QUANTUM FIELD BETWEEN MOVING MIRRORS:
A THREE DIMENSIONAL EXAMPLE

S. Hacyan, R. Jáuregui, C. Villarreal
Instituto de Física, U. N. A. M., Apdo. Postal 20-364,
México D.F. 01000,México

Abstract

The scalar quantum field between uniformily moving plates in three dimensional space is studied. Field equations for Dirichlet boundary conditions are solved exactly. Comparison of the resulting wavefunctions with their instantaneous static counterpart is performed via Bogolubov coefficients. Unlike the one dimensional problem, "particle" creation as well as squeezing may occur. The time dependent Casimir energy is also evaluated.

1 Introduction.

During the last twenty five years, much effort has been devoted to the understanding of quantum phenomena in systems under the influence of external conditions. In particular, Moore [1] initiated the study of the quantization of the electromagnetic field in a cavity with perfectly reflecting movable boundaries. Nowadays, it is recognized that this kind of system has several interesting nonclassical properties. Among them, there is the possibility of producing a nonadiabatic distortion of vacuum state leading to a modification of the field (Casimir) energy [2], along with the "creation" of photons [3]. It is also possible to obtain nonclassical statistical properties of the photons inside such a cavity: squeezing [4] and nonthermal distributions [5] are expected.

In order to avoid technical complications, most investigations of the field between moving plates have been restricted to the one dimensional case. However, it is not obvious whether all the results can be extrapolated to the three dimensional space. In this article, we study the quantum mechanics of a scalar massless field propagating between two plates which approach or recede each other with constant speed. The main results which follow are that the boundary conditions on the moving plates produce squeezed states and a nonzero vacuum expectation value of the particle number operator. These effects vanish in the one dimensional case [6]. The nonstationary Casimir energy is also evaluated.

2 Quantum field between the plates

Consider two parallel plates which are moving with a constant relative velocity. The natural coordinates for this problem are

$$t = \tau \cosh \zeta, \quad z = \tau \sinh \zeta, \tag{1}$$

where z and t are the Minkowski coordinates. Taking $-\infty < \tau, \zeta < \infty$, the Milne coordinates cover the entire past and future quadrants of the (z, τ) plane.

The equation for a massless scalar field ψ is:

$$\Box \psi \equiv -\frac{1}{\tau}\frac{\partial}{\partial \tau}\left(\tau \frac{\partial}{\partial \tau}\psi\right) + \left[\frac{1}{\tau^2}\frac{\partial^2}{\partial \zeta^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]\psi = 0 \, . \tag{2}$$

Now, the world-line of each plate can be taken as $\zeta = \pm\zeta_0$, where ζ_0 is the speed of each plate as seen in their center of velocity frame. Dirichlet boundary conditions on the plates take the simple form $\psi(\pm\zeta_0) = 0$. (It is also straightforward to impose Neumann boundary conditions.) It will be convenient to normalize the field in a box with fixed walls with separation a and b in the x and y directions

The general solution of the wave equation with the above boundary conditions can be decomposed as the product of a function of ζ and τ, and a plane wave solution propagating in the $\mathbf{r} \equiv \{x, y\}$ plane with wave vector $\mathbf{k} \equiv \pi\{n_x/a, n_y/b\}$. Explicitely,

$$\psi_{n,\mathbf{k}} = N_\psi \sin(k_x x) \, \sin(k_y y) \sin\left[\nu(\zeta + \zeta_0)\right] H^{(j)}_{i\nu}(k\tau) \, , \tag{3}$$

where

$$N_{\psi\pm} = \left(\frac{\pi}{ab\zeta_0}\right)^{1/2} e^{\pm\nu\pi/2} \, . \tag{4}$$

is a normalization factor, $H^{(j)}_{i\nu}$ are the standard Hankel functions ($j = 1, 2$), $k \equiv |\mathbf{k}|$, and we have defined $\nu \equiv n\pi/2\zeta_0$, n being a positive integer. In the future region, $t > 0$, $H^{(2)}_{i\nu}$ and $H^{(1)}_{i\nu}$ correspond to modes of positive and negative frequency respectively, while the opposite is true in the past region, $t < 0$ (see, e. g.: [7][8]). We will denote the positive (negative) frequency modes by ψ^+ (ψ^-).

