times G \times Y^*$, $\mathcal{Y} = Y \times X^*$, $\mathcal{Z} = Z \times \hat{Z}$, and $\mathcal{X}^h = X^h \times G^h \times (Y^*)^h$. (Recall that $\hat{Z}$ was introduced in (H18).) By the hypotheses on Z and $\hat{Z}$, we have that $\mathcal{Z}$ is compactly imbedded into $\mathcal{Y}$. Let $\mathcal{T} \in \mathcal{L}(\mathcal{Y};\mathcal{X})$ be defined in the following manner: $\mathcal{T}(\tilde{r}, \tilde{\tau}) = (\tilde{u}, \tilde{g}, \tilde{\xi})$ for $(\tilde{r}, \tilde{\tau}) \in \mathcal{Y}$ and $(\tilde{u}, \tilde{g}, \tilde{\xi}) \in \mathcal{X}$ if and only if

$$\tilde{u} + T\tilde{r} = 0\,, \qquad (3.10)$$

$$\tilde{\xi} + T^*\tilde{\tau} = 0\,, \qquad (3.11)$$

and

$$\tilde{g} - EK^*\tilde{\xi} = 0\,. \qquad (3.12)$$

Similarly, the operator $\mathcal{T}^h \in \mathcal{L}(\mathcal{Y};\mathcal{X}^h)$ is defined as follows: $\mathcal{T}^h(\tilde{r}, \tilde{\tau}) = (\tilde{u}^h, \tilde{g}^h, \tilde{\xi}^h)$ for $(\tilde{r}, \tilde{\tau}) \in \mathcal{Y}$ and $(\tilde{u}^h, \tilde{g}^h, \tilde{\xi}^h) \in \mathcal{X}^h$ if and only if

$$\tilde{u}^h + T^h\tilde{r} = 0\,, \qquad (3.13)$$

$$\tilde{\xi}^h + (T^*)^h\tilde{\tau} = 0\,, \qquad (3.14)$$

and

$$\tilde{g}^h - E^h K^*\tilde{\xi}^h = 0\,. \qquad (3.15)$$

The nonlinear mapping $\mathcal{G} : \Lambda \times \mathcal{X} \to \mathcal{Y}$ is defined as follows: $\mathcal{G}\big(\lambda, (\tilde{u}, \tilde{g}, \tilde{\xi})\big) = (\tilde{r}, \tilde{\tau})$ for $\lambda \in \Lambda$, $(\tilde{u}, \tilde{g}, \tilde{\xi}) \in \mathcal{X}$, and $(\tilde{r}, \tilde{\tau}) \in \mathcal{Y}$ if and only if

$$\tilde{r} = \lambda\, N(\tilde{u}) + \lambda\, K\tilde{g} \qquad (3.16)$$

and

$$\tilde{\tau} = \lambda\,[N'(\tilde{u})]^*\tilde{\xi} - \lambda\,\mathcal{F}'(\tilde{u})\,. \qquad (3.17)$$

It is evident that the optimality system (2.23)-(2.25) and its finite dimensional counterpart (3.1)-(3.3) can be written as

$$(u, g, \xi) + \mathcal{T}\mathcal{G}\big(\lambda, (u, g, \xi)\big) = 0$$

and

$$(u^h, g^h, \xi^h) + \mathcal{T}^h \mathcal{G}\big(\lambda, (u^h, g^h, \xi^h)\big) = 0\,,$$

respectively, i.e., with $\psi = (u, g, \xi)$ and $\psi^h = (u^h, g^h, \psi^h)$, in the form of (3.4) and (3.6), respectively.

Now we examine the approximation properties of $\mathcal{T}^h$.

LEMMA 3.2. *Let the operators* $\mathcal{T}$ *and* $\mathcal{T}^h$ *be defined by* (3.10)-(3.12) *and* (3.13)-(3.15), *respectively. Assume that the hypotheses* (H13)-(H15) *hold. Then,*

$$\lim_{h\to 0} \|(\mathcal{T} - \mathcal{T}^h)(r,\tau)\|_{\mathcal{X}} = 0 \quad \forall\,(r,\tau) \in \mathcal{Y}\,. \tag{3.18}$$

Proof: Let $(\tilde u, \tilde g, \tilde\xi) = \mathcal{T}(r,\tau)$, i.e., $(\tilde u, \tilde g, \tilde\xi)$ satisfies (3.10)-(3.12). Let $(\tilde u^h, \tilde g^h, \tilde\xi^h) = \mathcal{T}^h(r,\tau)$, i.e., $(\tilde u^h, \tilde g^h, \tilde\xi^h)$ satisfies (3.13)-(3.15). Subtracting the corresponding equations yields that

$$\|\tilde u - \tilde u^h\|_X = \|(T - T^h) r\|_X\,,$$

$$\|\tilde\xi - \tilde\xi^h\|_{Y^*} = \|(T^* - (T^*)^h)\tau\|_{Y^*}\,,$$

and

$$\begin{aligned}\|\tilde g - \tilde g^h\|_G &= \|(E - E^h) K^* \tilde\xi^h + E K^* (\tilde\xi - \tilde\xi^h)\|_G \\ &\le \|(E - E^h) K^* \tilde\xi^h\|_G + \|E K^*\|_{\mathcal{L}(Y^*;G)} \|(\tilde\xi - \tilde\xi^h)\|_G\,.\end{aligned}$$

Thus, for some constant $C > 0$,

$$\begin{aligned}&\|(\mathcal{T} - \mathcal{T}^h)(r,\tau)\|_{\mathcal{X}} \\ &\qquad \le C\left\{\|(T - T^h) r\|_X + \|(T^* - (T^*)^h)\tau\|_{Y^*} + \|(E - E^h) K^* \tilde\xi^h\|_G\right\}.\end{aligned}$$

Then, the result of the proposition follows from (H13)-(H15). $\Box$

Next, we examine the derivative of the mapping $\mathcal{G}$.

LEMMA 3.3. *Let the mapping* $\mathcal{G} : \Lambda \times \mathcal{X} \to \mathcal{Y}$ *be defined by* (3.16)-(3.17). *Assume that the hypotheses* (H9), (H16), *and* (H18)-(H19) *hold. Then, for every* $\lambda \in \Lambda$ *and every* $(u, g, \xi) \in \mathcal{X}$, *the operator* $\mathcal{G}_{(u,g,\xi)}\big(\lambda, (u, g, \xi)\big) \in \mathcal{L}(\mathcal{X}; \mathcal{Z})$.

Proof: A simple calculation shows that $\mathcal{G}_{(u,g,\xi)}\big(\lambda, (u, g, \xi)\big) \in \mathcal{L}(\mathcal{X}; \mathcal{Y})$ is given by

$$\mathcal{G}_{(u,g,\xi)}\big(\lambda, (u, g, \xi)\big) \cdot (\tilde u, \tilde g, \tilde\xi) = \lambda \begin{pmatrix} N'(u) \cdot \tilde u + K \tilde g \\ [N''(u) \cdot \tilde u]^* \cdot \xi + [N'(u)]^* \cdot \tilde\xi - \mathcal{F}''(u) \cdot \tilde u \end{pmatrix}.$$

Then, the result follows from (H9) and (H18)-(H19). $\Box$

A solution $\big(u(\lambda), g(\lambda), \xi(\lambda)\big)$ of the optimality system (2.23)-(2.25) is called *regular* if the system (for the unknowns $(\tilde u, \tilde g, \tilde\xi)$)

$$\tilde u + \lambda T N'(u) \tilde u + \lambda T K \tilde g = \tilde x\,, \tag{3.19}$$

$$\tilde\xi + \lambda T^* [N''(u)]^* \tilde u \cdot \xi + \lambda T^* [N'(u)]^* \tilde\xi - \lambda T^* \mathcal{F}''(u) \tilde u = \tilde y \tag{3.20}$$

and

$$\tilde g - E K^* \tilde\xi = \tilde z\,. \tag{3.21}$$

is uniquely solvable for any $(\tilde{x}, \tilde{z}, \tilde{y}) \in \mathcal{X} = X \times G \times Y^*$. (Note that the linear operator appearing on the left hand side of (3.19)-(3.21) is obtained by linearizing the optimality system (2.23)-(2.25) about (u, g, ξ).)

In the following theorem, we will assume that the solution $\big(u(\lambda), g(\lambda), \xi(\lambda)\big)$ of the optimality system (2.23)-(2.25) that we are trying to approximate is a regular solution. The assumptions we have made, in particular (H9), (H18)-(H19), are sufficient to guarantee that for almost all values of λ, this is indeed the case.

LEMMA 3.4. *Assume the hypotheses of Lemma 3.3. Then, for almost all λ, solutions $\big(u(\lambda), g(\lambda), \xi(\lambda)\big)$ of the optimality system (2.23)-(2.25) are regular.*

Proof: The system (3.19)-(3.21) is equivalent to

$$\Big(I + \lambda \mathcal{T} \mathcal{S}(u, g, \xi)\Big)(\tilde{u}, \tilde{g}, \tilde{\xi}) = (\tilde{x}, \tilde{z}, \tilde{y})\,, \tag{3.22}$$

where the linear operator $\mathcal{S}(u, g, \xi) : \mathcal{X} \to \mathcal{Y}$ is defined by

$$\begin{aligned} \mathcal{S}(u, g, \xi) \cdot (\tilde{u}, \tilde{g}, \tilde{\xi}) &\equiv \frac{1}{\lambda} \mathcal{G}_{(u,g,\xi)}\big(\lambda, (u, g, \xi)\big) \cdot (\tilde{u}, \tilde{g}, \tilde{\xi}) \\ &= \begin{pmatrix} N'(u) \cdot \tilde{u} + K\tilde{g} \\ [N''(u) \cdot \tilde{u}]^* \cdot \xi + [N'(u)]^* \cdot \tilde{\xi} - \mathcal{F}''(u) \cdot \tilde{u} \end{pmatrix} . \end{aligned}$$

Now, $\mathcal{T} \in \mathcal{L}(\mathcal{Y}; \mathcal{X})$, so that, by Lemma 3.3, $\big(I + \lambda \mathcal{T} \mathcal{S}(u, g, \xi)\big)$ is a compact perturbation of the identity operator from $\mathcal{X}$ to $\mathcal{X}$. Thus, for almost all λ, (3.22), or equivalently (3.19)-(3.21), is uniquely solvable, i.e., for almost all λ, the solution $\big(u(\lambda), g(\lambda), \xi(\lambda)\big)$ of the optimality system (2.23)-(2.25) is regular. $\square$

Using Theorem 3.1, we can now provide an error estimate for approximations of solutions of the abstract problem.

THEOREM 3.5. *Let $\big(u(\lambda), g(\lambda), \xi(\lambda)\big) \in \mathcal{X}$, for $\lambda \in \Lambda$, be a branch of regular solutions of the optimality system (2.23)-(2.25). Assume that the hypotheses (H13)-(H19) hold. Then, there exists a $\delta > 0$ and an $h_0 > 0$ such that for $h < h_0$, the discrete optimality system (3.1)-(3.3) has a unique solution $\big(u^h(\lambda), g^h(\lambda), \xi^h(\lambda)\big)$ satisfying*

$$\|\big(u(\lambda), g(\lambda), \xi(\lambda)\big) - \big(u^h(\lambda), g^h(\lambda), \xi^h(\lambda)\big)\|_{\mathcal{X}} < \delta\,.$$

Moreover,

$$\lim_{h \to 0} \|\big(u(\lambda), g(\lambda), \xi(\lambda)\big) - \big(u^h(\lambda), g^h(\lambda), \xi^h(\lambda)\big)\|_{\mathcal{X}} = 0 \tag{3.23}$$

uniformly in $\lambda \in \Lambda$ and there exists a constant C, independent of h and λ, such that

$$\begin{aligned} &\lim_{h \to 0} \|\big(u(\lambda), g(\lambda), \xi(\lambda)\big) - \big(u^h(\lambda), g^h(\lambda), \xi^h(\lambda)\big)\|_{\mathcal{X}} \\ &\qquad \le C\lambda \Big\{ \|(T^h - T)\big(N(u(\lambda)) + Kg(\lambda)\big)\|_X + \|(E^h - E)K^* \xi(\lambda)\|_G \\ &\qquad\qquad + \|\big((T^*)^h - T^*\big)\Big([N'(u(\lambda))]^* \xi - \mathcal{F}'(u(\lambda))\Big)\|_{Y^*} \Big\}\,. \end{aligned} \tag{3.24}$$

Proof: Assumptions (H16) and (H17) ensure that $\mathcal{G} \in C^2(\mathcal{X}, \mathcal{Y})$ and $D^2\mathcal{G}$ maps bounded sets of $\Lambda \times \mathcal{X}$ into bounded sets of $\mathcal{Y}$. By Lemma 3.3, assumptions (H18) and (H19) imply that (3.5) holds. By Lemma 3.2, assumptions (H13)-(H15) imply that (3.7) holds. Then, since $\mathcal{Z}$ is compactly imbedded into $\mathcal{Y}$, (3.7) implies that (3.8) holds. Thus, all the hypotheses of Theorem 3.1 are verified. Then, a direct application of Theorem 3.1 yields (3.23) and (3.24) follows from (3.9). □

It is easily seen that (3.23) and (3.24) are equivalent to:

$$\lim_{h\to 0}\Big\{\|u(\lambda)-u^h(\lambda)\|_X+\|g(\lambda)-g^h(\lambda)\|_G+\|\xi(\lambda)-\xi^h(\lambda)\|_{Y^*}\Big\}=0$$

uniformly in $\lambda\in\Lambda$ and that there exists a constant C, independent of h and λ, such that

$$\begin{aligned}\|u(\lambda)-u^h(\lambda)\|_X&+\|g(\lambda)-g^h(\lambda)\|_G+\|\xi(\lambda)-\xi^h(\lambda)\|_{Y^*}\\ &\le C\lambda\Big\{\|(T^h-T)\big(N(u(\lambda))+Kg(\lambda)\big)\|_X+\|(E^h-E)K^*\xi(\lambda)\|_G\\ &\qquad+\|\big((T^*)^h-T^*\big)\big([N'(u(\lambda))]^*\xi(\lambda)-\mathcal{F}'(u(\lambda))\big)\|_{Y^*}\Big\}.\end{aligned}$$

If, in (3.9), the operator $\mathcal{T}$ is invertible, we have, using (3.4), that

$$\|\psi^h(\lambda)-\psi(\lambda)\|_{\mathcal{X}}\le C\|(\mathcal{T}^h\mathcal{T}^{-1}-I)\psi(\lambda)\|_{\mathcal{X}}\quad\forall\,\lambda\in\Lambda\,.$$

Thus, if the operator T from Y to X is invertible, we have that (3.24) is equivalent to

$$\begin{aligned}\|u(\lambda)-u^h(\lambda)\|_X&+\|g(\lambda)-g^h(\lambda)\|_G+\|\xi(\lambda)-\xi^h(\lambda)\|_{Y^*}\\ &\le C\Big\{\|(T^hT^{-1}-I)u(\lambda)\|_X+\|(E^hE^{-1}-I)g(\lambda)\|_G\\ &\qquad+\|\big((T^*)^h(T^*)^{-1}-I\big)\xi(\lambda)\|_{Y^*}\Big\}.\end{aligned}\tag{3.25}$$

4. Applications

We now apply the framework and analyses developed in §2 and §3 to some concrete problems, all of which feature constraints on admissible states and controls that take the form of a system of nonlinear partial differential equations. In each application, we use a different control mechanism so that the discussion provided in this section illustrates the treatment of a variety of such mechanisms. However, one could use any of the control mechanisms discussed in any of the applications in any other application, or in fact, use any combination of such mechanisms.

Before examining any specific application, we establish some notation. Further notation will be established as needed when the individual applications are considered.

Throughout, C will denote a positive constant whose meaning and value changes with context. Also, $H^s(\mathcal{D})$ for $s\in\mathbb{R}$ denotes the standard real Sobolev space of order s with respect to the set $\mathcal{D}$, where $\mathcal{D}$ could either be a bounded domain $\Omega\in\mathbb{R}^d$, $d=2,3$, or part of the boundary Γ of such a domain. Of particular interest are the spaces $H^0(\mathcal{D})=L^2(\mathcal{D})$,

$$H^1(\mathcal{D})=\left\{\phi\in L^2(\mathcal{D})\;\Big|\;\frac{\partial\phi}{\partial x_j}\in L^2(\mathcal{D})\quad\text{for } j=1,\dots,d\right\}$$

and

$$H^2(\mathcal{D})=\left\{\phi\in L^2(\mathcal{D})\;\Big|\;\frac{\partial\phi}{\partial x_j},\,\frac{\partial^2\phi}{\partial x_j\partial x_k}\in L^2(\mathcal{D})\quad\text{for } j,k=1,\dots,d\right\}.$$

Also of interest is the subspace

$$H_0^1(\mathcal{D}) = \left\{ \phi \in H^1(\mathcal{D}) \;\middle|\; \phi = 0 \quad \text{on } \partial\mathcal{D} \right\},$$

where $\partial\mathcal{D}$ denotes the boundary of $\mathcal{D}$.

Dual spaces will be denoted by $(\cdot)^*$. Duality pairings between spaces and their duals will be denoted by $\langle\cdot,\cdot\rangle$. Norms of functions belonging to $H^s(\Omega)$ and $H^s(\Gamma)$ are denoted by $\|\cdot\|_s$ and $\|\cdot\|_{s,\Gamma}$, respectively. Of particular interest are the $L^2(\Omega)$-norm $\|\cdot\|_0$, the $H^1(\Omega)$-norm

$$\|\phi\|_1^2 = \sum_{j=1}^{d} \left\| \frac{\partial \phi}{\partial x_j} \right\|_0^2 + \|\phi\|_0^2,$$

and the $H^2(\Omega)$-norm

$$\|\phi\|_2^2 = \sum_{j,k=1}^{d} \left\| \frac{\partial^2 \phi}{\partial x_j \partial x_k} \right\|_0^2 + \|\phi\|_1^2.$$

Corresponding Sobolev spaces of real, vector-valued functions having r components will be denoted by $\mathbf{H}^s(\mathcal{D})$, e.g., $\mathbf{H}^1(\mathcal{D}) = [H^1(\mathcal{D})]^r$. Of particular interest will be the spaces $\mathbf{L}^2(\mathcal{D}) = \mathbf{H}^0(\mathcal{D}) = [L^2(\mathcal{D})]^r$,

$$\mathbf{H}^1(\mathcal{D}) = \left\{ v_j \in L^2(\mathcal{D}) \;\middle|\; \frac{\partial v_j}{\partial x_k} \in L^2(\mathcal{D}) \quad \text{for } j = 1,\dots,r \text{ and } k = 1,\dots,d \right\},$$

and

$$\mathbf{H}^2(\mathcal{D}) = \left\{ v_j \in L^2(\mathcal{D}) \;\middle|\; \frac{\partial v_j}{\partial x_k} \in L^2(\mathcal{D}),\, \frac{\partial^2 v_j}{\partial x_k \partial x_\ell} \in L^2(\Omega) \quad \text{for } j = 1,\dots,r \text{ and } k,\ell = 1,\dots,d \right\},$$

where v_j, $j = 1,\dots,r$, denote the components of $\mathbf{v}$. Also of interest is the subspace

$$\mathbf{H}_0^1(\mathcal{D}) = \left\{ \mathbf{v} \in \mathbf{H}^1(\mathcal{D}) \;\middle|\; v_j = 0 \text{ on } \partial\mathcal{D},\, j = 1,\dots,r \right\}.$$

Norms for spaces of vector-valued functions will be denoted by the same notation as that used for their scalar counterparts. For example,

$$\|\mathbf{v}\|_s^2 = \sum_{j=1}^{r} \|v_j\|_s^2 \quad \text{and} \quad \|\mathbf{v}\|_{s,\Gamma}^2 = \sum_{j=1}^{r} \|v_j\|_{s,\Gamma}^2.$$

We denote the $L^2(\Omega)$ and $\mathbf{L}^2(\Omega)$ inner products by $(\cdot,\cdot)$, i.e., for $p, q \in L^2(\Omega)$ and $\mathbf{u}, \mathbf{v} \in \mathbf{L}^2(\Omega)$

$$(p,q) = \int_\Omega pq\, d\Omega \quad \text{and} \quad (\mathbf{u},\mathbf{v}) = \int_\Omega \mathbf{u}\cdot\mathbf{v}\, d\Omega.$$

Similarly, we denote by $(\cdot,\cdot)_\Gamma$ the $L^2(\Gamma)$ and $\mathbf{L}^2(\Gamma)$ inner products, i.e., for $p, q \in L^2(\Gamma)$ and $\mathbf{u}, \mathbf{v} \in \mathbf{L}^2(\Gamma)$

$$(p,q)_\Gamma = \int_\Gamma pq\, d\Gamma \quad \text{and} \quad (\mathbf{u},\mathbf{v})_\Gamma = \int_\Gamma \mathbf{u}\cdot\mathbf{v}\, d\Gamma.$$

Since in all cases L^2-spaces will be used as pivot spaces, the above inner product notation can also be used to denote duality pairings between functions defined on H^s-spaces and their dual spaces.

