Coherent State Constructions of Bases for Some Physically Relevant Group Chains:

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Abstract

Rotor coherent state constructions are given for the Wigner supermultiplet SU(4)$\supset$ SU(2)xSU(2) and for the special irreducible representations [N0] of the SO(5)$\supset$ SO(3)$\supset$SO(2) group chain in exact parallel with the rotor coherent state construction for the SU(3)$\supset$ SO(3)$\supset$ SO(2) case given by Rowe, LeBlanc, and Repka. Matrix elements of the coherent state realizations of the group generators are given in all cases by very simple expressions in terms of angular momentum Wigner coefficients involving intrinsic projection labels K. The $\mathcal{K}$-matrix technique of vector coherent state theory is used to effectively elevate these K labels to the status of good quantum numbers. Analytic expressions are given for the $(\mathcal{K}\mathcal{K}^\dagger)$-matrices for many of the more important irreducible representations.

[redacted]

1 Introduction

In the past few years two types of coherent state constructions have been widely used to give very explicit matrix representations of many higher rank symmetry groups. In both, the irreducible representations of a larger group are constructed by an induction process from the irreducible representations of a simpler subgroup, hopefully with completely known Wigner-Racah calculus. In the more widely used first type of vector coherent state construction, [1], [2], [3], state vectors are mapped onto states of a multi-dimensional harmonic oscillator through a set of Bargmann variables, $\mathbf{z}$. This VCS construction has been widely used for many of the mathematically natural group chains such as U(n) $\supset$ U(n-1) x U(1) $\supset$ U(n-2) x U(1) $\supset$... for which the subgroup chain gives a complete labelling of the state vectors. In the more recent second type of coherent state construction rotor expansions are used which are particularly effective for many of the physically relevant group chains for which an SO(3) or SU(2) subgroup related to a physically meaningful angular momentum is the important subgroup in the group chain.

In this talk I want to focus on three group chains with particular relevance for nuclear structure problems: 1) The SU(3) $\supset$ SO(3) $\supset$ SO(2) chain of the 3-dimensional harmonic oscillator of the nuclear shell model with good orbital angular momentum; 2) The SU(4) $\supset$ SU(2) x SU(2) Wigner supermultiplet with good spin and isospin needed to complement the orbital functions of 1); and finally, the SO(5) $\supset$ SO(3) $\supset$ SO(2) chain needed e.g. for the 5-dimensional harmonic oscillator of the quadrupole phonon states of the Bohr-Mottelson collective model or for two of the important symmetry group chains of the interacting boson model of Iachello and Arima, [4]. Like all physically relevant group chains, all three suffer from a missing label problem. For all of them many solutions have been proposed for this problem, some of them highly practical, others quite elegant or numerically feasible; see e.g. the pioneering work of Moskinsky [5], [6], [7]. It is the purpose of this presentation to try to convince you that the new rotor coherent state constructions give a very elegant yet also very systematic and practical solution to the missing label problem. Moreover the solution is essentially exactly the same for all three examples.

2 The Rotor Coherent State Expansion for the SU(3) $\supset$ SO(3) Case

For the SU(3) scheme in a basis of good orbital angular momentum a coherent state rotor expansion has recently been given by Rowe, LeBlanc, and Repka, [8]. This construction is closely parallel to the seminal work of Elliott [9], [10], [11] in which an angular momentum projection label, K, the projection of the orbital angular momentum onto an intrinsic or body-fixed z'-axis is used in place of the missing quantum number in the SU(3) $\supset$ SO(3) $\supset$ SO(2) scheme. Only a brief synopsis of this work will be given, the details of the derivations being reserved for the second example.