At this point, we note that the field between plates separated a *fixed* distance L is given by

$$\phi_{n,\mathbf{k}} = N_\phi e^{i(\mathbf{k}\mathbf{r} - \omega t)} \sin\left[\frac{n\pi}{L}(z + L/2)\right] \, , \tag{5}$$

where $\omega = [\mathbf{k}^2 + (n\pi/L)^2]^{1/2}$ and the normalization coefficient is now

$$N_\phi = \left(\frac{4}{\omega abL}\right)^{1/2} \tag{6}$$

This coefficient follows from the scalar product in Minkowski coordinates:

$$(\phi_1, \phi_2)_{Ins} \equiv -i \int_{-L/2}^{L/2} dz \int dy \, dx \left(\phi_1^* \frac{\partial \phi_2}{\partial t} - \phi_2 \frac{\partial \phi_1^*}{\partial t}\right) \, , \tag{7}$$

where the subindex Ins refers to the instantaneous frame: the integration is taken over the volume enclosed by the fixed box at an arbitrary time t.

Hereafter, the field modes between moving plates will be called dynamical modes, whereas the modes between fixed plates will be called instantaneous modes. The crucial point is that between moving plates, the positive frequency modes of the dynamical field are a sum of both positive and

negative frequency modes of the instantaneous field between fixed plates. In general, any field mode ϕ_n can be expanded in terms of ψ_m as

$$\psi_m = \sum_n \alpha_{mn}\phi_n + \sum_n \beta_{mn}\phi_n^*, \tag{8}$$

where α_{mn} and β_{mn} are the Bogolubov coefficients, and the indices m and n describe the set of all parameters characterizing the modes. In the particular case we are interested in, take ϕ and ψ as the wave functions describing the fields between fixed and moving plates respectively. More precisely, consider a pair of plates of fixed separation $L = 2t\tanh\zeta_0$ which coincides instantaneously at Minkowski time t with a pair of plates moving with relative speed $v = \tanh(2\zeta_0)$ (Fig. 1). Now, the Bogolubov coefficients can be calculated taking the scalar product between the ψ_m and ϕ_n modes over the hypersurface $t = const.$ between the plates, (Note that for definitiveness we are considering the future region $t > 0$ where the plates are separating but it is straighforward to adapt the analysis to the past region.) Thus,

$$\alpha_{m,n,\mathbf{k},\mathbf{k}'} = (\phi_{n,\mathbf{k}}^+, \psi_{m,\mathbf{k}'}^+)_{Ins} \tag{9}$$

and

$$\beta_{m,n,\mathbf{k},\mathbf{k}'} = (\phi_{n,\mathbf{k}}^-, \psi_{m,\mathbf{k}'}^+)_{Ins}. \tag{10}$$

The integration involved in Eqs. (9) and (10) is to be performed over the hypersurface $t = const.$ bounded by the plates with separation $L = 2t\tanh\zeta_0$, with t interpreted as a parameter. At this point, we note that in any practical case, the velocities of the plates are non relativistic, that is $\zeta_0 \ll 1$. This permits to make a convenient approximation which, together with the change of variables $z = t\tanh\zeta$, simplifies Eq. (9) to the form:

$$\alpha_{m,n,\mathbf{k},\mathbf{k}'} \simeq i\frac{ab}{4}N_\phi N_\psi \delta_{\mathbf{k}\mathbf{k}'}\delta_{m,n}\zeta_0\, t\left[\frac{d}{dt}H_{i\nu}^{(2)}(kt) - i\omega H_{i\nu}^{(2)}(kt)\right]e^{i\omega t} \tag{11}$$