For details concerning the notation employed, one may consult, e.g., [1].

4.1. Distributed controls for the von Kármán plate equations

For this application we will use distributed controls, i.e., control is effected through a source term in the governing partial differential equations. Let Ω be a bounded, convex polygonal domain in $\mathbb{R}^2$ and let Γ denote the boundary of Ω. The von Kármán equations for a clamped plate are given by (see, e.g., [9] or [18])

$$\Delta^2\psi_1 + \frac{1}{2}[\psi_2, \psi_2] = 0 \quad \text{in } \Omega\,,$$

$$\Delta^2\psi_2 - [\psi_1, \psi_2] = \lambda g \quad \text{in } \Omega\,,$$

and

$$\psi_1 = \frac{\partial\psi_1}{\partial n} = \psi_2 = \frac{\partial\psi_2}{\partial n} = 0 \quad \text{on } \Gamma\,,$$

where

$$[\psi, \phi] = \frac{\partial^2\psi}{\partial x_1^2}\frac{\partial^2\phi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_2^2}\frac{\partial^2\phi}{\partial x_1^2} - 2\frac{\partial^2\psi}{\partial x_1 x_2}\frac{\partial^2\phi}{\partial x_1 x_2}\,.$$

Here, ψ_1 denotes the Airy stress function, ψ_2 the deflection of the plate in the direction normal to the plate, λg is an external load normal to the plate which depends on the loading parameter λ, and $\partial(\cdot)/\partial n$ the normal derivative in the direction of the outer normal to Γ.

By introducing appropriate rescalings, i.e., by replacing ψ_1 by $\lambda\psi_1$, ψ_2 by $\lambda\psi_2$, and g by λg, we can rewrite the von Kármán equations as follows:

$$\Delta^2\psi_1 + \frac{\lambda}{2}[\psi_2, \psi_2] = 0 \quad \text{in } \Omega\,, \tag{4.1}$$

$$\Delta^2\psi_2 - \lambda\,[\psi_1, \psi_2] = \lambda\, g \quad \text{in } \Omega \tag{4.2}$$

and

$$\psi_1 = \frac{\partial\psi_1}{\partial n} = \psi_2 = \frac{\partial\psi_2}{\partial n} = 0 \quad \text{on } \Gamma\,. \tag{4.3}$$

We introduce the spaces

$$H_0^2(\Omega) = \left\{\psi \in H^2(\Omega) \;\middle|\; \psi = 0\,,\ \frac{\partial\psi}{\partial n} = 0 \text{ on } \Gamma\right\}\,,$$

$$\mathbf{H}_0^2(\Omega) = [H_0^2(\Omega)]^2\,, \quad H^{-2}(\Omega) = \left(H_0^2(\Omega)\right)^*\,, \quad \text{and} \quad \mathbf{H}^{-2}(\Omega) = \left(\mathbf{H}_0^2(\Omega)\right)^*$$

and the bilinear form

$$a(\psi, \phi) = \int_\Omega \Delta\psi\,\Delta\phi\,d\Omega \quad \forall\ \psi, \phi \in H^2(\Omega)$$

in order to define the following weak formulation of the von Kármán equations (4.1)-(4.3): find $\boldsymbol{\psi} = (\psi_1, \psi_2) \in \mathbf{H}_0^2(\Omega)$ such that

$$a(\psi_1, \phi_1) + \frac{\lambda}{2}\big([\psi_2, \psi_2], \phi_1\big) = 0 \quad \forall\, \phi_1 \in H_0^2(\Omega) \tag{4.4}$$

and

$$a(\psi_2, \phi_2) - \lambda\big([\psi_1, \psi_2], \phi_2\big) = \lambda\langle g, \phi_2\rangle \quad \forall\, \phi_2 \in H_0^2(\Omega)\,. \tag{4.5}$$

Using the identity

$$\big([\psi, \phi], \zeta\big) = \big([\psi, \zeta], \phi\big) \quad \forall\, \psi, \phi, \zeta \in H_0^2(\Omega)\,, \tag{4.6}$$

one can show that for each $g \in H^{-2}(\Omega)$, (4.4)-(4.5) possesses at least one solution $\boldsymbol{\psi} = (\psi_1, \psi_2) \in \mathbf{H}_0^2(\Omega)$ and that all solutions of (4.4)-(4.5) satisfy the a priori estimate

$$\|\psi_1\|_2 + \|\psi_2\|_2 \leq C\|g\|_{-2}\,; \tag{4.7}$$

see, e.g., [18], for details. In the sequel a solution to (4.1)-(4.3) will be understood in the sense of (4.4)-(4.5).

Given a desired state $\boldsymbol{\psi}_0 = \big(\psi_{10}, \psi_{20}\big) \in \mathbf{L}^2(\Omega)$, we define for any $\boldsymbol{\psi} = (\psi_1, \psi_2) \in \mathbf{H}_0^2(\Omega)$ and $g \in L^2(\Omega)$ the functional

$$\begin{aligned} \mathcal{J}(\boldsymbol{\psi}, g) &= \mathcal{J}(\psi_1, \psi_2, g) \\ &= \frac{\lambda}{2}\int_\Omega \Big((\psi_1 - \psi_{10})^2 + (\psi_2 - \psi_{20})^2\Big)\, d\Omega + \frac{\lambda}{2}\int_\Omega g^2\, d\Omega\,. \end{aligned} \tag{4.8}$$

We then consider the following optimal control problem associated with the von Kármán plate equations:

$$\min\big\{\, \mathcal{J}(\boldsymbol{\psi}, g) \mid \boldsymbol{\psi} \in \mathbf{H}_0^2(\Omega)\,,\ g \in \Theta \,\big\} \quad \text{subject to} \quad (4.4)\text{-}(4.5)\,, \tag{4.9}$$

where Θ is a subset of $L^2(\Omega)$.

We define the spaces $X = \mathbf{H}_0^2(\Omega)$, $Y = \mathbf{H}^{-2}(\Omega)$, $G = L^2(\Omega)$, and $Z = \mathbf{L}^1(\Omega)$. By compact imbedding results, $Z \hookrightarrow\hookrightarrow Y$. For the time being, we assume that the admissible set Θ for the control g is a closed, convex subset of $G = L^2(\Omega)$.

Let the continuous linear operator $T \in \mathcal{L}(Y; X)$ be defined as follows: for $\mathbf{f} = (f_1, f_2) \in Y = \mathbf{H}^{-2}(\Omega)$, $T\mathbf{f} = \boldsymbol{\psi} \in X = \mathbf{H}_0^2(\Omega)$ is the unique solution of

$$a(\psi_1, \phi_1) = \langle f_1, \phi_1\rangle \quad \forall\, \phi_1 \in H_0^2(\Omega)$$

and

$$a(\psi_2, \phi_2) = \langle f_2, \phi_2\rangle \quad \forall\, \phi_2 \in H_0^2(\Omega)\,.$$

It can be easily verified that T is self-adjoint.

We define the (differentiable) nonlinear mapping $N : X \to Y$ by

$$N(\boldsymbol{\psi}) = \begin{pmatrix} \frac{1}{2}[\psi_2, \psi_2] \\ -[\psi_1, \psi_2] \end{pmatrix} \quad \forall\, \boldsymbol{\psi} \in X$$

or equivalently

$$\langle N(\boldsymbol{\psi}), \boldsymbol{\phi}\rangle = \frac{1}{2}\left([\psi_2, \psi_2], \phi_1\right) - \left([\psi_1, \psi_2], \phi_2\right) \quad \forall\, \boldsymbol{\phi} = (\phi_1, \phi_2) \in X$$

and define $K : g \in L^2(\Omega) \to Y$ by

$$Kg = -\begin{pmatrix} 0 \\ g \end{pmatrix} ,$$

or equivalently,

$$\langle Kg, \boldsymbol{\phi}\rangle = -\langle g, \phi_2\rangle \quad \forall\, \boldsymbol{\phi} = (\phi_1, \phi_2) \in X \,.$$

Clearly, the constraint equations (4.4)-(4.5) can be expressed as

$$\boldsymbol{\psi} + \lambda T N(\boldsymbol{\psi}) + \lambda T K g = 0 \,,$$

i.e., in the form (2.2). With the obvious definitions for $\mathcal{F}(\cdot)$ and $\mathcal{E}(\cdot)$, i.e.,

$$\mathcal{F}(\boldsymbol{\psi}) = \frac{1}{2}\int_\Omega \Big((\psi_1 - \psi_{10})^2 + (\psi_2 - \psi_{20})^2 \Big)\, d\Omega \quad \forall\, \boldsymbol{\psi} \in X$$

and

$$\mathcal{E}(g) = \frac{1}{2}\int_\Omega g^2\, d\Omega \quad \forall\, g \in G \,,$$

the functional (4.8) can be expressed as

$$\mathcal{J}(\boldsymbol{\psi}, g) = \lambda\, \mathcal{F}(\boldsymbol{\psi}) + \lambda\, \mathcal{E}(g) \,,$$

i.e., in the form (2.1). Thus, the minimization problem (4.9) is in the form of the minimization problem (2.3).

We are now in a position to verify, for the minimization problem (4.9), all the hypotheses of §2 and §3.

4.1.1. Verification of the hypotheses for the existence of optimal solutions.

We first verify that the hypotheses (H1)-(H6) hold in the current setting.

(H1) is obviously satisfied with a lower bound 0.

(H2) holds with $\alpha = 1$ and $\beta = 2$.

(H3) is verified with the choice $(\boldsymbol{\psi}^{(0)}, g^{(0)}) \in X \times \Theta$, where $g^{(0)}$ is an arbitrarily chosen element in Θ and $\boldsymbol{\psi}^{(0)} = \big(\psi_1^{(0)}, \psi_2^{(0)}\big)$ is a solution of

$$\Delta^2 \psi_1^{(0)} + \frac{\lambda}{2}[\psi_2^{(0)}, \psi_2^{(0)}] = 0 \quad \text{in } \Omega \,,$$

$$\Delta^2 \psi_2^{(0)} - \lambda\, [\psi_1^{(0)}, \psi_2^{(0)}] = \lambda\, g^{(0)} \quad \text{in } \Omega \,,$$

and

$$\psi_1^{(0)} = \frac{\partial \psi_1^{(0)}}{\partial n} = \psi_2^{(0)} = \frac{\partial \psi_2^{(0)}}{\partial n} = 0 \quad \text{on } \Gamma \,.$$

In order to verify (H4), we assume $\{g^{(n)}\} \subset \Theta$ is a sequence satisfying $g^{(n)} \rightharpoonup g$ in $L^2(\Omega)$; then, we have $g^{(n)} \rightharpoonup g$ in $H^{-2}(\Omega)$ so that $\lim_{n\to\infty}\langle g^{(n)}, z\rangle = \langle g, z\rangle$ for all $z \in H^2(\Omega)$, i.e., $Kg^{(n)} \rightharpoonup Kg$ in Y. Assume that the sequence $\{\boldsymbol{\psi}^{(n)}\} \subset \mathbf{H}_0^2(\Omega)$

satisfies $\boldsymbol{\psi}^{(n)} \rightharpoonup \boldsymbol{\psi}$ in $\mathbf{H}_0^2(\Omega)$; then, $(\partial^2\boldsymbol{\psi}^{(n)}/\partial x_i\partial x_j) \rightharpoonup (\partial^2\boldsymbol{\psi}/\partial x_i\partial x_j)$ in $\mathbf{L}^2(\Omega)$ and, by using a compact imbedding result, $\boldsymbol{\psi}^{(n)} \to \boldsymbol{\psi}$ in $\mathbf{L}^2(\Omega)$. Now, using the identity (4.6),

$$\begin{aligned}
\langle N(\boldsymbol{\psi}^{(n)}), \boldsymbol{\phi}\rangle &= \frac{1}{2}\Big([\psi_2^{(n)}, \psi_2^{(n)}], \phi_1\Big) - \Big([\psi_1^{(n)}, \psi_2^{(n)}], \phi_2\Big) \\
&= \frac{1}{2}\Big([\psi_2^{(n)}, \phi_1], \psi_2^{(n)}\Big) - \Big([\psi_1^{(n)}, \phi_2], \psi_2^{(n)}\Big) \\
&\to \frac{1}{2}\left([\psi_2, \phi_1], \psi_2\right) - \left([\psi_1, \phi_2], \psi_2\right) \\
&= \frac{1}{2}\left([\psi_2, \psi_2], \phi_1\right) - \left([\psi_1, \psi_2], \phi_2\right) = \langle N(\boldsymbol{\psi}), \boldsymbol{\phi}\rangle\,.
\end{aligned}$$

Hence, (H4) is verified.

The verification of (H5) follows directly from the observation that the mappings $\boldsymbol{\phi} \mapsto \mathcal{F}(\boldsymbol{\phi}) = (1/2)\|\boldsymbol{\phi} - \boldsymbol{\psi}_0\|_0^2$ and $g \mapsto \mathcal{E}(g) = (1/2)\|g\|_0^2$ are convex.

The verification of (H6) is a trivial consequence of the a priori estimate (4.7).

It is now just a matter of citing Theorem 2.1 to prove the existence of an optimal solution that minimizes (4.8) subject to (4.4)-(4.5).

THEOREM 4.1. *There exists a* $(\boldsymbol{\phi}, g) \in \mathbf{H}_0^2(\Omega) \times \Theta$ *such that* (4.8) *is minimized subject to* (4.4)-(4.5). □

4.1.2. Verification of the hypotheses for the existence of Lagrange multipliers.

We now assume $(\boldsymbol{\psi}, g)$ is an optimal solution and turn to the verification of hypotheses (H7)-(H9).

The validity of (H7) is obvious.

(H8) holds since the mapping $g \mapsto \mathcal{E}(g) = (1/2)\|g\|_0^2$ is convex.

(H9) can be verified as follows. For any $\boldsymbol{\psi} \in X$, the operator $N'(\boldsymbol{\psi}) : X \to Y$ is given by

$$N'(\boldsymbol{\psi}) \cdot \boldsymbol{\phi} = \begin{pmatrix} [\psi_2, \phi_2] \\ -[\psi_1, \phi_2] - [\psi_2, \phi_1] \end{pmatrix} \quad \forall\, \boldsymbol{\phi} = (\phi_1, \phi_2) \in X\,.$$

Thus, using the definition of $[\cdot,\cdot]$, we obtain that $N'(\boldsymbol{\psi}) \cdot \boldsymbol{\phi} \in \mathbf{L}^1(\Omega) = Z$.

The Lagrangian is given by

$$\begin{aligned}
\mathcal{L}(\boldsymbol{\psi}, g, \boldsymbol{\eta}, k) = k\,\mathcal{J}(\boldsymbol{\psi}, g) - \Big\{ & a(\psi_1, \eta_1) + \frac{\lambda}{2}([\psi_2, \psi_2], \eta_1) \\
& + a(\psi_2, \eta_2) - \lambda\big([\psi_1, \psi_2], \eta_2\big) - \lambda(g, \eta_2)\Big\}
\end{aligned}$$

for all $(\boldsymbol{\psi}, g, \boldsymbol{\eta}, k) \in X \times G \times X \times \mathbb{R} = \mathbf{H}_0^2(\Omega) \times L^2(\Omega) \times \mathbf{H}_0^2(\Omega) \times \mathbb{R}$. Note that in this form of the Lagrangian, the Lagrange multiplier $\boldsymbol{\eta} \in X = Y^*$ so that we have already introduced the change of variables indicated between (2.17)-(2.18) and (2.19)-(2.21).

Having verified the hypotheses (H7)-(H9), we may apply Theorem 2.4 to conclude that there exists a Lagrange multiplier $\boldsymbol{\eta} \in X = \mathbf{H}_0^2(\Omega)$ and a real number k such that

$$\boldsymbol{\eta} + \lambda\, T^*\big([N'(\boldsymbol{\psi})]^* \cdot \boldsymbol{\eta} - k\,\mathcal{J}_{\boldsymbol{\psi}}(\boldsymbol{\psi}, g)\big) = \mathbf{0} \tag{4.10}$$

and

$$\mathcal{L}(\boldsymbol{\psi}, g, \boldsymbol{\eta}, k) \le \mathcal{L}(\boldsymbol{\psi}, z, \boldsymbol{\eta}, k) \quad \forall\, z \in \Theta \tag{4.11}$$

and that for almost all values of λ, we may choose $k = 1$.

Recall that T is self-adjoint. Also, note that for any $\boldsymbol{\psi} \in X = \mathbf{H}_0^2(\Omega)$,

$$[N'(\boldsymbol{\psi})]^* \cdot \boldsymbol{\eta} = \begin{pmatrix} -[\psi_2, \eta_2] \\ [\psi_2, \eta_1] - [\psi_1, \eta_2] \end{pmatrix} \quad \forall\, \boldsymbol{\eta} = (\eta_1, \eta_2) \in X\,.$$

Thus, (4.10), with $k = 1$, can be rewritten as

$$a(\zeta_1, \eta_1) - \lambda\big([\psi_2, \eta_2], \zeta_1\big) = \lambda(\psi_1 - \psi_{10}, \zeta_1) \quad \forall\, \zeta_1 \in H_0^2(\Omega) \tag{4.12}$$

and

$$\begin{aligned} a(\zeta_2, \eta_2) + \lambda\big([\psi_2, \eta_1], \zeta_2\big) - \lambda\big([\psi_1, \eta_2], \zeta_2\big) \\ = \lambda(\psi_2 - \psi_{20}, \zeta_2) \quad \forall\, \zeta_2 \in H_0^2(\Omega)\,. \end{aligned} \tag{4.13}$$

Using the definition of the Lagrangian functional, (4.11), with $k = 1$, can be rewritten as

$$\frac{\lambda}{2}\,(z, z) + \lambda(z, \eta_2) - \frac{\lambda}{2}\,(g, g) - \lambda(g, \eta_2) \geq 0 \quad \forall\, z \in \Theta\,.$$

Note that, in the above expression, we have already employed hypothesis (H12) which in the current context is trivially satisfied with E the identity operator on $G^* = G = L^2(\Omega)$. For each $\epsilon \in (0, 1)$ and each $t \in \Theta$, set $z = \epsilon t + (1-\epsilon) g \in \Theta$ in the last equation to obtain

$$\frac{\epsilon^2}{2}\,(t - g, t - g) + \epsilon\,(t - g, g) + \epsilon\,(t - g, \eta_2) \geq 0 \quad \forall\, t \in \Theta$$

so that, after dividing by $\epsilon > 0$ and then letting $\epsilon \to 0^+$, we obtain

$$(t - g, g + \eta_2) \geq 0 \quad \forall\, t \in \Theta\,. \tag{4.14}$$

We see that for almost all values of λ, necessary conditions for an optimum are that (4.4)-(4.5) and (4.12)-(4.14) are satisfied. The system formed by these equations will be called an *optimality system.*

We now specialize to the case $\Theta = L^2(\Omega)$. Note that the hypothesis (H10) is satisfied. Then, using Theorem 2.5, we see that the inequality (4.14) becomes an equality and, by letting $z = t - g$ vary arbitrarily in $L^2(\Omega)$, we now have, instead of (4.14),

$$(z, g + \eta_2) = 0 \quad \forall\, z \in L^2(\Omega)\,. \tag{4.15}$$

Thus, according to that theorem, we have that for almost all λ, an optimality system of equations is now given by (4.4)-(4.5), (4.12)-(4.13), and (4.15). However, we can go further and verify that the hypothesis (H11)$'$ is valid, which in turn will justify the existence of a Lagrange multiplier satisfying the optimality system for *all* $\lambda \in \Lambda$. We now assume the domain Ω is a convex polygon with no angles greater than $126°$.