In the rotor coherent state construction for SU(3) an arbitrary state vector, $|\Psi>$, is transformed into its coherent state wave function, $\Psi(\Omega)$,

$$\Psi(\Omega) = \langle \phi_{(\lambda\mu)} | R(\Omega) | \Psi \rangle \tag{1}$$

where $|\phi_{(\lambda\mu)}>$ is the highest weight state in the SU(3) $\supset$ SU(2) x U(1) scheme. Here R(Ω) is a standard rotation operator

$$R(\Omega) = e^{i\alpha L_z} e^{i\beta L_y} e^{i\gamma L_z} \tag{2}$$

where α, β, γ are Euler angles, L_i are space-fixed components of the orbital angular momentum operator, and where the scalar product is defined in terms of the standard angular measure

$$\int d\Omega = \int\int\int d\alpha sin\beta d\beta d\gamma \tag{3}$$

(Note, however, that the conventions for the R(Ω) of ref [8] are somewhat different from the most widely used nuclear physics conventions [12].) If $|\Psi>$ is expanded in angular momentum eigenvectors $|\nu$;LM>, where ν is shorthand for all additional quantum numbers, these angular momentum base vectors are mapped into their coherent state realizations

$$\begin{aligned}\Psi_{\nu;LM}(\Omega) &= \langle\phi_{(\lambda\mu)}|R(\Omega)|\nu; LM\rangle \\ &= \sum_K \langle\phi_{(\lambda\mu)}|\nu; LK\rangle D^L_{KM}(\Omega) \\ &= \sum_K c_K \sqrt{\frac{2L+1}{8\pi^2}} D^L_{KM}(\Omega).\end{aligned} \tag{4}$$

That is, angular momentum eigenstates are mapped into a basis of (normalized) D-functions which form a simple orthonormal set with respect to the rotational measure of eq. (3). The symmetries of the c_K are such that the *symmetrized, orthonormal* rotor basis

$$\sqrt{\frac{(2L+1)}{16\pi^2(1+\delta_{K0})}}\{D^L_{KM}(\Omega) + (-1)^{\lambda+\mu+L} D^L_{-KM}(\Omega)\} \tag{5}$$

is most convenient. Operators, **O**, are then mapped into their coherent state realizations, $\Gamma(\mathbf{O})$, through

$$\Gamma(\mathbf{O})\Psi(\Omega) = \langle\phi_{(\lambda\mu)}|R(\Omega)\mathbf{O}|\Psi\rangle. \tag{6}$$

It will of course be convenient to express all operators in terms of spherical tensors of rank, r, such that

$$\begin{aligned}\Gamma(O^r_m)\Psi(\Omega) &= \langle\phi_{(\lambda\mu)}|R(\Omega)O^r_m|\Psi\rangle \\ &= \langle\phi_{(\lambda\mu)}|R(\Omega)O^r_m R(\Omega)^{-1} R(\Omega)|\Psi\rangle \\ &= \sum_k D^r_{km}(\Omega)\langle\phi_{(\lambda\mu)}|O^r_k R(\Omega)|\Psi\rangle.\end{aligned} \tag{7}$$

The SU(3) group generators are the 3 components of the orbital angular momentum operators, L^1_m, and the 5 components of the Elliott ($\lambda\mu$-preserving) quadrupole operator, Q^2_m. The rotor realizations $\Gamma(L^1_m)$ are given in terms of their usual Euler angle realizations

$$\Gamma(L_0) = \frac{1}{i}\frac{\partial}{\partial\gamma}, \qquad \Gamma(L_\pm) = e^{\pm i\gamma}\{i\cot\beta\frac{\partial}{\partial\gamma} \pm \frac{\partial}{\partial\beta}\}, \tag{8}$$

where $\Gamma(L_0)$ has eigenvalue M, while $\Gamma(L_+)$, ($\Gamma(L_-)$) are standard M-raising, (lowering) operators. Eq. (7) shows that we need both the standard (right-action) rotor realizations of operators, O^r_m, as well as their left-action version which will be denoted by a $\bar{\Gamma}$,

$$\bar{\Gamma}(O^r_k)\Psi(\Omega) = \langle\phi_{(\lambda\mu)}|O^r_k R(\Omega)|\Psi\rangle. \tag{9}$$