with an analogous equation for β. In this nonrelativistic approximation, it is very convenient to use the asymptotic forms of the Hankel functions which are valid for indices with large magnitudes[10]:

$$H_{i\nu}^{(2)}(kt) \simeq \sqrt{2/\pi}(\nu^2 + k^2t^2)^{-1/4}$$
$$\exp\left[-\nu\pi/2 - i(\nu^2 + k^2t^2)^{1/2} + i\nu\mathrm{Arsh}(\nu/kt) - i\pi/4\right]. \tag{12}$$

It finally follows that the Bogolubov coefficients can be approximated as

$$\alpha_{m,n,\mathbf{k},\mathbf{k}'} \simeq \delta_{\mathbf{k}\mathbf{k}'}\,\delta_{m,n}\left[1 + \frac{-i(kt)^2}{4[\nu^2 + (kt)^2]^{3/2}}\right]e^{-\pi i/4 + i\nu\mathrm{Arsh}(\nu/kt)} \tag{13}$$

and

$$\beta_{m,n,\mathbf{k},\mathbf{k}'} \simeq \delta_{\mathbf{k}\mathbf{k}'}\,\delta_{m,n}\left[\frac{i(kt)^2}{4[\nu^2 + (kt)^2]^{3/2}}\right]e^{-\pi i/4 - 2i[\nu^2 + (kt)^2]^{1/2} + i\nu\mathrm{Arsh}(\nu/kt)}. \tag{14}$$

It is also worth mentioning that in the case when there are no plates, the β coefficient turns out to be null when evaluated as a scalar product over the entire $\tau = const.$ hyperplane (the interested reader can check this point using the standard properties of the Hankel functions). This implies

63

that the Milne vacuum is equivalent to the Minkowski vacuum (see ref.[8] for a discussion of this point).

However, unlike the one dimensional case, the coefficient β is not null when the field is rectricted between the moving plates; thus the Fock space defined by the dynamical modes $\{\psi_{nk}\}$ is nontrivialy related to the instantaneous Fock space specified by the fixed modes $\{\phi_{nk}\}$. which, in principle, can be interpreted as the number of particles "created" by the motion of the plates; this point will be further discussed in the next section.

In particular, the dynamical vacuum state $|0>_{Dyn}$ has a nonzero expectation value of the number of "instantaneous" particles. Thus, the dynamical vacuum is a nontrivial distortion of the instantaneous vacuum state $|0>_{Ins}$. Explicitly, the "particle" number density is given by the distribution function

$$P_{\mathbf{k},n} = \sum_{\mathbf{k},n} |\beta_{\mathbf{k},n}|^2 = \frac{(kt)^4}{16[\nu^2 + (kt)^2]^3}, \tag{15}$$

Notice, however, that the real character of this particles is intrinsically related to its measurability.

The Bogolubov coefficients also relate coherent and squeezed sates. A state which is originally coherent according to a instantaneous configuration of the plates becomes a squeezed state with variances [2]:

$$\left\{ \begin{array}{c} \sigma_{x_i,x_j} \\ \sigma_{p_i,p_j} \end{array} \right\} = \frac{1}{2}\delta_{ij} + \mathrm{Re}\sum_n \left[\beta_{ni}^*\beta_{nj} \pm \frac{1}{2}(\alpha_{ni}\beta_{nj}^* + \alpha_{nj}\beta_{ni}^*) \right] \delta_{ij} \ ,$$

and

$$\sigma_{x_i,p_j} = \mathrm{Im}\sum_n \left[\beta_{ni}^*\beta_{nj} + \frac{1}{2}(\alpha_{ni}\beta_{nj}^* + \alpha_{nj}\beta_{ni}^*) \right] . \tag{16}$$