Let λ be given such that $1/\lambda$ is an eigenvalue of $-TN'(\boldsymbol{\psi})$, where $(\boldsymbol{\psi}, g) \in \mathbf{H}_0^2(\Omega) \times L^2(\Omega)$ is an optimal pair that minimizes (4.8) subject to (4.4)-(4.5). We wish to show that for each $\tilde{\mathbf{f}} \in \mathbf{H}^{-2}(\Omega)$, there exists a $\tilde{g} \in L^2(\Omega)$ and a $\tilde{\boldsymbol{\psi}} \in \mathbf{H}_0^2(\Omega)$ such that

$$\tilde{\boldsymbol{\psi}} + \lambda T N'(\boldsymbol{\psi}) \cdot \tilde{\boldsymbol{\psi}} + \lambda T K \tilde{g} = \tilde{\mathbf{f}}\,,$$

i.e.,

$$a(\tilde{\psi}_1, \phi_1) + \lambda\big([\psi_2, \tilde{\psi}_2], \phi_1\big) = \langle \tilde{f}_1, \phi_1 \rangle \quad \forall\, \phi_1 \in H_0^2(\Omega) \tag{4.16}$$

and

$$a(\tilde{\psi}_2,\phi_2) - \lambda\big([\tilde{\psi}_1,\psi_2],\phi_2\big) - \lambda\big([\psi_1,\tilde{\psi}_2],\phi_2\big) - \lambda(\tilde{g},\phi_2) \\ = \langle \tilde{f}_2,\phi_2\rangle \quad \forall\ \phi_2 \in H^2_0(\Omega)\,. \tag{4.17}$$

To show this, we first let $\tilde{\boldsymbol{\psi}} \in \mathbf{H}^2_0(\Omega)$ be a solution of

$$a(\tilde{\psi}_1,\phi_1) + \lambda\big([\psi_2,\tilde{\psi}_2],\phi_1\big) = \langle \tilde{f}_1,\phi_1\rangle \quad \forall\ \phi_1 \in H^2_0(\Omega)$$

and

$$a(\tilde{\psi}_2,\phi_2) - \lambda\big([\tilde{\psi}_1,\psi_2],\phi_2\big) = \langle \tilde{f}_2,\phi_2\rangle \quad \forall\ \phi_2 \in H^2_0(\Omega)\,.$$

The existence of such a $\tilde{\boldsymbol{\psi}}$ can be shown in a manner similar to that for showing the existence of a solution to the von Kármán equation; the key step is that by adding the two equations with the test function $\boldsymbol{\phi}$ replaced by $\tilde{\boldsymbol{\psi}}$, we have the a priori estimate

$$a(\tilde{\psi}_1,\tilde{\psi}_1) + a(\tilde{\psi}_2,\tilde{\psi}_2) = \langle \tilde{f}_1,\tilde{\psi}_1\rangle + \langle \tilde{f}_2,\tilde{\psi}_2\rangle\,.$$

Then, we choose $\tilde{g} = -[\psi_1,\tilde{\psi}_2]$. Note that regularity results for the biharmonic equation applied to (4.4)-(4.5) yield $\boldsymbol{\psi} \in \mathbf{H}^4(\Omega)$ (see [3]). Hence, using imbedding theorems we deduce that $\tilde{g} \in L^2(\Omega)$. It is obvious that $\tilde{g}$ and $\tilde{\boldsymbol{\psi}}$ satisfy (4.16)-(4.17), i.e., we have verified (H11)$'$. Hence we conclude that for *all* λ, the optimality system (4.4)-(4.5), (4.12)-(4.13), and (4.15) has a solution. Thus, we have Theorem 2.6 which, in the present context, is given as follows.

THEOREM 4.2. *Let* $(\boldsymbol{\psi},g) \in \mathbf{H}^2_0(\Omega)\times L^2(\Omega)$ *denote an optimal solution that minimizes* (4.8) *subject to* (4.5)-(4.6). *Then, for all* $\lambda \in \Lambda$, *there exists a nonzero Lagrange multiplier* $\boldsymbol{\eta} \in \mathbf{H}^2_0(\Omega)$ *satisfying the Euler equations* (4.12)-(4.13) *and* (4.15). □

4.1.3. Verification of the hypotheses for approximations and error estimates.
We finally verify the hypotheses (H13)-(H19) that are used in connection with approximations and error estimates.

A finite element discretization of the optimality system (4.4)-(4.5), (4.12)-(4.13), and (4.15) is defined in the usual manner. We first choose families of finite dimensional subspaces $X^h \subset \mathbf{H}^2_0(\Omega)$ and $G^h \subset L^2(\Omega)$ parameterized by a parameter h that tends to zero and satisfying the following approximation properties: there exists a constant C and an integer r such that

$$\inf_{\boldsymbol{\phi}^h \in X^h} \|\boldsymbol{\phi} - \boldsymbol{\phi}^h\|_2 \le Ch^m \|\boldsymbol{\phi}\|_{m+2}\,, \quad \forall\ \boldsymbol{\phi} \in \mathbf{H}^{m+2}(\Omega)\,,\ 1 \le m \le r \tag{4.18}$$

and

$$\inf_{z^h \in G^h} \|z - z^h\|_0 \le Ch^m \|z\|_m\,, \quad \forall\ z \in H^m(\Omega)\,,\ 1 \le m \le r\,. \tag{4.19}$$

One may consult, e.g., [8] for some finite element spaces satisfying (4.18) and (4.19). For example, one may choose $X^h = V^h \times V^h$ where V^h is the piecewise quintic-$C^1(\overline{\Omega})$ finite element space constrained to satisfy the given boundary conditions and defined with respect to a family of triangulations of Ω. In this case, h is a measure of the grid size. For simplicity, one may choose $G^h = V^h$.

Once the approximating spaces have been chosen, we may formulate the approximate problem for the optimality system (4.4)-(4.5), (4.12)-(4.13), and (4.15): seek $\boldsymbol{\psi}^h \in X^h$, $g^h \in G^h$, and $\boldsymbol{\eta}^h \in X^h$ such that

$$a(\psi_1^h, \phi_1^h) + \frac{\lambda}{2}\big([\psi_2^h, \psi_2^h], \phi_1^h\big) = 0 \quad \forall\, \phi_1^h \in V^h\,, \tag{4.20}$$

$$a(\psi_2^h, \phi_2^h) - \lambda\big([\psi_1^h, \psi_2^h], \phi_2^h\big) = (g, \phi_2^h) \quad \forall\, \phi_2^h \in V^h\,, \tag{4.21}$$

$$a(\zeta_1^h, \eta_1^h) - \lambda\big([\psi_2^h, \eta_2^h], \zeta_1^h\big) = \lambda(\psi_1^h - \psi_{10}, \zeta_1^h) \quad \forall\, \zeta_1^h \in V^h\,, \tag{4.22}$$

$$a(\zeta_2^h, \eta_2^h) + \lambda\big([\psi_2^h, \eta_1^h], \zeta_2^h\big) - \lambda\big([\psi_1^h, \eta_2^h], \zeta_2^h\big) = \lambda(\psi_2^h - \psi_{20}, \zeta_2^h) \quad \forall\, \zeta_2^h \in V^h\,, \tag{4.23}$$

and

$$(z^h, g^h + \eta_2^h) = 0 \quad \forall\, z^h \in G^h\,. \tag{4.24}$$

The operator $T^h \in \mathcal{L}(Y; X^h)$ is defined as follows: for $\mathbf{f} \in Y$, $T^h\mathbf{f} = \boldsymbol{\psi}^h \in X^h$ is the unique solution of

$$a(\psi_1^h, \phi_1^h) = \langle f_1, \phi_1^h\rangle \quad \forall\, \phi_1^h \in V^h$$

and

$$a(\psi_2^h, \phi_2^h) = \langle f_2, \phi_2^h\rangle \quad \forall\, \phi_2^h \in V^h\,.$$

Since $T = T^*$, we define $(T^*)^h = T^h$.

We define the operator $E^h : L^2(\Omega) \to G^h$ as the $L^2(\Omega)$-projection on G^h, i.e., for each $g \in L^2(\Omega)$,

$$(E^h g, \phi^h) = (g, \phi^h) \quad \forall\, \phi^h \in G^h\,.$$

Since $G = L^2(\Omega)$ is reflexive, E^h is in fact an operator from $G^* \to G^h$.

By the well-known results concerning the approximation of biharmonic equations (see, e.g., [2] or [8]), we obtain

$$\|(T - T^h)\mathbf{f}\|_X \to 0$$

as $h \to 0$, for all $\mathbf{f} \in Y$. This is simply a restatement of (H13).

(H14) follows trivially from (H13) and the fact that T is self-adjoint and we have chosen $(T^*)^h = T^h$.

(H15) follows from the best approximation property of $L^2(\Omega)$-projections and (4.19).

(H16) and (H17) follow from the fact that N and $\mathcal{F}$ are polynomials. Here we also use imbedding theorems and Cauchy inequalities.

We set $\hat{Z} = Z = \mathbf{L}^1(\Omega)$. For each $\boldsymbol{\eta} \in \mathbf{H}_0^2(\Omega)$ and $\boldsymbol{\zeta} \in \mathbf{H}_0^2(\Omega)$, Sobolev imbedding theorems imply that

$$[N'(\boldsymbol{\psi})]^* \cdot \boldsymbol{\eta} = \begin{pmatrix} -[\psi_2, \eta_2] \\ [\psi_2, \eta_1] - [\psi_1, \eta_2] \end{pmatrix} \in \hat{Z}\,,$$

$$\left([N''(\boldsymbol{\psi})]^* \cdot \boldsymbol{\zeta}\right) \cdot \boldsymbol{\eta} = \begin{pmatrix} -[\zeta_2, \eta_2] \\ [\zeta_2, \eta_1] - [\zeta_1, \eta_2] \end{pmatrix} \in \hat{Z}$$

and

$$\left(\mathcal{F}''(\boldsymbol{\psi}) \cdot \boldsymbol{\zeta}\right) \cdot \boldsymbol{\eta} = \begin{pmatrix} \eta_1 \zeta_1 \\ \eta_2 \zeta_2 \end{pmatrix} \in \hat{Z}\,.$$

These relations verify (H18).

From the definition of the operator K we see that K maps $L^2(\Omega)$ into $\mathbf{L}^1(\Omega)$, i.e., K maps G into Z. Thus (H19) is verified.

Hence, we are now in a position to apply Theorem 3.5 to derive error estimates for the approximate solutions of the optimality system (4.4)-(4.5), (4.12)-(4.13) and (4.15). It should be noted that Lemma 3.4 implies that for almost all values of λ, the solutions of the optimality system are regular.

THEOREM 4.3. *Assume that Λ is a compact interval of $\mathbb{R}_+$ and that there exists a branch $\{(\lambda, \boldsymbol{\psi}(\lambda), g(\lambda), \boldsymbol{\eta}(\lambda)) : \lambda \in \Lambda\}$ of regular solutions of the optimality system* (4.4)-(4.5), (4.12)-(4.13), *and* (4.15). *Assume that the finite element spaces X^h and G^h satisfy the hypotheses* (4.18)-(4.19). *Then, there exists a $\delta > 0$ and an $h_0 > 0$ such that for $h \le h_0$, the discrete optimality system* (4.20)-(4.24) *has a unique branch of solutions $\{\left(\lambda, \boldsymbol{\psi}^h(\lambda), g^h(\lambda), \boldsymbol{\eta}^h(\lambda)\right) : \lambda \in \Lambda\}$ satisfying*

$$\{\|\boldsymbol{\psi}^h(\lambda) - \boldsymbol{\psi}(\lambda)\|_2 + \|g^h(\lambda) - g(\lambda)\|_0 + \|\boldsymbol{\eta}^h(\lambda) - \boldsymbol{\eta}(\lambda)\|_2\} < \delta \qquad \text{for all } \lambda \in \Lambda.$$

Moreover,

$$\lim_{h\to 0} \{\|\boldsymbol{\psi}^h(\lambda) - \boldsymbol{\psi}(\lambda)\|_2 + \|g^h(\lambda) - g(\lambda)\|_0 + \|\boldsymbol{\eta}^h(\lambda) - \boldsymbol{\eta}(\lambda)\|_2\} = 0\,,$$

uniformly in $\lambda \in \Lambda$.

If, in addition, the solution of the optimality system satisfies $\left(\boldsymbol{\psi}(\lambda), g(\lambda), \boldsymbol{\eta}(\lambda)\right) \in \mathbf{H}^{m+2}(\Omega) \times H^m(\Omega) \times \mathbf{H}^{m+2}(\Omega)$ for $\lambda \in \Lambda$, then there exists a constant C, independent of h, such that

$$\begin{aligned} &\|\boldsymbol{\psi}(\lambda) - \boldsymbol{\psi}^h(\lambda)\|_2 + \|g(\lambda) - g^h(\lambda)\|_0 + \|\boldsymbol{\eta}(\lambda) - \boldsymbol{\eta}^h(\lambda)\|_2 \\ &\qquad \le C h^m \left(\|\boldsymbol{\psi}(\lambda)\|_{m+2} + \|g(\lambda)\|_m + \|\boldsymbol{\eta}(\lambda)\|_{m+2}\right), \end{aligned}$$

uniformly in $\lambda \in \Lambda$.

Proof: All results follow from Theorem 3.5. For the last result, we also use (3.25) and the estimates (see, e.g., [2] or [8])

$$\|(T^h T^{-1} - I)\boldsymbol{\psi}\|_2 \le C h^m \|\boldsymbol{\psi}\|_{m+2} \qquad \text{for } \boldsymbol{\psi} \in \mathbf{H}^{m+2}(\Omega)\,,$$

$$\|\left((T^*)^h (T^*)^{-1} - I\right)\boldsymbol{\eta}\|_2 = \|(T^h T^{-1} - I)\boldsymbol{\eta}\|_2 \le C h^m \|\boldsymbol{\eta}\|_{m+2} \qquad \text{for } \boldsymbol{\eta} \in \mathbf{H}^{m+2}(\Omega)\,,$$

and

$$\|(E^h E^{-1} - I) g\|_0 \le C h^m \|g\|_m \qquad \text{for } g \in H^m(\Omega)\,.$$

In these estimates, the constant C is independent of h, $\boldsymbol{\psi}$, g, $\boldsymbol{\eta}$, and λ. □

Remark. In fact, we obtain from (4.15) that $g = -\eta_2$ so that the term $\|g(\lambda)\|_m$ in the right-hand side of the error estimate is redundant. □

Remark. By using (4.15) again, along with (4.24) and the error estimate in Theorem 4.3, we have the following improved error estimate for the approximation of the control g:

$$\|g(\lambda) - g^h(\lambda)\|_2 = \|\eta_2(\lambda) - \eta_2^h(\lambda)\|_2 \le C h^m \left(\|\boldsymbol{\psi}(\lambda)\|_{m+2} + \|\boldsymbol{\eta}(\lambda)\|_{m+2}\right).$$

Of course, we also use the fact that we have chosen $G^h = V^h \subset H^2(\Omega)$. $\square$

4.2. Neumann boundary controls for the Ginzburg-Landau superconductivity equations

For this application we will use Neumann boundary controls, i.e., control is effected through the data in a Neumann boundary condition. Let Ω be a bounded open domain in $\mathbb{R}^d$, $d = 2$ or 3, and let Γ be its boundary. A simplified Ginzburg-Landau model for superconductivity is given by

$$-\Delta\psi_1 + \left(\psi_1^2 + \psi_2^2 + |\mathbf{A}|^2 - 1\right)\psi_1 - \nabla\cdot(\mathbf{A}\psi_2) - \mathbf{A}\cdot\nabla\psi_2 = 0 \quad \text{in } \Omega\,,$$

$$-\Delta\psi_2 + \left(\psi_1^2 + \psi_2^2 + |\mathbf{A}|^2 - 1\right)\psi_2 + \nabla\cdot(\mathbf{A}\psi_1) + \mathbf{A}\cdot\nabla\psi_1 = 0 \quad \text{in } \Omega\,,$$

$$\mathbf{n}\cdot(\nabla\psi_1 + \mathbf{A}\psi_2) = \lambda g_1 \quad \text{on } \Gamma\,,$$

and

$$\mathbf{n}\cdot(\nabla\psi_2 - \mathbf{A}\psi_1) = \lambda g_2 \quad \text{on } \Gamma\,.$$

Here, ψ_1 and ψ_2 denote the real and imaginary parts, respectively, of the complex-valued order parameter, $\mathbf{A}$ is a given real magnetic potential, g_1 and g_2 are related to the normal component of the current at the boundary, and $\lambda > 0$ is a "current loading" parameter. These equations are a special case of a more general model for superconductivity wherein $\mathbf{A}$ is also unknown; see, e.g., [22] for a derivation of the general model. It can be shown that in certain limits, e.g, high values of the applied field, the above simpler model is valid; see [7].

By introducing appropriate rescalings, i.e, by replacing ψ_j by $\lambda\psi_j$ and g_j by λg_j, $j = 1, 2$, we can rewrite the above Ginzburg-Landau equations as follows:

$$-\Delta\psi_1 + (|\mathbf{A}|^2 - 1)\psi_1 - \nabla\cdot(\mathbf{A}\psi_2) - \mathbf{A}\cdot\nabla\psi_2 + \lambda\left(\psi_1^2 + \psi_2^2\right)\psi_1 = 0 \quad \text{in } \Omega\,, \tag{4.25}$$

$$-\Delta\psi_2 + (|\mathbf{A}|^2 - 1)\psi_2 + \nabla\cdot(\mathbf{A}\psi_1) + \mathbf{A}\cdot\nabla\psi_1 + \lambda\left(\psi_1^2 + \psi_2^2\right)\psi_2 = 0 \quad \text{in } \Omega\,, \tag{4.26}$$

$$\mathbf{n}\cdot(\nabla\psi_1 + \mathbf{A}\psi_2) = \lambda g_1 \quad \text{on } \Gamma\,, \tag{4.27}$$

and

$$\mathbf{n}\cdot(\nabla\psi_2 - \mathbf{A}\psi_1) = \lambda g_2 \quad \text{on } \Gamma\,. \tag{4.28}$$

We introduce the bilinear forms

$$a(\psi,\phi) = \int_\Omega \Big(\nabla\psi\cdot\nabla\phi + (|\mathbf{A}|^2 - 1)\psi\phi\Big)\, d\Omega \quad \forall\, \psi,\phi \in H^1(\Omega)$$

and

$$b(\psi,\phi) = \int_\Omega \mathbf{A}\cdot(\psi\nabla\phi - \phi\nabla\psi)\; d\Omega \quad \forall\, \psi,\phi \in H^1(\Omega)\,.$$

We assume that $\mathbf{A} \in \mathbf{H}^1(\Omega)$. Note that

$$a(\psi,\phi) = a(\phi,\psi) \quad \text{and} \quad b(\psi,\phi) = -b(\phi,\psi)\,.$$

Then, a weak formulation of the Ginzburg-Landau equations (4.25)-(4.28) is defined as follows: seek $\boldsymbol{\psi} = (\psi_1, \psi_2) \in \mathbf{H}^1(\Omega)$ such that

$$a(\psi_1,\phi_1) + b(\psi_2,\phi_1) + \lambda\big((\psi_1^2+\psi_2^2)\psi_1,\phi_1\big) = \lambda\langle g_1,\phi_1\rangle_\Gamma \quad \forall\ \phi_1 \in H^1(\Omega) \tag{4.29}$$

and

$$a(\psi_2,\phi_2) - b(\psi_1,\phi_2) + \lambda\big((\psi_1^2+\psi_2^2)\psi_2,\phi_2\big) = \lambda\langle g_2,\phi_2\rangle_\Gamma \quad \forall\ \phi_2 \in H^1(\Omega)\,. \tag{4.30}$$

It can be shown that, for each $\mathbf{g} = (g_1, g_2) \in \mathbf{H}^{-1/2}(\Gamma)$, (4.29) and (4.30) possess at least one solution $\boldsymbol{\psi} \in \mathbf{H}^1(\Omega)$ and that all solutions of (4.29) and (4.30) satisfy the a priori estimate

$$\|\psi_1\|_1 + \|\psi_2\|_1 \le C\left(\|g_1\|_{-1/2,\Gamma} + \|g_2\|_{-1/2,\Gamma}\right)\ ; \tag{4.31}$$

see, e.g., [11], for details. In the sequel, a solution of (4.25)-(4.28) will be understood in the sense of (4.29)-(4.30).