The latter can be evaluated from the left action of the operator on the SU(3) highest weight state. For the angular momentum generators

$$\begin{aligned}
\bar{\Gamma}(L^1_k)\Psi_{\nu;LM}(\Omega) &= \langle\phi_{(\lambda\mu)}|L^1_k R(\Omega)|\nu; LM\rangle \\
&= \sum_K D^L_{KM}(\Omega)\langle\phi_{(\lambda\mu)}|L^1_k|\nu; LK\rangle \\
&= \sum_K D^L_{KM}(\Omega)\sqrt{L(L+1)}\langle LK1k|L(K+k)\rangle\langle\phi_{(\lambda\mu)}|\nu; L(K+k)\rangle \\
&= \sum_K D^L_{(K-k)M}(\Omega)\sqrt{L(L+1)}\langle L(K-k)1k|LK\rangle\langle\phi_{(\lambda\mu)}|\nu; LK\rangle,
\end{aligned} \tag{10}$$

so that $\bar{\Gamma}(L_+)$, ($\bar{\Gamma}(L_-)$) are now K-lowering, (raising) operators, a well known property of the intrinsic (body-fixed) components of angular momentum operators. The $\bar{\Gamma}(L_k)$ can therefore be given in terms of their Euler angle realizations through the well known rotor expressions for intrinsic components,

$$\bar{\Gamma}(L_0) = \frac{1}{i}\frac{\partial}{\partial\alpha}; \qquad \bar{\Gamma}(L_\pm) = e^{\mp i\alpha}\{\frac{i}{\sin\beta}\frac{\partial}{\partial\gamma} - i\cot\beta\frac{\partial}{\partial\alpha} \pm \frac{\partial}{\partial\beta}\}; \tag{11}$$

where $\bar{\Gamma}(L_0)$ has the simple eigenvalue K. The coherent state realizations of the quadrupole operator as given by Rowe, LeBlanc, and Repka [4] are

$$\begin{aligned}
\Gamma(Q^2_m) &= (2\lambda+\mu+3)D^2_{0m}(\Omega) - \tfrac{1}{2}[\mathbf{L}^2, D^2_{0m}(\Omega)] \\
&+\sqrt{\tfrac{3}{2}}\{D^2_{2m}(\Omega)(\mu - \bar{\Gamma}(L_0)) + D^2_{-2m}(\Omega)(\mu + \bar{\Gamma}(L_0))\},
\end{aligned} \tag{12}$$

i.e., these are expressed in terms of the very simple operators, $\mathbf{L}^2$, $\bar{\Gamma}(L_0)$, and simple D-functions. The well-known matrix elements of these D-functions in the orthonormal rotor basis of eq. (5) at once lead to the (standard) angular-momentum reduced matrix elements

$$\begin{aligned}
\langle K; L'||\Gamma(Q^2)||K; L\rangle &= \sqrt{(2L+1)}\{\langle LK20|L'K\rangle[(2\lambda+\mu+3) \\
&-\tfrac{1}{2}L'(L'+1) + \tfrac{1}{2}L(L+1)] + \delta_{K1}\langle L12-2|L'1\rangle\sqrt{\tfrac{3}{2}}(-1)^{L+\lambda+1}(\mu+1),\}
\end{aligned} \tag{13}$$

$$\langle (K\pm 2); L'||\Gamma(Q^2)||K; L\rangle = \sqrt{(2\mathrm{L}+1)}\langle LK2\pm 2|L'(K\pm 2)\rangle\sqrt{\tfrac{3}{2}}(\mu \mp K)\sigma_{KK'}, \tag{14}$$

with $\sigma_{KK'} = \sqrt{2}$ for either K or $K' = 0$, and $\sigma_{KK'} = 1$ otherwise. The simplicity of this result is negated partly by the fact that the $\Gamma(Q^2_m)$ are nonunitary realizations of these operators. In order to translate the above nonhermitian matrix elements into the hermitian matrix elements of Q^2_m in ordinary Hilbert space, the nonunitary realizations, $\Gamma(\mathbf{O})$, of coherent state constructions is converted to a unitary realization $\gamma(\mathbf{O})$ via the $\mathcal{K}$-operator equation