In our problem i denotes the set of variables $\mathbf{k}, n$. So that,

$$\left\{ \begin{array}{c} \sigma_{x_i,x_j} \\ \sigma_{p_i,p_j} \end{array} \right\} = \frac{1}{2}\delta_{ij} + \delta_{ij}\frac{(kt)^4}{16[\nu^2 + (kt)^2]^3}\left\{1 \mp \cos(2[\nu^2 + (kt)^2]^{1/2})\right\}$$

$$\pm \ \delta_{ij}\frac{(kt)^2}{4[\nu^2 + (kt)^2]^{3/2}}\sin(2[\nu^2 + (kt)^2]^{1/2}) \tag{17}$$

and

$$\sigma_{x_i,p_j} = \ - \ \delta_{ij}\frac{(kt)^4}{16[\nu^2 + (kt)^2]^3}\sin(2[\nu^2 + (kt)^2]^{1/2})$$

$$- \ \delta_{ij}\frac{(kt)^2}{4[\nu^2 + (kt)^2]^{3/2}}\cos(2[\nu^2 + (kt)^2]^{1/2}) . \tag{18}$$

Thus, the squeezed ellipse in phase space rotates with its ellipticity vanishing as an inverse power of time.

3 Casimir effect

Boundary conditions in any given system may alter its ground state. A well known example in quantum field theory is Casimir effect, i.e., the attractive force between two infinite conducting

plates in otherwise empty space. A direct consequence of the existence of Casimir forces is that mantaining even uniform relative motion of a pair of conducting plates requires external forces. It is also expected that the Casimir energy for nonstationary boundaries differs from the stationary case. In fact, one could think that the creation of particles with distribution (4.3) or the squeezing (4.5) of originally coherent states take place at the expense of the Casimir energy between the plates [2]. Notice, however, that such an interpretation is not obvious because the distribution of particles (4.3) diverges when integrated over all momenta. This is a consequence of dealing with idealized conducting plates.

The energy density for our nonstationary problem is given by [11]

$$\epsilon = \frac{1}{\pi} \int_0^\infty d\omega \omega^2 [\tilde{D}^+(\omega, \tau) + \tilde{D}^-(\omega, \tau)] \tag{19}$$

where $\tilde{D}^\pm$ denotes the Fourier transform of Wightman $D^\pm$ functions:

$$\tilde{D}^\pm(\omega, \tau) = \int_{-\infty}^\infty d\sigma e^{i\omega\sigma} D^\pm(\tau + \frac{1}{2}\sigma, \tau - \frac{1}{2}\sigma) \tag{20}$$

$$D^\pm(\tau + \frac{1}{2}\sigma, \tau - \frac{1}{2}\sigma) = _{Dyn}\!< 0|\psi(\tau \pm \frac{1}{2}\sigma, x, y, \zeta)\psi(\tau \mp \frac{1}{2}\sigma, x, y, \zeta)|0 >_{Dyn} \tag{21}$$

The free Wightman function in Milne coordinates is given by

$$D_F^\pm(x, x') = -\frac{1}{4\pi^2} \frac{1}{-\tau^2 - \tau'^2 + 2\tau\tau'\cosh(\zeta \pm \zeta') + (y - y')^2 + (x - x')^2} \tag{22}$$

The boundary conditions in our problem can be easily imposed by image method. So that, for two infinite plates

$$D^\pm(x, x') = -\frac{1}{4\pi^2} \sum_{n=-\infty}^\infty \frac{1}{-\tau^2 - \tau'^2 + 2\tau\tau'\cosh(\zeta \pm \zeta' - 4\zeta_0 n) + (y - y')^2 + (x - x')^2} \tag{23}$$

The energy of the field between plates per unit area is

$$E = \frac{1}{ab} \int_{-\zeta_0}^{\zeta_0} d\zeta \, \tau \, \epsilon \tag{24}$$