Given a desired state $\boldsymbol{\psi}_0 = (\psi_{10}, \psi_{20}) \in \mathbf{L}^2(\Omega)$, we define for any $\boldsymbol{\psi} = (\psi_1, \psi_2) \in \mathbf{H}^1(\Omega)$ and $\mathbf{g} = (g_1, g_2) \in \mathbf{L}^2(\Gamma)$ the functional

$$\mathcal{J}(\boldsymbol{\psi},\mathbf{g}) = \frac{\lambda}{2}\int_\Omega \Big((\psi_1-\psi_{10})^2 + (\psi_2-\psi_{20})^2\Big)\,d\Omega + \frac{\lambda}{2}\int_\Gamma (g_1^2+g_2^2)\,d\Gamma\,. \tag{4.32}$$

We then consider the following optimal control problem associated with the Ginzburg-Landau equations for superconductivity:

$$\min\big\{\,\mathcal{J}(\boldsymbol{\psi},\mathbf{g}) \mid \boldsymbol{\psi}\in\mathbf{H}^1(\Omega)\,,\ \mathbf{g}\in\Theta\,\big\} \quad \text{subject to} \quad (4.29)\text{ and }(4.30)\,, \tag{4.33}$$

where Θ is a subset of $\mathbf{L}^2(\Gamma)$.

We define the spaces $X = \mathbf{H}^1(\Omega)$, $Y = (\mathbf{H}^1(\Omega))^*$, $G = \mathbf{L}^2(\Gamma)$, and $Z = [\mathbf{H}^{1/2+\epsilon}(\Omega)]^*$ where $\epsilon \in (0, 1/2)$ is chosen such that $\mathbf{H}^1(\Omega) \hookrightarrow\hookrightarrow \mathbf{H}^{1/2+\epsilon}(\Omega) \hookrightarrow \mathbf{L}^4(\Omega)$. By compact imbedding results, $\mathbf{L}^{4/3}(\Omega) \hookrightarrow Z \hookrightarrow\hookrightarrow Y$. For the time being, we assume that the admissible set Θ for the control $\mathbf{g}$ is a closed convex subset of $G = \mathbf{L}^2(\Gamma)$.

Let the continuous linear operator $T \in \mathcal{L}(Y; X)$ be defined as follows: for each $\mathbf{f} = (f_1, f_2) \in Y = (\mathbf{H}^1(\Omega))^*$, $T\mathbf{f} = \boldsymbol{\psi} \in X = \mathbf{H}^1(\Omega)$ is the unique solution of

$$a(\psi_1,\phi_1) + b(\psi_2,\phi_1) = \langle f_1,\phi_1\rangle \quad \forall\ \phi_1 \in H^1(\Omega)$$

and

$$a(\psi_2,\phi_2) - b(\psi_1,\phi_2) = \langle f_2,\phi_2\rangle \quad \forall\ \phi_2 \in H^1(\Omega)\,.$$

It can be easily verified that T is self-adjoint. Also, it can be shown that for most choices of $\mathbf{A}$, the operator T is well defined; see [11].

We define the (differentiable) nonlinear mapping $N : X \to Y$ by

$$N(\boldsymbol{\psi}) = \begin{pmatrix} (\psi_1^2+\psi_2^2)\psi_1 \\ (\psi_1^2+\psi_2^2)\psi_2 \end{pmatrix} \quad \forall\ \boldsymbol{\psi} \in X$$

or equivalently

$$\langle N(\boldsymbol{\psi}),\boldsymbol{\phi}\rangle = \big((\psi_1^2+\psi_2^2)\psi_1,\phi_1\big) + \big((\psi_1^2+\psi_2^2)\psi_2,\phi_2\big) \quad \forall\ \boldsymbol{\phi} = (\phi_1,\phi_2) \in X$$

and define $K : \mathbf{H}^{-1/2}(\Gamma) \to Y$ as the injection mapping:

$$\langle K\mathbf{z}, \mathbf{v}\rangle = -\langle \mathbf{z}, \mathbf{v}\rangle_\Gamma \quad \forall\, \mathbf{z} \in \mathbf{H}^{-1/2}(\Gamma)\,, \forall\, \mathbf{v} \in \mathbf{H}^1(\Omega)\,.$$

Clearly, the constraint equations (4.29)-(4.30) can be expressed as

$$\boldsymbol{\psi} + \lambda T N(\boldsymbol{\psi}) + \lambda T K \mathbf{g} = 0\,,$$

i.e., in the form (2.2). With the obvious definitions for $\mathcal{F}(\cdot)$ and $\mathcal{E}(\cdot)$, i.e.,

$$\mathcal{F}(\boldsymbol{\psi}) = \frac{1}{2}\int_\Omega \Big((\psi_1 - \psi_{10})^2 + (\psi_2 - \psi_{20})^2 \Big)\, d\Omega \quad \forall\, \boldsymbol{\psi} \in X$$

and

$$\mathcal{E}(\mathbf{g}) = \frac{1}{2}\int_\Gamma (g_1^2 + g_2^2)\, d\Gamma \quad \forall\, \mathbf{g} \in G\,,$$

the functional (4.32) can be expressed as

$$\mathcal{J}(\boldsymbol{\psi}, g) = \lambda\, \mathcal{F}(\boldsymbol{\psi}) + \lambda\, \mathcal{E}(\mathbf{g})\,,$$

i.e., in the form (2.1). Thus, the minimization problem (4.33) is in the form of the minimization problem (2.3).

We are now in a position to verify, for the minimization problem (4.33), all the hypotheses of §2 and §3.

4.2.1. Verification of the hypotheses for the existence of optimal solutions.

We first verify that the hypotheses (H1)-(H6) hold in the current setting.

(H1) is obviously satisfied with a lower bound 0.

(H2) holds with $\alpha = 1$ and $\beta = 2$.

(H3) is verified since $\boldsymbol{\psi} = \mathbf{0}$ and $\mathbf{g} = \mathbf{0}$ is obviously a solution of (4.29)-(4.30).

In order to verify (H4), we assume $\{\mathbf{g}^{(n)}\} \subset \Theta \subset \mathbf{L}^2(\Gamma)$ is a sequence satisfying $\mathbf{g}^{(n)} \rightharpoonup \mathbf{g}$ in $\mathbf{L}^2(\Gamma)$; then, we have $\mathbf{g}^{(n)} \rightharpoonup \mathbf{g}$ in $\mathbf{H}^{-1/2}(\Gamma)$ so that $\lim_{n\to\infty}\langle \mathbf{g}^{(n)}, \mathbf{v}\rangle_\Gamma = \langle \mathbf{g}, \mathbf{v}\rangle_\Gamma$ for all $\mathbf{v} \in \mathbf{H}^1(\Omega)$, i.e., $K\mathbf{g}^{(n)} \rightharpoonup K\mathbf{g}$ in Y. Assume that the sequence $\{\boldsymbol{\psi}^{(n)}\} \subset \mathbf{H}^1(\Omega)$ satisfies $\boldsymbol{\psi}^{(n)} \rightharpoonup \boldsymbol{\psi}$ in $\mathbf{H}^1(\Omega)$; then, by using the compact imbedding $\mathbf{H}^1(\Omega) \hookrightarrow\hookrightarrow \mathbf{L}^4(\Omega)$, $\boldsymbol{\psi}^{(n)} \to \boldsymbol{\psi}$ in $\mathbf{L}^4(\Omega)$. Now,

$$\begin{aligned}\langle N(\boldsymbol{\psi}^{(n)}), \boldsymbol{\phi}\rangle &= \Big(((\psi_1^{(n)})^2 + (\psi_2^{(n)})^2)\psi_1^{(n)}, \phi_1 \Big) + \Big(((\psi_1^{(n)})^2 + (\psi_2^{(n)})^2)\psi_2^{(n)}, \phi_2 \Big) \\ &\to \Big((\psi_1^2 + \psi_2^2)\psi_1, \phi_1 \Big) + \Big((\psi_1^2 + \psi_2^2)\psi_2, \phi_2 \Big) = \langle N(\boldsymbol{\psi}), \boldsymbol{\phi}\rangle\,.\end{aligned}$$

Hence, (H4) is verified.

The verification of (H5) follows directly from the observation that the mappings $\boldsymbol{\phi} \mapsto \mathcal{F}(\boldsymbol{\phi}) = (1/2)\|\boldsymbol{\phi} - \boldsymbol{\psi}_0\|_0^2$ and $\mathbf{g} \mapsto \mathcal{E}(\mathbf{g}) = (1/2)\|\mathbf{g}\|_{0,\Gamma}^2$ are convex.

The verification of (H6) is a trivial consequence of the a priori estimate (4.31).

It is now just a matter of citing Theorem 2.1 to prove the existence of an optimal solution that minimizes (4.32) subject to (4.29)-(4.30).

THEOREM 4.4. *There exists a* $(\boldsymbol{\phi}, \mathbf{g}) \in \mathbf{H}^1(\Omega) \times \Theta$ *such that* (4.32) *is minimized subject to* (4.29)-(4.30). □

4.2.2. Verification of the hypotheses for the existence of Lagrange multipliers.

We now assume $(\boldsymbol{\psi}, \mathbf{g})$ is an optimal solution and turn to the verification of hypotheses (H7)-(H9).

The validity of (H7) is obvious.

(H8) holds since the mapping $\mathbf{g} \mapsto \mathcal{E}(\mathbf{g}) = (1/2)\int_\Gamma |\mathbf{g}|^2\, d\Gamma$ is convex.

(H9) can be verified as follows. For any $\boldsymbol{\psi} \in X$, the operator $N'(\boldsymbol{\psi}) : X \to Y$ is given by

$$N'(\boldsymbol{\psi}) \cdot \boldsymbol{\phi} = \begin{pmatrix} (3\psi_1^2 + \psi_2^2)\phi_1 + (2\psi_1\psi_2)\phi_2 \\ (3\psi_2^2 + \psi_1^2)\phi_2 + (2\psi_1\psi_2)\phi_1 \end{pmatrix} \quad \forall\, \boldsymbol{\phi} = (\phi_1, \phi_2) \in X\,.$$

Thus, we obtain that $N'(\boldsymbol{\psi}) \cdot \boldsymbol{\phi} \in \mathbf{L}^{4/3}(\Omega) \hookrightarrow [\mathbf{H}^{1/2+\epsilon}(\Omega)]^* = Z$.

The Lagrangian is given by

$$\begin{aligned} \mathcal{L}(\boldsymbol{\psi}, \mathbf{g}, \boldsymbol{\eta}, k) =& k\, \mathcal{J}(\boldsymbol{\psi}, \mathbf{g}) \\ &- \Big\{ a(\psi_1, \eta_1) + b(\psi_2, \phi_1) + \lambda\big((\psi_1^2 + \psi_2^2)\psi_1, \eta_1\big) - \lambda(g_1, \eta_1)_\Gamma \\ &\quad + a(\psi_2, \eta_2) - b(\psi_1, \phi_2) - \lambda\big((\psi_1^2 + \psi_2^2)\psi_2, \eta_2\big) - \lambda(g_2, \eta_2)_\Gamma \Big\} \end{aligned}$$

for all $(\boldsymbol{\psi}, \mathbf{g}, \boldsymbol{\eta}, k) \in X \times G \times X \times \mathbb{R} = \mathbf{H}^1(\Omega) \times \mathbf{L}^2(\Gamma) \times \mathbf{H}^1(\Omega) \times \mathbb{R}$. Note that in this form of the Lagrangian, the Lagrange multiplier $\boldsymbol{\eta} \in X = Y^*$ so that we have already introduced the change of variables indicated between (2.17)-(2.18) and (2.19)-(2.21).

Having verified the hypotheses (H7)-(H9), we may apply Theorem 2.4 to conclude that there exists a Lagrange multiplier $\boldsymbol{\eta} \in X = \mathbf{H}^1(\Omega)$ and a real number k such that

$$\boldsymbol{\eta} + \lambda\, T^*\big([N'(\boldsymbol{\psi})]^* \cdot \boldsymbol{\eta} - k\, \mathcal{J}_{\boldsymbol{\psi}}(\boldsymbol{\psi}, g)\big) = \mathbf{0} \tag{4.34}$$

and

$$\mathcal{L}(\boldsymbol{\psi}, g, \boldsymbol{\eta}, k) \le \mathcal{L}(\boldsymbol{\psi}, z, \boldsymbol{\eta}, k) \quad \forall\, z \in \Theta \tag{4.35}$$

and that for almost all values of λ, we may choose $k = 1$.

Recall that T is self-adjoint. Also, note that for any $\boldsymbol{\psi} \in X = \mathbf{H}^1(\Omega)$,

$$[N'(\boldsymbol{\psi})]^* \cdot \boldsymbol{\eta} = \begin{pmatrix} (3\psi_1^2 + \psi_2^2)\eta_1 + (2\psi_1\psi_2)\eta_2 \\ (3\psi_2^2 + \psi_1^2)\eta_2 + (2\psi_1\psi_2)\eta_1 \end{pmatrix} \quad \forall\, \boldsymbol{\eta} = (\eta_1, \eta_2) \in X\,.$$

Thus, $N'(\boldsymbol{\psi})$ is self-adjoint as well and (4.34), with $k = 1$, can be rewritten as

$$\begin{aligned} a(\zeta_1, \eta_1) - b(\zeta_1, \eta_2) &+ \lambda\big((3\psi_1^2 + \psi_2^2)\eta_1, \zeta_1\big) \\ &+ \lambda\big((2\psi_1\psi_2)\eta_2, \zeta_1\big) = \lambda(\psi_1 - \psi_{10}, \zeta_1) \quad \forall\, \zeta_1 \in H^1(\Omega) \end{aligned} \tag{4.36}$$

and

$$\begin{aligned} a(\zeta_2, \eta_2) + b(\zeta_2, \eta_1) &+ \lambda\big((3\psi_2^2 + \psi_1^2)\eta_2, \zeta_2\big) \\ &+ \lambda\big((2\psi_1\psi_2)\eta_1, \zeta_2\big) = \lambda(\psi_2 - \psi_{20}, \zeta_2) \quad \forall\, \zeta_2 \in H^1(\Omega)\,. \end{aligned} \tag{4.37}$$

Using the definition of the Lagrangian functional, (4.35), with $k = 1$, can be rewritten as

$$\frac{\lambda}{2}\,(\mathbf{z}, \mathbf{z})_\Gamma + \lambda(\mathbf{z}, \boldsymbol{\eta})_\Gamma - \frac{\lambda}{2}\,(\mathbf{g}, \mathbf{g})_\Gamma - \lambda(\mathbf{g}, \boldsymbol{\eta})_\Gamma \ge 0 \quad \forall\, \mathbf{z} \in \Theta\,.$$

Note that, in the above expression, we have already employed hypothesis (H12) which in the current context is trivially satisfied with E the identity operator on $G^* = G = \mathbf{L}^2(\Gamma)$. For each $\epsilon \in (0,1)$ and each $\mathbf{t} \in \Theta$, set $\mathbf{z} = \epsilon\mathbf{t} + (1-\epsilon)\mathbf{g} \in \Theta$ in the last equation to obtain

$$\frac{\epsilon^2}{2}(\mathbf{t}-\mathbf{g},\mathbf{t}-\mathbf{g})_\Gamma + \epsilon\,(\mathbf{t}-\mathbf{g},\mathbf{g})_\Gamma + \epsilon\,(\mathbf{t}-\mathbf{g},\boldsymbol{\eta})_\Gamma \geq 0 \quad \forall\, \mathbf{t} \in \Theta$$

so that, after dividing by $\epsilon > 0$ and then letting $\epsilon \to 0^+$, we obtain

$$(\mathbf{t}-\mathbf{g},\mathbf{g}+\boldsymbol{\eta})_\Gamma \geq 0 \quad \forall\, \mathbf{t} \in \Theta\,. \tag{4.38}$$

We see that for almost all values of λ, necessary conditions for an optimum are that (4.29)-(4.30) and (4.36)-(4.38) are satisfied. Again, the system formed by these equations will be called an *optimality system.*

We now specialize to the case $\Theta = \mathbf{L}^2(\Gamma)$. Note that the hypothesis (H10) is satisfied. Then, using Theorem 2.5, we see that the inequality (4.38) becomes an equality and, by letting $\mathbf{z} = \mathbf{t} - \mathbf{g}$ vary arbitrarily in $\mathbf{L}^2(\Gamma)$, we now have, instead of (4.38),

$$(\mathbf{z},\mathbf{g}+\boldsymbol{\eta})_\Gamma = 0 \quad \forall\, \mathbf{z} \in \mathbf{L^2}(\Gamma)\,. \tag{4.39}$$

Thus, according to that theorem, we have that for almost all λ, an optimality system of equations is now given by (4.29)-(4.30), (4.36)-(4.37), and (4.39). However, we can go further and verify that the hypothesis (H11) is valid, which in turn will justify the existence of a Lagrange multiplier satisfying the optimality system for *all* $\lambda \in \Lambda$.

To verify (H11), we first note that, through the change of variable $\xi = T^* v$, that assumption can be equivalently stated as follows:

if $\xi \in Y^*$ *satisfies* $\big(I + \lambda T^*[N'(u)]^*\big)\xi = 0$ *and* $K^*\xi = 0$, *then* $\xi = 0$.

To verify this version of (H11), we assume that $\boldsymbol{\xi} \in Y^* = \mathbf{H}^1(\Omega)$ satisfies $\big(I + \lambda T^*[N'(\boldsymbol{\psi})]^*\big)\boldsymbol{\xi} = \mathbf{0}$ and $K^*\boldsymbol{\xi} = 0$, i.e.,

$$a(\zeta_1,\xi_1) - b(\zeta_1,\xi_2) + \lambda\left((3\psi_1^2+\psi_2^2)\xi_1,\zeta_1\right) + \lambda\left((2\psi_1\psi_2)\xi_2,\zeta_1\right) = 0 \quad \forall\, \zeta_1 \in H^1(\Omega)\,,$$

$$a(\zeta_2,\xi_2) + b(\zeta_2,\xi_1) + \lambda\left((3\psi_2^2+\psi_1^2)\xi_2,\zeta_2\right) + \lambda\left((2\psi_1\psi_2)\xi_1,\zeta_2\right) = 0 \quad \forall\, \zeta_2 \in H^1(\Omega)\,,$$

and

$$\boldsymbol{\xi} = \mathbf{0} \quad \text{on } \Gamma\,.$$

(Note that $K^*\boldsymbol{\xi} = \boldsymbol{\xi}|_\Gamma$.) Let Ω' be a smooth extension of Ω such that $\overline{\Omega}$ is a compact subset of Ω'. We then define $\boldsymbol{\xi}'$, $\boldsymbol{\psi}'$ and $\mathbf{A}'$ to be the extension, by zero outside Ω, of $\boldsymbol{\xi}$, $\boldsymbol{\psi}$ and $\mathbf{A}$, respectively. Let the forms $a'(\cdot,\cdot)$, $b'(\cdot,\cdot)$, and $(\cdot,\cdot)'$ defined over Ω' be the analogues of corresponding forms defined over Ω. We may show from the last three equations that

$$\boldsymbol{\xi}' \in \mathbf{H}^1(\Omega')\,, \qquad \boldsymbol{\psi}' \in \mathbf{L}^6(\Omega')\,,$$

$$a'(\zeta_1,\xi_1') - b'(\zeta_1,\xi_2') + \lambda\left((3{\psi_1'}^2+{\psi_2'}^2)\xi_1',\zeta_1\right)' + \lambda\left((2\psi_1'\psi_2')\xi_2',\zeta_1\right)' = 0 \quad \forall\, \zeta_1 \in \mathbf{H}_0^1(\Omega')\,,$$

and

$$a'(\zeta_2,\xi_2') + b^{'}(\zeta_2,\xi_1') + \lambda\left((3{\psi_2'}^2 + {\psi_1'}^2)\xi_2',\zeta_2\right)' + \lambda\left((2\psi_1'\psi_2')\xi_1',\zeta_2\right)' = 0 \quad \forall\,\zeta_2\in\mathbf{H}^1_0(\Omega^{'})\,.$$

In the sense of distribution, $\boldsymbol{\xi}'$ satisfies

$$-\Delta\xi_1' - 2\mathbf{A}'\cdot\nabla\xi_2' + \left(|\mathbf{A}'|^2 + \lambda\,(3{\psi_1'}^2 + {\psi_2'}^2) - 1\right)\xi_1' - (\nabla\cdot\mathbf{A}' - 2\lambda\,\psi_1'\psi_2')\xi_2' = 0 \quad \text{in } \Omega^{'} \tag{4.40}$$

and

$$-\Delta\xi_2' + 2\mathbf{A}'\cdot\nabla\xi_1' + (\nabla\cdot\mathbf{A}' + 2\lambda\,\psi_1'\psi_2')\xi_1' + \left(|\mathbf{A}'|^2 + \lambda\,(3{\psi_2'}^2 + {\psi_1'}^2) - 1\right)\xi_2' = 0 \quad \text{in } \Omega^{'}\,. \tag{4.41}$$

We now quote the following unique continuation result whose proof can be found in [17]. See also [12] and [19].