$$\gamma(\mathbf{O}) = \mathcal{K}^{-1}\Gamma(\mathbf{O})\mathcal{K}. \tag{15}$$

Matrix elements of the $\mathcal{K}$ and $\mathcal{K}^{-1}$ operators can then be used to convert the nonhermitian matrix elements of $\Gamma(\mathbf{O})$ to hermitian form $\gamma(\mathbf{O})$ and hence directly to hermitian form in ordinary Hilbert space. Thus

$$\langle\nu'; L'||Q^2||\nu; L\rangle = \sum_{K,K'}(\mathcal{K}^{-1}(L'))_{\nu'K'}\langle K'; L'||\Gamma(Q^2)||K; L\rangle(\mathcal{K}(L))_{K\nu} \tag{16}$$

where the new quantum numbers, ν, are defined through the eigenvalues of the hermitian matrix $(\mathcal{K}\mathcal{K}^\dagger)$ which can be calculated in coherent state theory by simple recursion techniques through the known matrix elements of the group generators $\Gamma(Q^2)$. The $(\mathcal{K}\mathcal{K}^\dagger)_{KK'}$ matrix elements, moreover, can be given in simple analytic form [13] as functions of λ, μ, and L. As a simple example, the $(\mathcal{K}\mathcal{K}^\dagger)$ matrix for the irreducible representations $(\lambda 2)$ with λ-L = even is 2-dimensional, with K=2 or 0, in the basis of eq. (5), with

$$(\mathcal{K}\mathcal{K}^\dagger)_{22} = \tfrac{1}{2}[2(\lambda+3)^2 - L(L+1)]C$$
$$(\mathcal{K}\mathcal{K}^\dagger)_{00} = [2(\lambda+2)^2 - L(L+1)]C$$
$$(\mathcal{K}\mathcal{K}^\dagger)_{20} = \sqrt{\tfrac{1}{2}(L-1)L(L+1)(L+2)}C, \tag{17}$$

with

$$C = N/(\lambda+2-L)!!(\lambda+L+3)!!, \tag{18}$$

with

$$N = \frac{(\lambda-1)!!(\lambda+4)!!}{2(\lambda+3)} \quad for \quad \lambda = odd; \qquad N = \frac{\lambda!!(\lambda+3)!!}{2(\lambda+2)} \quad for \quad \lambda = even, \tag{19}$$

where $\lambda!! = \lambda\ (\lambda\text{-}2)\ \ldots\ 2$ (or 1). The $(\mathcal{K}\mathcal{K}^\dagger)$-matrix can be converted into the needed matrix elements of $\mathcal{K}$ and $\mathcal{K}^{-1}$ through the unitary matrix, U, which diagonalizes the hermitian matrix $\mathcal{K}\mathcal{K}^\dagger$

$$(U(\mathcal{K}\mathcal{K}^\dagger)U^\dagger)_{\nu'\nu} = \Lambda_\nu \delta_{\nu'\nu} \tag{20}$$

with

$$(\mathcal{K})_{K\nu} = U^\dagger_{K\nu}\sqrt{\Lambda_\nu}; \qquad (\mathcal{K}^{-1})_{\nu K} = \frac{1}{\sqrt{\Lambda_\nu}} U_{\nu K} \tag{21}$$

defined for all states ν with non zero eigenvalue, Λ_ν. Note, that a zero eigenvalue Λ_ν signals a forbidden state. The matrix of eq. (17), e.g., has one zero eigenvalue for L = λ + 2; so that there is but a single allowed state for this maximum L-value. For L > (λ+2) the matrix elements of $\Gamma(Q^2)$ insure that all matrix elements of $\mathcal{K}\mathcal{K}^\dagger$ are zero. The $\mathcal{K}$-matrix technique of coherent state theory thus effectively converts the Elliott K-projection label to the status of a good quantum number.