When performing the ζ integration two different contributions in the energy density arise. The first one has terms independent of the ζ_0 value. It is formed by the D^+ term and by the zero mode term of D^-. The Fourier tranform of the latter is the well known $\omega/2$ which gives rise to infinite vacuum expectation value in free space. The second kind of terms correspond to the Casimir energy per unit area, which is explicitely given by

$$E_C = -\frac{1}{4\pi^2} \frac{\zeta_0}{(2\tau)^3} \left[\sum_n \frac{1}{sh^4(2n\zeta_0)} + \sum_n \frac{1}{sh^2(2n\zeta_0)} \right]. \tag{25}$$

In the nonrelativistic limit:

$$E_C \simeq -\frac{1}{4\pi^2} \frac{\zeta_0}{(2\tau)^3} \left[\sum_n \frac{1}{(2\zeta_0 n)^4} + \sum_n \frac{1}{(2\zeta_0 n)^2} \right]. \tag{26}$$

and the instantaneous separation of the plates is

$$L \simeq 2\tau\zeta_0 - \frac{1}{3}\tau\zeta_0^3 \qquad (27)$$

Thus, we recover the static Casimir energy and find the first order correction due to the movement of the plates.

4   Concluding remarks

From the results obtained above, it is clear that the three dimensional case contains many features which are not present in one dimension. Roughly speaking, the one dimensional case corresponds to the limit $k = 0$ of our formulas, that is, when there are no modes propagating parallel to the plates.

The first thing to notice is that there is a squeezing of quantum states between the moving plates, although with peculiar oscillating variances.

The other important result concerns the possibilty of creating "photons". If one believes the standard interpretation of particle number (see, e. g. [3]), the motion of the plates creates new particles with a distribution function given by Eq. (4.3). This interpretation is *qualitatively* consistent with the change in Casimir energy due to the movement of the plates. In fact, whether real particles are created is a question which can be settled only when an operational definition of particle is given, for instance in terms of the interaction of the field with a well defined detector, e.g. an atom.

The results presented here are still preliminary since we have analized only a scalar field. The case of an electromagnetic field will be studied in a forthcoming publication. We expect that by considering a more realistic field, several problems will become clearer. Among them, the detectability of "created" particles by an incoming atom originally in an ordinary stationary state. In any case, the problem seems to be sufficiently rich to deserve further considerations.

References

[1] G.T. Moore, J. Math. Phys. **11**,2979 (1970).

[2] V. I. Man'ko, *Symmetries in Physics*, A. Frank and K.B. Wolf Editors, Springer-Verlag,Berlin, 1992, p. 311.

[3] N. Birrel and P. Davies, *Quantum Fields in Curved Space*, Cambridge University Press, Cambridge, England, 1982.

[4] V. V. Dodonov, A. B. Klimov and V.I. Man'ko, Phys. Lett. A **149**, 225 (1990); J. Sov. Laser Research **12**,439 (1991).

[5] S.Sarkar, J. Phys. **A21**,971 (1988).

[6] R. I. Baranov, Yu M. Shirokov, Soviet Phys. JETP **26** 1199 (1968); S. A. Fulling and P. C. W. Davies Proc. R. Soc. Lond. A **348**,393 (1976).

[7] A. di Sessa, J. Math. Phys. **15**,1892 (1974); Padmanabhan, Phys. Rev. Lett. **64**,2471 (1990).

[8] R. C. Arcuri, B. F. Svaiter and N. F. Svaiter, Mod. Phys. Lett. A **9**,19 (1994).

[9] S. Hacyan, R. Jauregui, F. Soto and C. Villarreal J. Math. Phys. A **23**, 2401 (1990).

[10] W. A. Watson, *A treatise on the Theory of Bessel Functions*, 2nd ed. Cambridge University Press, Cambridge, England, 1944.

[11] S. Hacyan, A. Sarmiento, G. Cocho and F. Soto, Phys. Rev. **D32**,914 (1985).