LEMMA 4.5. *Let $\Omega^{'}$ be an open and connected subset of $\mathbb{R}^d$, $d=2$ or 3. Let the functions $\mathbf{V}\in[L^q_{\rm loc}(\Omega^{'})]^{d\times d}$ for some $q\geq 2$ and $\mathbf{W}\in[L^{2d-1}_{\rm loc}(\Omega^{'})]^{d\times d\times d}$ be given. If $\boldsymbol{\xi}\in\mathbf{H}^1_{\rm loc}(\Omega^{'})$, $-\Delta\xi_i + \sum_{j=1}^d\sum_{k=1}^d W_{ijk}(\partial\xi_k/\partial x_j) + \sum_{j=1}^d V_{ij}\xi_j = 0$ (in the sense of distributions), $i=1,...,d$, and $\boldsymbol{\xi}=\mathbf{0}$ on an open, non-empty subset of $\Omega^{'}$, then $\boldsymbol{\xi}=\mathbf{0}$ on $\Omega^{'}$.* □

Since $\mathbf{A}\in\mathbf{H}^1(\Omega)$ and $\boldsymbol{\psi}\in\mathbf{H}^1(\Omega)$, it is easy to see that the coefficients in (4.40)-(4.41) satisfy the regularity requirements of Lemma 4.5. Also note that $\boldsymbol{\xi}'=\mathbf{0}$ on $(\Omega^{'}\setminus\Omega)$ which contains an open set. Thus we obtain that $\boldsymbol{\xi}'=\mathbf{0}$ in $\Omega^{'}$, or $\boldsymbol{\xi}=\mathbf{0}$ in Ω, i.e., (H11) is verified.

Hence we conclude that for *all* λ, the optimality system (4.29)-(4.30), (4.36)-(4.37), and (4.39) has a solution. Thus, we have Theorem 2.6 which, in the present context, is given as follows.

THEOREM 4.6. *Let $(\boldsymbol{\psi},\mathbf{g})\in\mathbf{H}^1(\Omega)\times\mathbf{L}^2(\Gamma)$ denote an optimal solution that minimizes (4.32) subject to (4.29)-(4.30). Then, for all $\lambda\in\Lambda$, there exists a nonzero Lagrange multiplier $\boldsymbol{\eta}\in\mathbf{H}^1(\Omega)$ satisfying the Euler equations (4.36)-(4.37) and (4.39).* □

4.2.3. Verification of the hypotheses for approximations and error estimates. We finally verify the hypotheses (H13)-(H19) that are used in connection with approximations and to derive error estimates.

A finite element discretization of the optimality system (4.29)-(4.30), (4.36)-(4.37), and (4.39) is defined in the usual manner. We first choose families of finite dimensional subspaces $X^h\subset\mathbf{H}^1(\Omega)$ and $G^h\subset\mathbf{L}^2(\Gamma)$ parameterized by a parameter h that tends to zero and satisfying the following approximation properties: there exists a constant C and an integer r such that

$$\inf_{\boldsymbol{\phi}^h\in X^h}\|\boldsymbol{\phi}-\boldsymbol{\phi}^h\|_1 \leq Ch^m\|\boldsymbol{\phi}\|_{m+1}\,, \quad \forall\,\boldsymbol{\phi}\in\mathbf{H}^{m+1}(\Omega)\,,\ 1\leq m\leq r \tag{4.42}$$

and

$$\inf_{z^h\in G^h}\|\mathbf{z}-\mathbf{z}^h\|_{0,\Gamma} \leq Ch^m \inf_{\mathbf{v}\in\mathbf{H}^{m+1/2}(\Omega),\,\mathbf{v}|_\Gamma=\mathbf{z}}\|\mathbf{v}\|_{m+1/2}\,, \quad \forall\,\mathbf{z}\in\mathbf{H}^{m+1/2}(\Omega)|_\Gamma\,,\ 1\leq m\leq r\,. \tag{4.43}$$

One may consult, e.g., [8] and [15], for some finite element spaces satisfying (4.42) and (4.43). For example, one may choose $X^h = V^h \times V^h$ where V^h is the piecewise linear or quadratic finite element space defined with respect to a family of triangulations of Ω. In this case, h is a measure of the grid size. For simplicity we may choose $G^h = (X^h)|_\Gamma$, i.e., the functions in G^h are the restrictions to the boundary Γ of functions belonging to X^h.

Once the approximating spaces have been chosen, we may formulate the approximate problem for the optimality system (4.29)-(4.30), (4.36)-(4.37), and (4.39): seek $\boldsymbol{\psi}^h \in X^h$, $\mathbf{g}^h \in G^h$, and $\boldsymbol{\eta}^h \in X^h$ such that

$$(4.44)\quad a(\psi_1^h,\phi_1^h) + b(\psi_2^h,\phi_1^h) + \lambda\big\{\big((\psi_1^h)^2 + (\psi_2^h)^2\big)\psi_1^h,\phi_1^h\big\} = \lambda\langle g_1^h,\phi_1^h\rangle_\Gamma \quad \forall\,\phi_1^h \in V^h\,,$$

$$(4.45)\quad a(\psi_2^h,\phi_2^h) - b(\psi_1^h,\phi_2^h) + \lambda\big\{\big((\psi_1^h)^2 + (\psi_2^h)^2\big)\psi_2^h,\phi_2^h\big\} = \lambda\langle g_2^h,\phi_2^h\rangle_\Gamma \quad \forall\,\phi_2^h \in V^h\,,$$

$$(4.46)\quad \begin{aligned} a(\zeta_1^h,\eta_1^h) - b(\zeta_1^h,\eta_2^h) &+ \lambda\big((3(\psi_1^h)^2 + (\psi_2^h)^2)\eta_1^h,\zeta_1^h\big)\\ &+ \lambda\big((2\psi_1^h\psi_2^h)\eta_2^h,\zeta_1^h\big) = \lambda(\psi_1^h - \psi_{10},\zeta_1^h) \quad \forall\,\zeta_1^h \in V^h\,, \end{aligned}$$

$$(4.47)\quad \begin{aligned} a(\zeta_2^h,\eta_2^h) + b(\zeta_2^h,\eta_1^h) &+ \lambda\big((3(\psi_2^h)^2 + (\psi_1^h)^2)\eta_2^h,\zeta_2^h\big)\\ &+ \lambda\big((2\psi_1^h\psi_2^h)\eta_1^h,\zeta_2^h\big) = \lambda(\psi_2^h - \psi_{20},\zeta_2^h) \quad \forall\,\zeta_2^h \in V^h\,, \end{aligned}$$

and

$$(4.48)\qquad (\mathbf{z}^h, \mathbf{g}^h + \boldsymbol{\eta}^h)_\Gamma = 0 \quad \forall\,\mathbf{z}^h \in G^h\,.$$

The operator $T^h \in \mathcal{L}(Y;X^h)$ is defined as follows: for $\mathbf{f} \in Y$, $T^h\mathbf{f} = \boldsymbol{\psi}^h \in X^h$ is the solution for

$$a(\psi_1^h,\phi_1^h) + b(\psi_2^h,\phi_1^h) = \langle f_1,\phi_1^h\rangle \quad \forall\,\phi_1^h \in V^h$$

and

$$a(\psi_2^h,\phi_2^h) - b(\psi_1^h,\phi_2^h) = \langle f_2,\phi_2^h\rangle \quad \forall\,\phi_2^h \in V^h\,.$$

Since $T = T^*$, we define $(T^*)^h = T^h$.

We define the operator $E^h : \mathbf{L}^2(\Gamma) \to G^h$ as the $\mathbf{L}^2(\Gamma)$-projection on G^h, i.e., for each $\mathbf{g} \in \mathbf{L}^2(\Gamma)$,

$$(E^h\mathbf{g},\mathbf{z}^h)_\Gamma = (\mathbf{g},\mathbf{z}^h)_\Gamma \quad \forall\,\mathbf{z}^h \in G^h\,.$$

Since $G = \mathbf{L}^2(\Gamma)$ is reflexive, E^h is in fact an operator from $G^* \to G^h$.

By results concerning the approximation of the Ginzburg-Landau equations (see, e.g., [11]), we obtain

$$\|(T - T^h)\mathbf{f}\|_X \to 0$$

as $h \to 0$, for all $\mathbf{f} \in Y$. This is simply a restatement of (H13).

(H14) follows trivially from (H13) and the fact that T is self-adjoint and we have chosen $(T^*)^h = T^h$.

(H15) follows from the best approximation property of $\mathbf{L}^2(\Gamma)$-projections and (4.43).

(H16) and (H17) follow from the fact that N and $\mathcal{F}$ are polynomials. Here we also use imbedding theorems and Cauchy inequalities.

Setting $\hat{Z} = Z = \mathbf{H}^{1/2+\epsilon}(\Omega)$, we have that $\hat{Z} \hookrightarrow\hookrightarrow [\mathbf{H}^1(\Omega)]^* = X^*$. For each $\boldsymbol{\eta} \in \mathbf{H}^1(\Omega)$ and $\boldsymbol{\zeta} \in \mathbf{H}^1(\Omega)$, Sobolev imbedding theorems imply that

$$[N'(\boldsymbol{\psi})]^* \cdot \boldsymbol{\eta} = \begin{pmatrix} (3\psi_1^2 + \psi_2^2)\eta_1 + (2\psi_1\psi_2)\eta_2 \\ (3\psi_2^2 + \psi_1^2)\eta_2 + (2\psi_1\psi_2)\eta_1 \end{pmatrix} \in \mathbf{L}^{4/3}(\Omega) \subset \hat{Z}\,,$$

$$\big([N''(\boldsymbol{\psi})]^* \cdot \boldsymbol{\zeta}\big) \cdot \boldsymbol{\eta} = \begin{pmatrix} (6\psi_1\zeta_1 + 2\psi_2\zeta_2)\eta_1 + (2\psi_1\zeta_2)\eta_2 + (2\zeta_1\psi_2)\eta_2 \\ (6\psi_2\zeta_2 + 2\psi_1\zeta_1)\eta_2 + (2\psi_1\zeta_2)\eta_1 + (2\zeta_1\psi_2)\eta_1 \end{pmatrix} \in \mathbf{L}^{4/3}(\Omega) \subset \hat{Z}\,,$$

and

$$\big(\mathcal{F}''(\boldsymbol{\psi}) \cdot \boldsymbol{\zeta}\big) \cdot \boldsymbol{\eta} = \begin{pmatrix} \eta_1\zeta_1 \\ \eta_2\zeta_2 \end{pmatrix} \in \mathbf{L}^{4/3}(\Omega) \subset \hat{Z}\,.$$

These relations verify (H18).

From the definition of the operator K we see that K maps $\mathbf{L}^2(\Gamma)$ into $[\mathbf{H}^{1/2+\epsilon}(\Omega)]^*$, i.e., K maps G into Z. Thus (H19) is verified.

Hence, we are now in a position to apply Theorem 3.5 to derive error estimates for the approximate solutions of the optimality system (4.29)-(4.30), (4.36)-(4.37), and (4.39). It should be noted that Lemma 3.4 implies that for almost all values of λ, the solutions of the optimality system are regular.

THEOREM 4.7. *Assume that Λ is a compact interval of $\mathbb{R}_+$ and that there exists a branch $\{(\lambda, \boldsymbol{\psi}(\lambda), \mathbf{g}(\lambda), \boldsymbol{\eta}(\lambda)) : \lambda \in \Lambda\}$ of regular solutions of the optimality system* (4.29)-(4.30), (4.36)-(4.37), *and* (4.39). *Assume that the finite element spaces X^h and G^h satisfy the hypotheses* (4.42)-(4.43). *Then, there exists a $\delta > 0$ and a $h_0 > 0$ such that for $h \le h_0$, the discrete optimality system* (4.44)-(4.48) *has a unique branch of solutions $\{\big(\lambda, \boldsymbol{\psi}^h(\lambda), g^h(\lambda), \boldsymbol{\eta}^h(\lambda)\big) : \lambda \in \Lambda\}$ satisfying*

$$\{\|\boldsymbol{\psi}^h(\lambda) - \boldsymbol{\psi}(\lambda)\|_1 + \|\mathbf{g}^h(\lambda) - \mathbf{g}(\lambda)\|_{0,\Gamma} + \|\boldsymbol{\eta}^h(\lambda) - \boldsymbol{\eta}(\lambda)\|_1\} < \delta \qquad \textit{for all } \lambda \in \Lambda.$$

Moreover,

$$\lim_{h\to 0}\{\|\boldsymbol{\psi}^h(\lambda) - \boldsymbol{\psi}(\lambda)\|_1 + \|\mathbf{g}^h(\lambda) - \mathbf{g}(\lambda)\|_{0,\Gamma} + \|\boldsymbol{\eta}^h(\lambda) - \boldsymbol{\eta}(\lambda)\|_1\} = 0\,,$$

uniformly in $\lambda \in \Lambda$.

If, in addition, the solution of the optimality system satisfies $\big(\boldsymbol{\psi}(\lambda), \mathbf{g}(\lambda), \boldsymbol{\eta}(\lambda)\big) \in \mathbf{H}^{m+1}(\Omega) \times \mathbf{H}^{m+1/2}(\Omega)|_\Gamma \times \mathbf{H}^{m+1}(\Omega)$ for $\lambda \in \Lambda$, then there exists a constant C, independent of h, such that

$$\begin{aligned}&\|\boldsymbol{\psi}(\lambda) - \boldsymbol{\psi}^h(\lambda)\|_1 + \|\mathbf{g}(\lambda) - \mathbf{g}^h(\lambda)\|_{0,\Gamma} + \|\boldsymbol{\eta}(\lambda) - \boldsymbol{\eta}^h(\lambda)\|_1 \\ &\quad \le Ch^m\big(\|\boldsymbol{\psi}(\lambda)\|_{m+1} + \inf_{\mathbf{v}\in\mathbf{H}^{m+1/2}(\Omega),\mathbf{v}|_\Gamma=\mathbf{g}} \|\mathbf{v}\|_{m+1/2} + \|\boldsymbol{\eta}(\lambda)\|_{m+1}\big)\,,\end{aligned}$$

uniformly in $\lambda \in \Lambda$.

Proof: All results follow from Theorem 3.5. For the last result, we also use (3.25) and the estimates (see [11])

$$\|(T^hT^{-1} - I)\boldsymbol{\psi}\|_1 \le Ch^m\|\boldsymbol{\psi}\|_{m+1} \qquad \text{for } \boldsymbol{\psi} \in \mathbf{H}^{m+1}(\Omega)\,,$$

$$\|\big((T^*)^h(T^*)^{-1} - I\big)\boldsymbol{\eta}\|_1 = \|(T^hT^{-1} - I)\boldsymbol{\eta}\|_1 \le Ch^m\|\boldsymbol{\eta}\|_{m+1} \qquad \text{for } \boldsymbol{\eta} \in \mathbf{H}^{m+1}(\Omega)\,,$$

and (see, e.g., [2], [8] and [15])

$$\|(E^h E^{-1} - I)\mathbf{g}\|_{0,\Gamma} \le C h^m \inf_{\mathbf{v}\in\mathbf{H}^{m+1/2}(\Omega), \mathbf{v}|_\Gamma = \mathbf{g}} \|\mathbf{v}\|_{m+1/2} \qquad \text{for } \mathbf{g} \in \mathbf{H}^{m+1/2}(\Omega)|_\Gamma \,.$$

In these estimates, the constant C is independent of h, $\boldsymbol{\psi}$, $\mathbf{g}$, $\boldsymbol{\eta}$, and λ. □

Remark. In fact, we obtain from (4.39) that $\mathbf{g} = -\boldsymbol{\eta}|_\Gamma$ which implies

$$\inf_{\mathbf{v}\in\mathbf{H}^{m+1/2}(\Omega), \mathbf{v}|_\Gamma = \mathbf{g}} \|\mathbf{v}\|_{m+1/2} \le \|\boldsymbol{\eta}\|_{m+1/2} \le \|\boldsymbol{\eta}\|_{m+1}$$

so that the term $\left(\inf_{\mathbf{v}\in\mathbf{H}^{m+1/2}(\Omega), \mathbf{v}|_\Gamma = \mathbf{g}} \|\mathbf{v}\|_{m+1/2}\right)$ in the right-hand side of the error estimate is redundant. □

Remark. By using (4.39) again, along with (4.48) and the error estimate in Theorem 4.7, we have the following improved error estimate for the approximation of the control $\mathbf{g}$:

$$\|\mathbf{g}(\lambda) - \mathbf{g}^h(\lambda)\|_{1/2,\Gamma} \le C\|\boldsymbol{\eta}(\lambda) - \boldsymbol{\eta}^h(\lambda)\|_1 \le C h^m \left(\|\boldsymbol{\psi}(\lambda)\|_{m+1} + \|\boldsymbol{\eta}(\lambda)\|_{m+1}\right) .$$

Of course, we also use the fact that we have chosen $G^h = (X^h)|_\Gamma \subset H^{1/2}(\Gamma)$. □

4.3. Dirichlet boundary control for the Navier-Stokes equations of incompressible, viscous flow

For this application we will use Dirichlet boundary controls, i.e., control is effected through the data in a Dirichlet boundary condition. Let Ω denote a bounded domain in $\mathbb{R}^d$, $d = 2$ or 3 with a boundary denoted by Γ. Let $\mathbf{u}$ and p denote the velocity and pressure fields in Ω. The Navier-Stokes equations for a viscous, incompressible flow are given by (see, e.g., [13], [14], or [20])

$$-\nu\,\nabla\cdot\left((\nabla\mathbf{u}) + (\nabla\mathbf{u})^T\right) + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega\,,$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega$$

and

$$\mathbf{u} = \mathbf{b} + \mathbf{g} \quad \text{on } \Gamma\,,$$

where $\mathbf{f}$ is a given body force, $\mathbf{b}$ and $\mathbf{g}$ are boundary velocity data with $\int_\Gamma \mathbf{b}\cdot\mathbf{n}\,d\Gamma = 0$ and $\int_\Gamma \mathbf{g}\cdot\mathbf{n}\,d\Gamma = 0$, and ν denotes the (constant) kinematic viscosity. We have absorbed the constant density into the pressure and the body force. If the variables in these equations are nondimensionalized, then ν is simply the inverse of the Reynolds number Re.

Setting $\lambda = 1/\nu = Re$ and replacing p with p/λ, $\mathbf{b}$ with $\lambda\mathbf{b}$, and $\mathbf{g}$ with $\lambda\mathbf{g}$, we may write the Navier-Stokes equations in the form

$$-\nabla\cdot\left((\nabla\mathbf{u}) + (\nabla\mathbf{u})^T\right) + \nabla p + \lambda\mathbf{u}\cdot\nabla\mathbf{u} = \lambda\mathbf{f} \quad \text{in } \Omega\,, \tag{4.49}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega\,, \tag{4.50}$$

and

$$\mathbf{u} = \lambda(\mathbf{b} + \mathbf{g}) \quad \text{on } \Gamma\,. \tag{4.51}$$

We introduce the subspaces

$$L^2_0(\Omega) = \left\{ p \in L^2(\Omega) \;\Big|\; \int_\Omega p\, d\Omega = 0 \right\}$$

and

$$\mathbf{H}^1_n(\Gamma) = \left\{ \mathbf{g} \in \mathbf{H}^1(\Gamma) \;\Big|\; \int_\Gamma \mathbf{g}\cdot\mathbf{n}\, d\Gamma = 0 \right\}.$$

We also introduce the bilinear forms

$$a(\mathbf{u},\mathbf{v}) = \frac{1}{2}\int_\Omega \left((\nabla\mathbf{u}) + (\nabla\mathbf{u})^T\right) : \left((\nabla\mathbf{v}) + (\nabla\mathbf{v})^T\right) d\Omega \quad \forall\, \mathbf{u},\mathbf{v} \in \mathbf{H}^1(\Omega)$$

and

$$b(\mathbf{v},q) = -\int_\Omega q\,\nabla\cdot\mathbf{v}\, d\Omega \quad \forall\, \mathbf{v}\in\mathbf{H}^1(\Omega) \text{ and } \forall\, p \in L^2(\Omega)$$

and the trilinear form

$$c(\mathbf{u},\mathbf{v},\mathbf{w}) = \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w}\, d\Omega \quad \forall\, \mathbf{u},\mathbf{v},\mathbf{w} \in \mathbf{H}^1(\Omega).$$

These forms are continuous over the spaces of definition indicated above. Moreover, we have the coercivity properties

$$a(\mathbf{v},\mathbf{v}) \geq C_a\|\mathbf{v}\|_1^2 \quad \forall\, \mathbf{v}\in\mathbf{H}^1_0(\Omega) \tag{4.52}$$

and

$$\sup_{\mathbf{0}\neq\mathbf{v}\in\mathbf{H}^1_0(\Omega)} \frac{b(\mathbf{v},q)}{\|\mathbf{v}\|_1} \geq C_b\|q\|_0 \quad \forall\, q \in L^2_0(\Omega) \tag{4.53}$$

for some constants C_a and $C_b > 0$. For details concerning the notation employed and/or for (4.52)-(4.53), one may consult [13], [14], and [20].

We recast the Navier-Stokes equations (4.49)-(4.51) into the following particular weak form (see, e.g., [15]): seek $(\mathbf{u},p,\mathbf{t}) \in \mathbf{H}^1(\Omega)\times L^2_0(\Omega)\times\mathbf{H}^{-1/2}(\Gamma)$ such that

$$a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) - \langle\mathbf{t},\mathbf{v}\rangle_\Gamma + \lambda\, c(\mathbf{u},\mathbf{u},\mathbf{v}) = \lambda\,\langle\mathbf{f},\mathbf{v}\rangle \quad \forall\, \mathbf{v}\in\mathbf{H}^1(\Omega), \tag{4.54}$$

$$b(\mathbf{u},q) = 0 \quad \forall\, q \in L^2_0(\Omega), \tag{4.55}$$

and

$$\langle\mathbf{s},\mathbf{u}\rangle_\Gamma - \lambda\,\langle\mathbf{s},\mathbf{g}\rangle_\Gamma = \lambda\,\langle\mathbf{s},\mathbf{b}\rangle_\Gamma \quad \forall\, \mathbf{s}\in\mathbf{H}^{-1/2}(\Gamma). \tag{4.56}$$

Formally we have

$$\mathbf{t} = \left[-p\mathbf{n} + \left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right)\cdot\mathbf{n}\right]_\Gamma,$$

i.e., $\mathbf{t}$ is the stress force on the boundary. The existence of a solution $(\mathbf{u},p,\mathbf{t})$ for the system (4.54)-(4.56) was established in [15].

Given a desired velocity field $\mathbf{u}_0$, we define for any $(\mathbf{u}, p, \mathbf{t}) \in \mathbf{H}^1(\Omega) \times L^2_0(\Omega) \times \mathbf{H}^{-1/2}(\Gamma)$ and $\mathbf{g} \in \mathbf{H}^1_n(\Gamma)$ the functional

$$\mathcal{J}(\mathbf{u}, p, \mathbf{t}, \mathbf{g}) = \frac{\lambda}{4} \int_\Omega |\mathbf{u} - \mathbf{u}_0|^4 \, d\Omega + \frac{\lambda}{2} \int_\Gamma \left(|\nabla_s \mathbf{g}|^2 + |\mathbf{g}|^2 \right) d\Gamma \,, \tag{4.57}$$

where ∇_s denotes the surface gradient.

We define the spaces $X = \mathbf{H}^1(\Omega) \times L^2_0(\Omega) \times \mathbf{H}^{-1/2}(\Gamma)$, $Y = [\mathbf{H}^1(\Omega)]^* \times L^2_0(\Omega) \times \mathbf{H}^{1/2}(\Gamma)$, $G = \mathbf{H}^1_n(\Gamma)$, and $Z = \mathbf{L}^{3/2}(\Omega) \times \{0\} \times \mathbf{H}^1(\Gamma)$. By compact imbedding results, Z is compactly imbedded into Y. For the time being, we assume that the admissible set Θ for the control $\mathbf{g}$ is a closed, convex subset of $G = \mathbf{H}^1_n(\Gamma)$.

We then consider the following optimal control problem associated with the Navier-Stokes equations:

$$\min\{\mathcal{J}(\mathbf{u}, p, \mathbf{t}, \mathbf{g}) : (\mathbf{u}, p, \mathbf{t}) \in X, \mathbf{g} \in \Theta\} \quad \text{subject to} \quad (4.54)\text{-}(4.56)\,. \tag{4.58}$$

We define the continuous linear operator $T \in \mathcal{L}(Y; X)$ as follows: for each $(\boldsymbol{\zeta}, \eta, \boldsymbol{\kappa}) \in Y$, $T(\boldsymbol{\zeta}, \eta, \boldsymbol{\kappa}) = (\tilde{\mathbf{u}}, \tilde{p}, \tilde{\mathbf{t}}) \in X$ is the unique solution of

$$a(\tilde{\mathbf{u}}, \mathbf{v}) + b(\mathbf{v}, \tilde{p}) - \langle \tilde{\mathbf{t}}, \mathbf{v} \rangle_\Gamma = \langle \boldsymbol{\zeta}, \mathbf{v} \rangle \quad \forall\, \mathbf{v} \in \mathbf{H}^1(\Omega)\,,$$

$$b(\tilde{\mathbf{u}}, q) = (\eta, q) \quad \forall\, q \in L^2_0(\Omega)$$

and

$$\langle \mathbf{s}, \tilde{\mathbf{u}} \rangle_\Gamma = \langle \mathbf{s}, \boldsymbol{\kappa} \rangle_\Gamma \quad \forall\, \mathbf{s} \in \mathbf{H}^{-1/2}(\Gamma)\,.$$

It can be easily verified that T is self-adjoint.

We define the (differentiable) nonlinear mapping $N : X \to Y$ by

$$N(\mathbf{u}, p, \mathbf{t}) = -\begin{pmatrix} \mathbf{f} - \mathbf{u} \cdot \nabla \mathbf{u} \\ 0 \\ \mathbf{b} \end{pmatrix}$$

or equivalently

$$\langle N(\mathbf{u}, p, \mathbf{t}), (\mathbf{v}, q, \mathbf{s}) \rangle = -(\mathbf{f}, \mathbf{v}) + c(\mathbf{u}, \mathbf{u}, \mathbf{v}) - \langle \mathbf{s}, \mathbf{b} \rangle_\Gamma \quad \forall\, (\mathbf{v}, q, \mathbf{s}) \in X$$

and define $K : \mathbf{H}^{1/2}(\Gamma) \to Y$ by

$$K\mathbf{g} = -\begin{pmatrix} 0 \\ 0 \\ \mathbf{g} \end{pmatrix}$$

or equivalently

$$\langle K\mathbf{g}, (\mathbf{v}, q, \mathbf{s}) \rangle = -\langle \mathbf{s}, \mathbf{g} \rangle_\Gamma \quad \forall\, \mathbf{g} \in \mathbf{H}^{1/2}(\Gamma)\,, \forall\, (\mathbf{v}, q, \mathbf{s}) \in X\,.$$

Clearly, the constraint equations (4.54)-(4.56) can be expressed as

$$(\mathbf{u}, p, \mathbf{t}) + \lambda T N(\mathbf{u}, p, \mathbf{t}) + \lambda T K \mathbf{g} = 0\,,$$

i.e., in the form (2.2). With the obvious definitions for $\mathcal{F}(\cdot)$ and $\mathcal{E}(\cdot)$, i.e.,

$$\mathcal{F}(\mathbf{u},p,\mathbf{t}) = \frac{1}{4}\int_\Omega |\mathbf{u}-\mathbf{u}_0|^4\,d\Omega \quad \forall\,(\mathbf{u},p,\mathbf{t})\in X$$

and

$$\mathcal{E}(\mathbf{g}) = \frac{1}{2}\int_\Gamma \left(|\nabla_s \mathbf{g}|^2 + |\mathbf{g}|^2\right) d\Gamma\,,$$

the functional (4.57) can be expressed as

$$\mathcal{J}(\mathbf{u},p,\mathbf{t},\mathbf{g}) = \lambda\,\mathcal{F}(\mathbf{u},p,\mathbf{t}) + \lambda\,\mathcal{E}(\mathbf{g})\,,$$

i.e., in the form (2.3).

We are now in a position to verify, for the minimization problem (4.58), all the hypotheses of §2 and §3.

4.3.1. Verification of the hypotheses for the existence of optimal solutions.

We first verify the that the hypotheses (H1)-(H6) hold in the current setting.

(H1) is obviously satisfied with a lower bound 0.

(H2) holds with $\alpha = 1$ and $\beta = 2$.

(H3) is verified with the choice $(\mathbf{u}^{(0)}, p^{(0)}, \mathbf{t}^{(0)}, \mathbf{0}) \in X\times\Theta$ where $(\mathbf{u}^{(0)}, p^{(0)})$ is a solution to the Navier-Stokes equations with Dirichlet boundary conditions, and $\mathbf{t}^{(0)} = \left[-p^{(0)}\mathbf{n} + \left(\nabla\mathbf{u}^{(0)} + (\nabla\mathbf{u}^{(0)})^T\right)\cdot\mathbf{n}\right]_\Gamma$; see, e.g., [13] or [20].

In order to verify (H4), we assume $\{\mathbf{g}^{(n)}\}\subset\Theta\subset\mathbf{H}^1_n(\Gamma)$ is a sequence satisfying $\mathbf{g}^{(n)}\rightharpoonup\mathbf{g}$ in $\mathbf{H}^1(\Gamma)$; then we have $\mathbf{g}^{(n)}\rightharpoonup\mathbf{g}$ in $\mathbf{H}^{1/2}(\Gamma)$ so that $\lim_{n\to\infty}\langle\mathbf{g}^{(n)},\mathbf{v}\rangle_\Gamma = \langle\mathbf{g},\mathbf{v}\rangle_\Gamma$ for all $\mathbf{v}\in\mathbf{H}^1(\Omega)$, i.e., $K\mathbf{g}^{(n)}\rightharpoonup K\mathbf{g}$ in Y. Assume that the sequence $\{\mathbf{u}^{(n)}\}\subset\mathbf{H}^1(\Omega)$ satisfies $\mathbf{u}^{(n)}\rightharpoonup\mathbf{u}$ in $\mathbf{H}^1(\Omega)$; then $\mathbf{u}^{(n)}\to\mathbf{u}$ in $\mathbf{L}^4(\Omega)$ by the compactness of the imbedding $\mathbf{H}^1(\Omega)\hookrightarrow\hookrightarrow\mathbf{L}^4(\Omega)$. Now,

$$\begin{aligned}\langle N(\mathbf{u}^{(n)}),\mathbf{v}\rangle &= c(\mathbf{u}^{(n)},\mathbf{u}^{(n)},\mathbf{v}) = c(\mathbf{u},\mathbf{u}^{(n)},\mathbf{v}) + c(\mathbf{u}^{(n)}-\mathbf{u},\mathbf{u}^{(n)},\mathbf{v})\\ &\to c(\mathbf{u},\mathbf{u},\mathbf{v}) + 0 = \langle N(\mathbf{u}),\mathbf{v}\rangle \qquad \text{as } n\to\infty\,.\end{aligned}$$

Hence, (H4) is verified.

The verification of (H5) follows directly from the observation that the mappings $(\mathbf{u},p,\mathbf{t})\mapsto\mathcal{F}(\mathbf{u},p,\mathbf{t}) = (1/4)\,\|\mathbf{u}-\mathbf{u}\|^4_{\mathbf{L}^4(\Omega)}$ and $\mathbf{g}\mapsto\mathcal{E}(\mathbf{g}) = (1/2)\,\|\mathbf{g}\|^2_{1,\Gamma}$ are convex.

To verify (H6), we combine a priori estimates obtained from the constraint equations and the functional. Let $\{\mathbf{u}^{(k)}, p^{(k)}, \mathbf{t}^{(k)}, \mathbf{g}^{(k)}\}\subset\mathbf{H}^1(\Omega)\times L^2_0(\Omega)\times\mathbf{H}^{-1/2}(\Gamma)\times\mathbf{H}^1_n(\Gamma)$ be a sequence such that

$$\mathcal{J}(\mathbf{u}^{(k)},\mathbf{g}^{(k)}) \le C\,, \tag{4.59}$$

$$a(\mathbf{u}^{(k)},\mathbf{v}) + b(\mathbf{v},p^{(k)}) - \langle\mathbf{t}^{(k)},\mathbf{v}\rangle_\Gamma + \lambda\,c(\mathbf{u}^{(k)},\mathbf{u}^{(k)},\mathbf{v}) = \lambda\,\langle\mathbf{f},\mathbf{v}\rangle \quad \forall\,\mathbf{v}\in\mathbf{H}^1(\Omega)\,, \tag{4.60}$$

$$b(\mathbf{u}^{(k)},q) = 0 \quad \forall\,q\in L^2_0(\Omega)\,, \tag{4.61}$$

and

$$\langle\mathbf{s},\mathbf{u}^{(k)}\rangle_\Gamma - \lambda\langle\mathbf{s},\mathbf{g}^{(k)}\rangle_\Gamma = \lambda\langle\mathbf{s},\mathbf{b}\rangle_\Gamma \quad \forall\,\mathbf{s}\in\mathbf{H}^{-1/2}(\Gamma)\,. \tag{4.62}$$

First, (4.59) implies that $(\mathbf{u}^{(k)},\mathbf{g}^{(k)})$ is uniformly bounded in $\mathbf{L}^4(\Omega)\times\mathbf{H}^1(\Gamma)$. For each $\mathbf{g}^{(k)}$, we may choose a $(\mathbf{w}^{(k)},r^{(k)})\in\mathbf{H}^1(\Omega)\times L^2_0(\Omega)$ that satisfies the Stokes problem

$$a(\mathbf{w}^{(k)},\mathbf{v})+b(\mathbf{v},r^{(k)})=\langle\mathbf{f},\mathbf{v}\rangle\quad\forall\,\mathbf{v}\in\mathbf{H}^1_0(\Omega)\,,\tag{4.63}$$

$$b(\mathbf{w}^{(k)},q)=0\quad\forall\,q\in L^2_0(\Omega)\tag{4.64}$$

and

$$\mathbf{w}^{(k)}=\lambda(\mathbf{g}^{(k)}+\mathbf{b})\quad\text{on }\Gamma\,.\tag{4.65}$$

Furthermore, there holds the estimate

$$\|\mathbf{w}^{(k)}\|_1\le C(\|\mathbf{f}\|_0+\|\mathbf{b}\|_{1/2,\Gamma}+\|\mathbf{g}^{(k)}\|_{1,\Gamma})\,.\tag{4.66}$$

By subtracting (4.63) from (4.60) with $\mathbf{v}=\mathbf{u}^{(k)}-\mathbf{w}^{(k)}$, also using (4.61) and (4.64), we obtain

$$\begin{aligned}a(\mathbf{u}^{(k)}-\mathbf{w}^{(k)},\mathbf{u}^{(k)}-\mathbf{w}^{(k)})&=-\lambda\,c(\mathbf{u}^{(k)},\mathbf{u}^{(k)},\mathbf{u}^{(k)}-\mathbf{w}^{(k)})\\&=\lambda\,c(\mathbf{u}^{(k)},\mathbf{u}^{(k)}-\mathbf{w}^{(k)},\mathbf{u}^{(k)})\,.\end{aligned}\tag{4.67}$$

Note that

$$\begin{aligned}&|c(\mathbf{u}^{(k)},\mathbf{u}^{(k)}-\mathbf{w}^{(k)},\mathbf{u}^{(k)})|\\&=\frac12\left|\int_\Omega\mathbf{u}^{(k)}\cdot\left((\nabla(\mathbf{u}^{(k)}-\mathbf{w}^{(k)}))+(\nabla(\mathbf{u}^{(k)}-\mathbf{w}^{(k)}))^T\right)\cdot\mathbf{u}^{(k)}\,d\Omega\right|\\&\le C\left\|(\nabla(\mathbf{u}^{(k)}-\mathbf{w}^{(k)}))+(\nabla(\mathbf{u}^{(k)}-\mathbf{w}^{(k)}))^T\right\|_0\left\|\mathbf{u}^{(k)}\right\|_{\mathbf{L}^4(\Omega)}\\&\le\frac{1}{4\lambda}\left\|(\nabla(\mathbf{u}^{(k)}-\mathbf{w}^{(k)}))+(\nabla(\mathbf{u}^{(k)}-\mathbf{w}^{(k)}))^T\right\|_0^2+C_\lambda\left\|\mathbf{u}^{(k)}\right\|^4_{\mathbf{L}^4(\Omega)}\end{aligned}$$

so that, using (4.67), we have that

$$\frac14\left\|\left(\nabla(\mathbf{u}^{(k)}-\mathbf{w}^{(k)})\right)+\left(\nabla(\mathbf{u}^{(k)}-\mathbf{w}^{(k)})\right)^T\right\|_0^2\le C_\lambda\left\|\mathbf{u}^{(k)}\right\|^4_{\mathbf{L}^4(\Omega)}\,.$$

Then, by (4.66) and the triangle inequality, we have that

$$\left\|(\nabla\mathbf{u}^{(k)})+(\nabla\mathbf{u}^{(k)})^T\right\|_0\le C\{\|\mathbf{f}\|_0+\|\mathbf{b}\|_{1/2,\Gamma}+\|\mathbf{g}^{(k)}\|_{1,\Gamma}+\|\mathbf{u}^{(k)}\|^2_{\mathbf{L}^4(\Omega)}\}\,.$$

Thus,

$$\begin{aligned}&\|(\nabla\mathbf{u}^{(k)})+(\nabla\mathbf{u}^{(k)})^T\|_0+\|\mathbf{u}^{(k)}\|_{0,\Gamma}\\&\le\|(\nabla\mathbf{u}^{(k)})+(\nabla\mathbf{u}^{(k)})^T\|_0+\|\mathbf{b}\|_{0,\Gamma}+\|\mathbf{g}^{(k)}\|_{0,\Gamma}\\&\le C(\|\mathbf{f}\|_0+\|\mathbf{b}\|_{1/2,\Gamma}+\|\mathbf{g}^{(k)}\|_{1,\Gamma}+\|\mathbf{u}^{(k)}\|^2_{\mathbf{L}^4(\Omega)})\,.\end{aligned}$$

Since the mapping $\mathbf{u}\mapsto\|\nabla\mathbf{u}+(\nabla\mathbf{u})^T\|_0+\|\mathbf{u}\|_{0,\Gamma}$ defines a norm on $\mathbf{H}^1(\Omega)$ equivalent to the standard $\mathbf{H}^1(\Omega)$-norm, we have that

$$\|\mathbf{u}^{(k)}\|_1\le C\{\|\mathbf{f}\|_0+\|\mathbf{b}\|_{1/2,\Gamma}+\|\mathbf{g}^{(k)}\|_{1,\Gamma}+\|\mathbf{u}^{(k)}\|^2_{\mathbf{L}^4(\Omega)}\}\,,$$

and, since $\|\mathbf{u}^{(k)}\|_{\mathbf{L}^4(\Omega)}$ and $\|\mathbf{g}^{(k)}\|_{1,\Gamma}$ are uniformly bounded, we conclude that $\|\mathbf{u}^{(k)}\|_1$ is uniformly bounded as well. One easily concludes from (4.60) that $\|\mathbf{t}^{(k)}\|_{-1/2,\Gamma}$ is uniformly bounded. Thus (H6) is verified.

It is now just a matter of citing Theorem 2.1 to conclude the existence of an optimal solution that minimizes (4.57) subject (4.54)–(4.56).

THEOREM 4.8. *There exists a* $(\mathbf{u}, p, \mathbf{t}, \mathbf{g}) \in \mathbf{H}^1(\Omega) \times L^2_0(\Omega) \times \mathbf{H}^{-1/2}(\Omega) \times \Theta$ *such that* (4.57) *is minimized subject to* (4.54)-(4.56). □

4.3.2. Verification of the hypotheses for the existence of Lagrange multipliers.
We now assume $(\mathbf{u}, p, \mathbf{t}, \mathbf{g})$ is an optimal solution and turn to the verification of hypotheses (H7)-(H9).

The validity of (H7) is obvious.

(H8) holds since the mapping $\mathbf{z} \mapsto \mathcal{E}(\mathbf{g}) = (1/2)\int_\Gamma \left(|\nabla_s \mathbf{g}|^2 + |\mathbf{g}|^2\right) d\Gamma$ is convex.

(H9) can be verified as follows. For any $(\mathbf{u}, p, \mathbf{t}) \in X$, the operator $N'(\mathbf{u}, p, \mathbf{t}) : X \to Y$ is given by

$$N'(\mathbf{u}, p, \mathbf{t}) \cdot (\mathbf{v}, q, \mathbf{s}) = -\begin{pmatrix} \mathbf{u}\cdot\nabla\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{u} \\ 0 \\ \mathbf{0} \end{pmatrix}$$

for all $(\mathbf{v}, q, \mathbf{s}) \in \mathbf{H}^1(\Omega) \times L^2_0(\Omega) \times \mathbf{H}^{-1/2}(\Gamma)$. Thus we obtain $N'(\mathbf{u}, p, \mathbf{t}) \cdot (\mathbf{v}, q, \mathbf{s}) \in \mathbf{L}^{3/2}(\Omega) \times \{0\} \times \mathbf{H}^1(\Gamma) = Z$.

The Lagrangian is given by

$$\begin{aligned} &\mathcal{L}(\mathbf{u}, p, \mathbf{t}, \mathbf{g}, \boldsymbol{\nu}, \phi, \boldsymbol{\tau}, k) \\ &= k\,\mathcal{J}(\mathbf{u}, \mathbf{g}) - \{a(\mathbf{u}, \boldsymbol{\nu}) + \lambda\, c(\mathbf{u}, \mathbf{u}, \boldsymbol{\nu}) + b(\boldsymbol{\nu}, p) + b(\mathbf{u}, \phi) - \langle \boldsymbol{\tau}, \mathbf{u}\rangle_\Gamma \\ &\qquad - \langle \mathbf{t}, \boldsymbol{\nu}\rangle_\Gamma - \lambda\,\langle \mathbf{f}, \boldsymbol{\nu}\rangle_\Gamma + \lambda\,\langle \boldsymbol{\tau}, \mathbf{b}\rangle_\Gamma + \lambda\,\langle \boldsymbol{\tau}, \mathbf{g}\rangle_\Gamma\} \end{aligned}$$

for all $(\mathbf{u}, p, \mathbf{t}, \mathbf{g}, \boldsymbol{\nu}, \phi, \boldsymbol{\tau}, k) \in X \times G \times X \times \mathbb{R} = \mathbf{H}^1(\Omega) \times L^2_0(\Omega) \times \mathbf{H}^{-1/2}(\Gamma) \times \mathbf{H}^1_n(\Gamma) \times \mathbf{H}^1(\Omega) \times L^2_0(\Omega) \times \mathbf{H}^{-1/2}(\Gamma) \times \mathbb{R}$. Note that in this form of the Lagrangian, the Lagrange multiplier $(\boldsymbol{\nu}, \phi, \boldsymbol{\tau}) \in X = Y^*$ so that we have already introduced the change of variables indicated between (2.17)-(2.18) and (2.19)-(2.21).

Having verified the hypotheses (H7)-(H9), we may apply Theorem 2.4 to conclude that there exists a Lagrange multiplier $(\boldsymbol{\nu}, \phi, \boldsymbol{\tau}) \in X = \mathbf{H}^1(\Omega) \times L^2_0(\Omega) \times \mathbf{H}^{-1/2}(\Gamma)$ and a real number k such that

$$(\boldsymbol{\nu}, \phi, \boldsymbol{\tau}) + \lambda T^*\Big([N'(\mathbf{u}, p, \mathbf{t})]^* \cdot (\boldsymbol{\nu}, \phi, \boldsymbol{\tau}) - k\,\mathcal{J}_{(\mathbf{u},p,\mathbf{t})}(\mathbf{u}, p, \mathbf{t}, \mathbf{g})\Big) = 0 \tag{4.68}$$

and

$$\mathcal{L}(\mathbf{u}, p, \mathbf{t}, \mathbf{z}, \boldsymbol{\nu}, \phi, \boldsymbol{\tau}, k) \le \mathcal{L}(\mathbf{u}, p, \mathbf{t}, \mathbf{g}, \boldsymbol{\nu}, \phi, \boldsymbol{\tau}, k) \quad \forall\, \mathbf{z} \in \Theta \tag{4.69}$$

and that for almost all values of λ, we may choose $k = 1$.

Recall that $T^* = T$. Also, note that for $(\mathbf{u}, p, \mathbf{t}) \in X = \mathbf{H}^1(\Omega) \times L^2_0(\Omega) \times \mathbf{H}^{-1/2}(\Gamma)$, the operator $[N'(\mathbf{u}, p, \mathbf{t})]^* : X \to Y$ is given by

$$[N'(\mathbf{u}, p, \mathbf{t})]^* \cdot (\mathbf{v}, q, \mathbf{s}) = \begin{pmatrix} -\mathbf{u}\cdot\nabla\mathbf{v} + \mathbf{v}\cdot(\nabla\mathbf{u})^T \\ 0 \\ \mathbf{0} \end{pmatrix} \quad \forall\, (\mathbf{v}, q, \mathbf{s}) \in X.$$

Thus, (4.68), with $k=1$, can be rewritten as

$$\begin{aligned} a(\mathbf{w},\boldsymbol{\nu}) + \lambda\, c(\mathbf{w},\mathbf{u},\boldsymbol{\nu}) + \lambda\, c(\mathbf{u},\mathbf{w},\boldsymbol{\nu}) + b(\mathbf{w},\phi) - \langle \boldsymbol{\tau},\mathbf{w}\rangle_\Gamma \\ = \lambda\left((\mathbf{u}-\mathbf{u}_0)^3,\mathbf{w}\right) \quad \forall\, \mathbf{w}\in \mathbf{H}^1(\Omega)\,, \end{aligned} \tag{4.70}$$

$$b(\boldsymbol{\nu},r) = 0 \quad \forall\, r\in L^2_0(\Omega)\,, \tag{4.71}$$

and

$$\langle \mathbf{y},\boldsymbol{\nu}\rangle_\Gamma = 0 \quad \forall\, \mathbf{y}\in \mathbf{H}^{-1/2}(\Gamma)\,. \tag{4.72}$$

In the right-hand side of (4.70), we use the notation $(\mathbf{v}^3,\mathbf{w}) = \sum_{j=1}^d (v_j^3, w_j)$.

Using the definition of the Lagrangian functional, (4.69), with $k=1$, can be rewritten as

$$\begin{aligned} \frac{\lambda}{2}(\nabla_s\mathbf{z},\nabla_s\mathbf{z})_\Gamma + \frac{\lambda}{2}(\mathbf{z},\mathbf{z})_\Gamma - \frac{\lambda}{2}(\nabla_s\mathbf{g},\nabla_s\mathbf{g})_\Gamma \\ -\frac{\lambda}{2}(\mathbf{g},\mathbf{g})_\Gamma - \lambda\langle\boldsymbol{\tau},\mathbf{z}\rangle_\Gamma + \lambda\langle\boldsymbol{\tau},\mathbf{g}\rangle_\Gamma \ge 0 \quad \forall\, \mathbf{z}\in\Theta\,. \end{aligned}$$

For each $\epsilon\in(0,1)$ and each $\mathbf{z}\in\Theta$, by plugging $\epsilon\mathbf{z}+(1-\epsilon)\mathbf{g}\in\Theta$ into the last inequality we obtain

$$\begin{aligned} \epsilon\left(\nabla_s\mathbf{g},\nabla_s(\mathbf{z}-\mathbf{g})\right)_\Gamma + \epsilon\,(\mathbf{g},\mathbf{z}-\mathbf{g})_\Gamma + \frac{\epsilon^2}{2}\left(\nabla_s(\mathbf{z}-\mathbf{g}),\nabla_s(\mathbf{z}-\mathbf{g})\right)_\Gamma \\ +\frac{\epsilon^2}{2}(\mathbf{z}-\mathbf{g},\mathbf{z}-\mathbf{g})_\Gamma - \epsilon\langle\boldsymbol{\tau},\mathbf{z}-\mathbf{g}\rangle_\Gamma \ge 0 \quad \forall\, \mathbf{z}\in\Theta \end{aligned}$$

so that, after dividing by $\epsilon>0$ and then letting $\epsilon\to 0^+$, we obtain

$$\left(\nabla_s\mathbf{g},\nabla_s(\mathbf{z}-\mathbf{g})\right)_\Gamma + (\mathbf{g},\mathbf{z}-\mathbf{g})_\Gamma - \langle\boldsymbol{\tau},\mathbf{z}\rangle_\Gamma \ge 0 \quad \forall\, \mathbf{z}\in\Theta\,. \tag{4.73}$$

We see that for almost all values of λ, necessary conditions for an optimum are that (4.54)-(4.56), (4.70)-(4.72) and (4.73) are satisfied. Again, the system formed by these equations will be called an *optimality system*.

We now specialize to the case $\Theta = \mathbf{H}^1_n(\Gamma)$. Note that the hypothesis (H10) is satisfied. Then using Theorem 2.5, we see that the inequality (4.73) becomes an equality and, by letting $\mathbf{z}=\mathbf{k}-\mathbf{g}$ vary arbitrarily in $\mathbf{H}^1_n(\Gamma)$, we now have, instead of (4.73),

$$(\nabla_s\mathbf{g},\nabla_s\mathbf{z})_\Gamma + (\mathbf{g},\mathbf{z})_\Gamma - \langle\boldsymbol{\tau},\mathbf{z}\rangle_\Gamma = 0 \quad \forall\, \mathbf{z}\in\Theta = \mathbf{H}^1_n(\Gamma)\,. \tag{4.74}$$

Thus, according to that theorem, we have that for almost all λ, an optimality system of equations is now given by (4.54)-(4.56), (4.70)-(4.72) and (4.74). However, we can go further and verify that the hypothesis (H11) is valid, which in turn will justify the existence of a Lagrange multiplier satisfying the optimality system for *all* $\lambda\in\Lambda$.

We now verify (H11) which we again note can be equivalently stated as follows:

if $\xi\in Y^*$ *satisfies* $\left(I+\lambda T^*[N'(u)]^*\right)\xi = 0$ *and* $K^*\xi = 0$, *then* $\xi = 0$.

To verify this hypothesis, we assume that $(\boldsymbol{\xi},\sigma,\boldsymbol{\theta}) \in Y^* = \mathbf{H}^1(\Omega)\times L^2(\Omega)\times \mathbf{H}^{-1/2}(\Gamma)$ satisfies $\left(I+\lambda\, T^*[N'(\mathbf{u},p,\mathbf{t})]^*\right)(\boldsymbol{\xi},\sigma,\boldsymbol{\theta}) = (\mathbf{0},0,\mathbf{0})$ and $K^*(\boldsymbol{\xi},\sigma,\boldsymbol{\theta}) = \mathbf{0}$, i.e.,

$$a(\mathbf{w},\boldsymbol{\xi}) + \lambda\, c(\mathbf{w},\mathbf{u},\boldsymbol{\xi}) + \lambda\, c(\mathbf{u},\mathbf{w},\boldsymbol{\xi}) + b(\mathbf{w},\sigma) - \langle \boldsymbol{\theta},\mathbf{w}\rangle_\Gamma = 0 \quad \forall\, \mathbf{w}\in \mathbf{H}^1(\Omega)\,,$$

$$b(\boldsymbol{\xi},r) = 0 \quad \forall\, r\in L^2_0(\Omega)\,,$$

$$\langle \mathbf{y},\boldsymbol{\xi}\rangle_\Gamma = 0 \quad \forall\, \mathbf{y}\in \mathbf{H}^{-1/2}(\Gamma)\,,$$

and

$$\boldsymbol{\theta} = \mathbf{0} \quad \text{on } \Gamma\,.$$

(Note that $K^*(\boldsymbol{\xi},\sigma,\boldsymbol{\theta}) = \boldsymbol{\theta}$.) Let $\Omega^{'}$ be a smooth extension of Ω such that $\overline{\Omega}$ is a compact subset of $\Omega^{'}$. We then set $\boldsymbol{\xi}'$, σ' and $\mathbf{u}'$ to be the extension, by zero outside Ω, of $\boldsymbol{\xi}$, σ and $\mathbf{u}$, respectively. We may show from the last four equations that

$$\boldsymbol{\xi}' \in \mathbf{H}^1(\Omega^{'})\,, \qquad \sigma' \in L^2_0(\Omega^{'})\,,$$

$$a'(\mathbf{w},\boldsymbol{\xi}') + \lambda\, c'(\mathbf{w},\mathbf{u}',\boldsymbol{\xi}') + \lambda\, c'(\mathbf{u}',\mathbf{w},\boldsymbol{\xi}') + b^{'}(\mathbf{w},\sigma') = 0 \quad \forall\, \mathbf{w}\in \mathbf{H}^1_0(\Omega^{'})\,, \tag{4.75}$$

and

$$b^{'}(\boldsymbol{\xi}',r) = 0 \quad \forall\, r\in L^2_0(\Omega^{'})\,, \tag{4.76}$$

where the forms $a'(\cdot,\cdot)$, $b^{'}(\cdot,\cdot)$ and $c'(\cdot,\cdot,\cdot)$ defined over $\Omega^{'}$ are the analogues of corresponding forms defined over Ω. Using a unique continuation result for the system (4.75)-(4.76) that was established in [16] or [17], we obtain $\boldsymbol{\xi}' = \mathbf{0}$ and $\sigma' = 0$ in $\Omega^{'}$, or $\boldsymbol{\xi} = \mathbf{0}$ and $\sigma = 0$ in Ω. Thus (H11) is verified.

Hence we conclude that for *all* λ, the optimality system (4.54)-(4.56), (4.70)-(4.72), and (4.74) has a solution. Thus, we have Theorem 2.6 which, in the present context, is given as follows.

THEOREM 4.9. *Let $(\mathbf{u},p,\mathbf{t},\mathbf{g}) \in \mathbf{H}^1(\Omega)\times L^2_0(\Omega)\times \mathbf{H}^{-1/2}(\Gamma)\times \mathbf{H}^1_n(\Gamma)$ denote an optimal solution that minimizes* (4.57) *subject to* (4.54)-(4.56). *Then, for all $\lambda\in\Lambda$, there exists a nonzero Lagrange multiplier $(\boldsymbol{\nu},\phi,\boldsymbol{\tau}) \in \mathbf{H}^1(\Omega)\times L^2_0(\Omega)\times \mathbf{H}^{-1/2}(\Gamma)$ satisfying the Euler equations* (4.70)-(4.72) *and* (4.74). □

Note that, in the above expression, we have already employed hypothesis (H12) which in the current context is easily seen to be satisfied with $E: G\to G^*$ defined by

$$\langle E\mathbf{g},\mathbf{z}\rangle = \int_\Gamma (\nabla_s \mathbf{g}\cdot\nabla_s \mathbf{z} + \mathbf{g}\cdot\mathbf{z})\, d\Gamma \quad \forall\, \mathbf{z}\in \mathbf{H}^1_n(\Gamma) = G\,.$$

We also note that for each fixed $\boldsymbol{\tau}$, (4.74) with $\mathbf{g}\in \mathbf{H}^1_n(\Gamma)$ is equivalent to

$$(\nabla_s \mathbf{g},\nabla_s \mathbf{k})_\Gamma + (\mathbf{g},\mathbf{k})_\Gamma + \gamma \int_\Gamma \mathbf{k}\cdot\mathbf{n}\, d\Gamma = \langle \boldsymbol{\tau},\mathbf{k}\rangle_\Gamma \quad \forall\, \mathbf{k}\in \mathbf{H}^1(\Gamma) \tag{4.77}$$

and

$$\int_\Gamma \mathbf{g}\cdot\mathbf{n}\, d\Gamma = 0\,, \tag{4.78}$$

where $\gamma \in \mathbb{R}$ is an additional unknown constant that accounts for the single integral constraint of (4.78). The equivalence can be shown as follows. First, an application of Lax-Milgram Lemma to (4.74) on the space $\mathbf{H}^1_n(\Gamma)$ guarantees the existence and uniqueness of a solution $\mathbf{g} \in \mathbf{H}^1_n(\Gamma)$ to (4.74); this solution $\mathbf{g}$ clearly satisfies (4.77)-(4.78) with $\gamma = \int_\Gamma \big(\boldsymbol{\tau} \cdot \mathbf{n} - \nabla_s \mathbf{g} : \nabla_s \mathbf{n} - \mathbf{g} \cdot \mathbf{n}\big)\, d\Gamma$. Conversely, any solution $(\mathbf{g}, \gamma)$ of (4.77)-(4.78) trivially satisfies (4.74). Although (4.74) and (4.77)-(4.78) are equivalent, the latter is more easily discretized.

4.3.3. Verification of the hypotheses for approximations and error estimates.

We finally verify the hypotheses (H13)-(H19) that are used in connection with approximations and error estimates.

A finite element discretization of the optimality system (4.54)-(4.56), (4.70)-(4.72), and (4.74) is defined as follows. First, one chooses families of finite dimensional subspaces $\mathbf{V}^h \subset \mathbf{H}^1(\Omega)$ and $S^h \subset L^2(\Omega)$. These families are parameterized by the parameter h that tends to zero; commonly, this parameter is chosen to be some measure of the grid size in a subdivision of Ω into finite elements. We let $S^h_0 = S^h \cap L^2_0(\Omega)$ and $\mathbf{V}^h_0 = \mathbf{V}^h \cap \mathbf{H}^1_0(\Omega)$.

One may choose any pair of subspaces $\mathbf{V}^h$ and S^h that can be used for finding finite element approximations of solutions of the Navier-Stokes equations. Thus, concerning these subspaces, we make the following standard assumptions which are exactly those employed in well-known finite element methods for the Navier-Stokes equations. First, we have the approximation properties: there exist an integer k and a constant C, independent of h, $\mathbf{v}$ and q, such that

$$\inf_{\mathbf{v}^h \in \mathbf{V}^h} \|\mathbf{v} - \mathbf{v}^h\|_1 \le C h^m \|\mathbf{v}\|_{m+1} \quad \forall\, \mathbf{v} \in \mathbf{H}^{m+1}(\Omega)\,,\ 1 \le m \le k \tag{4.79}$$

and

$$\inf_{q^h \in S^h_0} \|q - q^h\|_0 \le C h^m \|q\|_m \quad \forall\, q \in H^m(\Omega) \cap L^2_0(\Omega)\,,\ 1 \le m \le k\,. \tag{4.80}$$

Next, we assume the *inf-sup condition*, or *Ladyzhenskaya-Babuska-Brezzi condition*: there exists a constant C, independent of h, such that

$$\inf_{0 \ne q^h \in S^h_0}\ \sup_{\mathbf{0} \ne \mathbf{v}^h \in \mathbf{V}^h} \frac{b(\mathbf{v}^h, q^h)}{\|\mathbf{v}^h\|_1\, \|q^h\|_0} \ge C\,. \tag{4.81}$$

This condition assures the stability of finite element discretizations of the Navier-Stokes equations. For thorough discussions of the approximation properties (4.79)-(4.80), see, e.g., [2] or [8], and for like discussions of the stability condition (4.81), see, e.g., [13] or [14]. The latter references may also be consulted for a catalogue of finite element subspaces that meet the requirements of (4.79)-(4.81).

Next, let $\mathbf{P}^h = \mathbf{V}^h|_\Gamma$, i.e., $\mathbf{P}^h$ consists of the restriction, to the boundary Γ, of functions belonging to $\mathbf{V}^h$. For all choices of conforming finite element spaces $\mathbf{V}^h$, e.g., Lagrange type finite element spaces, we have that $\mathbf{P}^h \subset \mathbf{H}^{-1/2}(\Gamma)$. For the subspaces $\mathbf{P}^h = \mathbf{V}^h|_\Gamma$, we can show the following approximation property: there exist an integer k and a constant C, independent of h and $\mathbf{s}$, such that

$$\begin{aligned} &\inf_{\mathbf{s}^h \in \mathbf{P}^h} \|\mathbf{s} - \mathbf{s}^h\|_{-1/2,\Gamma} \\ &\qquad \le C h^m \inf_{\mathbf{v} \in \mathbf{H}^m(\Omega), \mathbf{v}|_\Gamma = \mathbf{s}} \|\mathbf{v}\|_m \qquad \forall\, \mathbf{s} \in \mathbf{H}^m(\Omega)|_\Gamma\,,\ 1 \le m \le k\,. \end{aligned} \tag{4.82}$$

We also use the following inverse assumption: there exists a constant C, independent of h and $\mathbf{s}^h$, such that

$$\|\mathbf{s}^h\|_{s,\Gamma} \le Ch^{s-q}\|\mathbf{s}^h\|_{q,\Gamma} \quad \forall\, \mathbf{s}^h \in \mathbf{P}^h\,,\ -1/2 \le q \le s \le 1/2\,. \tag{4.83}$$

See [2] or [8] for details concerning (4.82) and (4.83). See also [15] for (4.82).

Now, let $\mathbf{Q}^h = \mathbf{V}^h|_\Gamma$, i.e., $\mathbf{Q}^h$ consists of the restriction, to the boundary Γ, of functions belonging to $\mathbf{V}^h$. Again, for all choices of conforming finite element spaces $\mathbf{V}^h$ we then have that $\mathbf{Q}^h \subset \mathbf{H}^1(\Gamma)$. We can show the approximation property: there exist an integer k and a constant C, independent of h and $\mathbf{k}$, such that for $1 \le m \le k$, $0 \le s \le 1$ and $\mathbf{k} \in \mathbf{H}^{m+1}(\Omega)|_\Gamma$,

$$\inf_{\mathbf{k}^h \in \mathbf{Q}^h} \|\mathbf{k} - \mathbf{k}^h\|_{s,\Gamma} \le Ch^{m-s+\frac{1}{2}} \inf_{\mathbf{v}\in \mathbf{H}^{m+1}(\Omega), \mathbf{v}|_\Gamma = \mathbf{k}} \|\mathbf{v}\|_{m+1}\,. \tag{4.84}$$

This property follows from (4.79), once one notes that the same type of polynomials are used in $\mathbf{Q}^h$ as are used in $\mathbf{V}^h$. We set $G^h = \mathbf{Q}^h \cap \mathbf{H}^1_n(\Gamma)$.

Once the approximating subspaces have been chosen we seek $\mathbf{u}^h \in \mathbf{V}^h$, $p^h \in S^h_0$, $\mathbf{t}^h \in \mathbf{P}^h$, $\mathbf{g}^h \in \mathbf{Q}^h$, $\boldsymbol{\nu}^h \in \mathbf{V}^h$, $\phi^h \in S^h_0$, $\boldsymbol{\tau}^h \in \mathbf{P}^h$, and $\gamma^h \in \mathbb{R}$ such that

$$a(\mathbf{u}^h, \mathbf{v}^h) + \lambda\, c(\mathbf{u}^h, \mathbf{u}^h, \mathbf{v}^h) + b(\mathbf{v}^h, p^h) - \langle \mathbf{v}^h, \mathbf{t}^h\rangle_\Gamma = \lambda\, \langle \mathbf{f}, \mathbf{v}^h\rangle \quad \forall\, \mathbf{v}^h \in \mathbf{V}^h\,, \tag{4.85}$$

$$b(\mathbf{u}^h, q^h) = 0 \quad \forall\, q^h \in S^h_0\,, \tag{4.86}$$

$$\langle \mathbf{u}^h, \mathbf{s}^h\rangle_\Gamma - \lambda \langle \mathbf{g}^h, \mathbf{s}^h\rangle_\Gamma = \lambda \langle \mathbf{b}, \mathbf{s}^h\rangle_\Gamma \quad \forall\, \mathbf{s}^h \in \mathbf{P}^h\,, \tag{4.87}$$

$$(\nabla_s \mathbf{g}^h, \nabla_s \mathbf{k}^h)_\Gamma + \langle \mathbf{g}^h, \mathbf{k}^h\rangle_\Gamma + \gamma^h \int_\Gamma \mathbf{k}^h \cdot \mathbf{n}\, d\Gamma = \langle \boldsymbol{\tau}^h, \mathbf{k}^h\rangle_\Gamma \quad \forall\, \mathbf{k}^h \in \mathbf{Q}^h\,, \tag{4.88}$$

$$\int_\Gamma \mathbf{g}^h \cdot \mathbf{n}\, d\Gamma = 0\,, \tag{4.89}$$

$$\begin{aligned} a(\mathbf{w}^h, \boldsymbol{\nu}^h) + \lambda\, c(\mathbf{w}^h, \mathbf{u}^h, \boldsymbol{\nu}^h) + \lambda\, c(\mathbf{u}^h, \mathbf{w}^h, \boldsymbol{\nu}^h) + b(\mathbf{w}^h, \phi^h) - \langle \mathbf{w}^h, \boldsymbol{\tau}^h\rangle_\Gamma \\ = \lambda \left((\mathbf{u}^h - \mathbf{u}_0)^3, \mathbf{w}^h\right) \quad \forall\, \mathbf{w}^h \in \mathbf{V}^h\,, \end{aligned} \tag{4.90}$$

$$b(\boldsymbol{\nu}^h, r^h) = 0 \quad \forall\, r^h \in S^h_0\,, \tag{4.91}$$

and

$$\langle \boldsymbol{\nu}^h, \mathbf{y}^h\rangle = 0 \quad \forall\, \mathbf{y}^h \in \mathbf{P}^h\,. \tag{4.92}$$

Note that if (4.85)-(4.92) are satisfied, then necessarily $\mathbf{g}^h \in G^h$. Also, in the right-hand side of (4.90), we use a notation similar to that used in the right-hand side of (4.70).

The operator $T^h \in \mathcal{L}(Y, X^h)$ is defined as the solution operator for

$$a(\mathbf{u}^h, \mathbf{v}^h) + b(\mathbf{v}^h, p^h) - \langle \mathbf{v}^h, \mathbf{t}^h \rangle_\Gamma = \langle \mathbf{f}, \mathbf{v}^h \rangle \quad \forall\, \mathbf{v}^h \in \mathbf{V}^h ,$$

$$b(\mathbf{u}^h, q^h) = 0 \quad \forall\, q^h \in S_0^h ,$$

and

$$\langle \mathbf{u}^h, \mathbf{s}^h \rangle_\Gamma = \langle \mathbf{b}, \mathbf{s}^h \rangle_\Gamma \quad \forall\, \mathbf{s}^h \in \mathbf{P}^h ;$$

i.e, for each $\mathbf{f} \in Y$, $T^h \mathbf{f} = \boldsymbol{\psi}^h \in X^h$ is the solution of the above system of equations.

Since $T = T^*$, we define $(T^*)^h = T^h$.

We define the operator $E^h : G^* \to G^h$ as follows. For each $\boldsymbol{\tau} \in G^*$, $\mathbf{g}^h = E^h \boldsymbol{\tau}$ if and only if

$$(\nabla_s \mathbf{g}^h, \nabla_s \mathbf{z}^h)_\Gamma + \langle \mathbf{g}^h, \mathbf{z}^h \rangle_\Gamma + \gamma^h \int_\Gamma \mathbf{z}^h \cdot \mathbf{n}\, d\Gamma = \langle \boldsymbol{\tau}^h, \mathbf{z}^h \rangle_\Gamma \quad \forall\, \mathbf{z}^h \in \mathbf{Q}^h$$

and

$$\int_\Gamma \mathbf{g}^h \cdot \mathbf{n}\, d\Gamma = 0 .$$

The existence and uniqueness of a solution $(\mathbf{g}^h, \gamma^h) \in \mathbf{Q}^h \times \mathbb{R}$ is guaranteed by the Brezzi theory for mixed finite element methods (see [4] or [5]) and the inequalities

$$(\nabla_s \mathbf{k}^h, \nabla_s \mathbf{k}^h)_\Gamma + (\mathbf{k}^h, \mathbf{k}^h)_\Gamma \geq C \|\mathbf{k}^h\|_{1,\Gamma}^2 \quad \forall\, \mathbf{k}^h \in \mathbf{Q}^h \subset \mathbf{H}^1(\Gamma) \tag{4.93}$$

and

$$\sup_{\mathbf{0} \neq \mathbf{k}^h \in \mathbf{Q}^h} \frac{\gamma^h \int_\Gamma \mathbf{k}^h \cdot \mathbf{n}\, d\Gamma}{\|\mathbf{k}^h\|_{1,\Gamma}} \geq C |\gamma^h| \quad \forall\, \gamma^h \in \mathbb{R} . \tag{4.94}$$

The solution necessarily satisfies $\mathbf{g}^h \in G^h$. Thus the operator E^h is well defined.

With these definitions we see that (4.85)-(4.92) can be written in the form (3.1)-(3.3).

By results concerning the approximation of the Navier-Stokes equations with inhomogeneous boundary conditions (see [15]), we obtain

$$\|(T - T^h) f\|_X \to 0$$

as $h \to 0$, for all $f = (\boldsymbol{\zeta}, \eta, \boldsymbol{\kappa}) \in Y$. This is simply a restatement of (H13).

(H14) follows trivially from (H13), the fact that T is self-adjoint, and the choice $(T^*)^h = T^h$.

To verify (H15), we note that the nondiscretized version of (4.93)-(4.94) certainly also holds, i.e.,

$$(\nabla_s \mathbf{k}, \nabla_s \mathbf{k})_\Gamma + (\mathbf{k}, \mathbf{k})_\Gamma \geq C \|\mathbf{k}\|_{1,\Gamma}^2 \quad \forall\, \mathbf{k} \in \mathbf{H}^1(\Gamma)$$

and

$$\sup_{\mathbf{0} \neq \mathbf{k} \in \mathbf{H}^1(\Gamma)} \frac{\gamma \int_\Gamma \mathbf{k} \cdot \mathbf{n}\, d\Gamma}{\|\mathbf{k}\|_{1,\Gamma}} \geq C |\gamma| \quad \forall\, \gamma \in \mathbb{R} .$$

Using the Brezzi theory for mixed finite element method (see [4] or [5]), we obtain that

$$\|(E - E^h)\boldsymbol{\tau}\|_{1,\Gamma} \to 0 \quad \text{as } h \to 0 ,$$

which verifies (H15).

(H16) and (H17) follow from the fact that N and $\mathcal{F}$ are polynomials. Here we also use imbedding theorems and Cauchy inequalities.

We set $\hat{Z} = \mathbf{L}^{3/2}(\Omega) \times \{0\} \times \{\mathbf{0}\}$. For each $(\mathbf{v}, q, \mathbf{s}) \in X = \mathbf{H}^1(\Omega) \times L^2_0(\Omega) \times \mathbf{H}^{-1/2}(\Gamma)$ and $(\mathbf{w}, r, \mathbf{k}) \in X = \mathbf{H}^1(\Omega) \times L^2_0(\Omega) \times \mathbf{H}^{-1/2}(\Gamma)$, Sobolev imbedding theorems imply that

$$[N'(\mathbf{u}, p, \mathbf{t})]^* \cdot (\mathbf{v}, q, \mathbf{s}) = -\begin{pmatrix} -(\mathbf{u}\cdot\nabla)\mathbf{v} + \mathbf{v}\cdot(\nabla\mathbf{u})^T \\ 0 \\ \mathbf{0} \end{pmatrix} \in \hat{Z}\,,$$

$$\big([N''(\mathbf{u}, p, \mathbf{t})]^* \cdot (\mathbf{v}, q, \mathbf{s})\big)\cdot(\mathbf{w}, r, \mathbf{k}) = -\begin{pmatrix} -(\mathbf{w}\cdot\nabla)\mathbf{v} + \mathbf{v}\cdot(\nabla\mathbf{w})^T \\ 0 \\ \mathbf{0} \end{pmatrix} \in \hat{Z}\,,$$

and

$$\big(\mathcal{F}''(\mathbf{u}, p, \mathbf{t}) \cdot (\mathbf{v}, q, \mathbf{s})\big)\cdot(\mathbf{w}, r, \mathbf{k}) = \begin{pmatrix} \begin{pmatrix} 3(u_1 - u_{01})^2 w_1 v_1 \\ \vdots \\ 3(u_d - u_{0d})^2 w_d v_d \end{pmatrix} \\ 0 \\ \mathbf{0} \end{pmatrix} \in \hat{Z}\,,$$

where d (= 2 or 3) is the space dimension. These relations verify (H18).

From the definition of the operator K we see that K maps $\mathbf{H}^1_n(\Gamma)$ into $\mathbf{L}^{3/2}(\Omega) \times \{0\} \times \mathbf{H}^1(\Gamma)$, i.e., K maps G into Z. Thus (H19) is verified.

Hence, we are now in a position to apply Theorem 3.5 to derive error estimates for the approximate solutions of the optimality system (4.54)-(4.56), (4.70)-(4.72) and (4.74). It should be noted that Lemma 3.4 implies that for almost all values of λ, the solutions of the optimality system are regular.

THEOREM 4.10. *Assume that Λ is a compact interval of $\mathbb{R}_+$ and that there exists a branch $\{(\lambda, \mathbf{u}(\lambda), p(\lambda), \mathbf{t}(\lambda), \mathbf{g}(\lambda), \boldsymbol{\nu}(\lambda), \phi(\lambda), \boldsymbol{\tau}(\lambda)) : \lambda \in \Lambda\}$ of regular solutions of the optimality system* (4.54)-(4.56), (4.70)-(4.72), *and* (4.74). *Assume that the finite element spaces X^h and G^h satisfy the hypotheses* (4.79)-(4.84). *Then, there exists a $\delta > 0$ and an $h_0 > 0$ such that for $h \le h_0$, the discrete optimality system* (4.85)-(4.92) *has a unique branch of solutions $\{(\lambda, \mathbf{u}^h(\lambda), p^h(\lambda), \mathbf{t}^h(\lambda), \mathbf{g}^h(\lambda), \boldsymbol{\nu}^h(\lambda), \phi^h(\lambda), \boldsymbol{\tau}^h(\lambda)) : \lambda \in \Lambda\}$ satisfying*

$$\Big(\|\mathbf{u}(\lambda) - \mathbf{u}^h(\lambda)\|_1 + \|p(\lambda) - p^h(\lambda)\|_0 + \|\mathbf{t}(\lambda) - \mathbf{t}^h(\lambda)\|_{-1/2,\Gamma} + \|\mathbf{g}(\lambda) - \mathbf{g}^h(\lambda)\|_{1,\Gamma} + \|\boldsymbol{\nu}(\lambda) - \boldsymbol{\nu}^h(\lambda)\|_1 + \|\phi(\lambda) - \phi^h(\lambda)\|_0 + \|\boldsymbol{\tau}(\lambda) - \boldsymbol{\tau}^h(\lambda)\|_{-1,2,\Gamma}\Big) < \delta \qquad \text{for all } \lambda \in \Lambda.$$

Moreover,

$$\lim_{h\to 0}\Big(\|\mathbf{u}(\lambda) - \mathbf{u}^h(\lambda)\|_1 + \|p(\lambda) - p^h(\lambda)\|_0 + \|\mathbf{t}(\lambda) - \mathbf{t}^h(\lambda)\|_{-1/2,\Gamma} + \|\mathbf{g}(\lambda) - \mathbf{g}^h(\lambda)\|_{1,\Gamma} + \|\boldsymbol{\nu}(\lambda) - \boldsymbol{\nu}^h(\lambda)\|_1 + \|\phi(\lambda) - \phi^h(\lambda)\|_0 + \|\boldsymbol{\tau}(\lambda) - \boldsymbol{\tau}^h(\lambda)\|_{-1,2,\Gamma}\Big) = 0$$

uniformly in $\lambda \in \Lambda$.

If, in addition, the solution satisfies $\big(\mathbf{u}(\lambda), p(\lambda), \mathbf{t}(\lambda), \mathbf{g}(\lambda), \boldsymbol{\nu}(\lambda), \phi(\lambda), \boldsymbol{\tau}(\lambda)\big) \in \mathbf{H}^{m+1}(\Omega) \times H^m(\Omega) \times \mathbf{H}^m(\Omega)|_\Gamma \times \mathbf{H}^{m+1}(\Omega)|_\Gamma \times \mathbf{H}^{m+1}(\Omega) \times H^m(\Omega) \times \mathbf{H}^m(\Omega)|_\Gamma$ *for* $\lambda \in \Lambda$, *then there exists a constant* C, *independent of* h, *such that*

$$
\begin{aligned}
&\Big(\|\mathbf{u}(\lambda) - \mathbf{u}^h(\lambda)\|_1 + \|p(\lambda) - p^h(\lambda)\|_0 + \|\mathbf{t}(\lambda) - \mathbf{t}^h(\lambda)\|_{-1/2,\Gamma} + \|\mathbf{g}(\lambda) - \mathbf{g}^h(\lambda)\|_{1,\Gamma} \\
&\quad + \|\boldsymbol{\nu}(\lambda) - \boldsymbol{\nu}^h(\lambda)\|_1 + \|\phi(\lambda) - \phi^h(\lambda)\|_0 + \|\boldsymbol{\tau}(\lambda) - \boldsymbol{\tau}^h(\lambda)\|_{-1,2,\Gamma} \Big) \\
&\quad \le C h^{m-1/2} \Big(\|\mathbf{u}(\lambda)\|_{m+1} + \|p(\lambda)\|_m + \inf_{\mathbf{v}\in\mathbf{H}^m(\Omega), \mathbf{v}|_\Gamma = \mathbf{t}} \|\mathbf{v}\|_m \\
&\qquad + \inf_{\mathbf{v}\in\mathbf{H}^{m+1}(\Omega), \mathbf{v}|_\Gamma = \mathbf{g}} \|\mathbf{v}\|_{m+1} + \|\boldsymbol{\nu}(\lambda)\|_{m+1} + \|\phi(\lambda)\|_m + \inf_{\mathbf{w}\in\mathbf{H}^m(\Omega), \mathbf{w}|_\Gamma = \boldsymbol{\tau}} \|\mathbf{w}\|_m \Big),
\end{aligned}
$$

uniformly in $\lambda \in \Lambda$.

Proof: All results follow from Theorem 3.5. For the last result, we also use (3.25) and the estimates (see, e.g., [16] or [17])

$$
\|(T^h T^{-1} - I)(\mathbf{u}, p, \mathbf{t})\|_X \le C h^m (\|\mathbf{u}\|_{m+1} + \|p\|_m + \inf_{\mathbf{v}\in\mathbf{H}^m(\Omega), \mathbf{v}|_\Gamma = \mathbf{t}} \|\mathbf{v}\|_m)
$$
$$
\text{for } \mathbf{u} \in \mathbf{H}^{m+1}(\Omega),\ p \in H^m(\Omega), \text{ and } \mathbf{t} \in \mathbf{H}^m(\Omega)|_\Gamma ,
$$

$$
\begin{aligned}
\|\big((T^*)^h (T^*)^{-1} - I\big)(\boldsymbol{\nu}, \phi, \boldsymbol{\tau})\|_{Y^*} &= \|(T^h T^{-1} - I)(\boldsymbol{\nu}, \phi, \boldsymbol{\tau})\|_X \\
&\le C h^m (\|\boldsymbol{\nu}\|_{m+1} + \|\phi\|_m + \inf_{\mathbf{w}\in\mathbf{H}^m(\Omega), \mathbf{w}|_\Gamma = \boldsymbol{\tau}} \|\mathbf{w}\|_m) \\
&\quad \text{for } \boldsymbol{\nu} \in \mathbf{H}^{m+1}(\Omega),\ \phi \in H^m(\Omega), \text{ and } \boldsymbol{\tau} \in \mathbf{H}^m(\Omega)|_\Gamma ,
\end{aligned}
$$

and

$$
\|(E^h E^{-1} - I)\mathbf{g}\|_{1,\Gamma} \le C h^{m-1/2} \inf_{\mathbf{v}\in\mathbf{H}^{m+1}(\Omega), \mathbf{v}|_\Gamma = \mathbf{g}} \|\mathbf{v}\|_{m+1} \qquad \text{for } \mathbf{g} \in \mathbf{H}^{m+1}(\Omega)|_\Gamma .
$$

In these estimates, the constant C is independent of h, $\mathbf{u}$, p, $\mathbf{t}$, $\mathbf{g}$, $\boldsymbol{\nu}$, ϕ, $\boldsymbol{\tau}$, and λ. □

Remark. If the control $\mathbf{g} \in \mathbf{H}^{m+3/2}(\Omega)|_\Gamma$, then the exponent of h in the error estimate of Theorem 4.10 can be increased from $(m - 1/2)$ to m. □

5. Conclusions

We have set up an abstract framework for the analysis and approximation of a class of nonlinear optimal control and optimization problems. Nonlinearities can occur in both the objective functional and in the constraints. Within the framework we have defined an abstract nonlinear optimization problem posed on infinite dimensional spaces, defined an approximate problem posed on finite dimensional spaces, and listed a number of hypotheses concerning the two problems. We then have shown that optimal solutions exist and that Lagrange multipliers may be used to enforce the constraints. We then used the Lagrange multiplier rule to derive an optimality system from which optimal states and controls may be deduced. We then derived existence results and error estimates for solutions of the approximate problem. The abstract framework and the results derived from that framework were then applied to three concrete control or optimization problems and their approximation by finite element methods. The first involves the von Kármán plate equations of nonlinear elasticity, the second the Ginzburg-Landau equations of superconductivity, and the third the

Navier-Stokes equations for incompressible, viscous flows. It is certainly possible to apply the abstract results that we have derived to a variety of optimal control problems arising in other settings.

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