It should, however, be stressed that the coherent state construction outlined here is very closely related to the Elliott angular momentum projection technique [10]. The matrix elements of Q^2_m in the form of eqs. 13) and 14) have essentially been given by Elliott in ref. [10]. Except for an overall normalization, (see eq. (19), which is related to the fact that the 1-dimensional $(\mathcal{K}\mathcal{K}^\dagger)$ for the minimum L-value of 0 (or 1) is chosen to be unity in the coherent state construction), the $(\mathcal{K}\mathcal{K}^\dagger)$-matrix elements are given by the overlap matrix of Elliott (see e.g., eq. (A.3) for $A(KLK')$ of ref. [11]; and the specific analytic functions given by Vergados for the lower μ-values in table 2A of ref. [14]).

What then are the advantages of the coherent state rotor construction? By mapping the SU(3) angular momentum eigenstates onto the orthonormal basis, eq. (5), of the rotor expansion the construction of matrix elements is split into two clearly separated simple steps: In step 1, matrix elements of $\Gamma(\mathbf{O})$ are given very simply in the orthonormal rotor basis where K defines the orthonormal states. In step 2, which is the unitarization process, K is converted to the quantum number ν in ordinary Hilbert space. By relating ν to the non-zero eigenvalues of $(\mathcal{K}\mathcal{K}^\dagger)$ an essentially author-independent choice can be made for the quantum number ν. Although some numerical work is required in the determination of the U-matrix elements which diagonalize the multi-dimensional $(\mathcal{K}\mathcal{K}^\dagger)$-matrices; no arbitrary choices are made in a Gram-Schmidt orthonormalization process, as in the Vergados basis [14], which is an attempt to make the Hilbert space quantum number, (the κ of ref. ([14]), as close as possible to the Elliott projection label K. (In this connection, it is interesting to note that both κ and ν tend to pure K-values in the limit λ >> L as a glance at the special example of eq. (17) will verify.

3 A Double Rotor Coherent State Expansion for the Wigner Supermultiplet SU(4) ⊃ SU(2) x SU(2).

A complete labelling scheme for the Wigner supermultiplet has been achieved by Draayer [15] who used the Elliott angular momentum projection technique to augment the spin and isospin quantum numbers (SM_S), (TM_T) with the projection labels K_S and K_T. In order to calculate the generator matrix elements and SU(4) reduced Wigner coefficients in this fully labelled but nonorthogonal basis, however, Draayer first calculates the transformation coefficients to the canonical fully specified orthonormal U(4) ⊃ U(3) ⊃ U(2) ⊃ U(1) basis, leading to a somewhat laborious calculational algorithm. This example therefore will fully illustrate the power of the rotor coherent state construction which leads in a very simple and direct way to the desired results.

The supermultiplet scheme is based on the four spin-charge states of a single nucleon, $|m_s m_t\rangle$, with nucleon, $|m_s\ m_t>$, with

$$\begin{aligned} |a\rangle &= |+\tfrac{1}{2}+\tfrac{1}{2}>, \qquad |b\rangle = |-\tfrac{1}{2}-\tfrac{1}{2}>, \\ |c\rangle &= |+\tfrac{1}{2}-\tfrac{1}{2}>, \qquad |d\rangle = |-\tfrac{1}{2}+\tfrac{1}{2}>. \end{aligned} \tag{22}$$

To gain the most convenient double rotor expansion it will be useful to define the basis states $|i\rangle$, $i = 1,\ldots,4$, by

$$\begin{aligned} |a\rangle &= \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle), \qquad |b\rangle = \frac{1}{\sqrt{2}}(-|1\rangle + |2\rangle), \\ |c\rangle &= \frac{1}{\sqrt{2}}(|3\rangle + |4\rangle), \qquad |d\rangle = \frac{1}{\sqrt{2}}(-|3\rangle + |4\rangle), \end{aligned} \tag{23}$$

and define the 15 supermultiplet generators [17], **S**,**T**, and $\mathbf{E} = \sigma\tau$ in terms of U(4) generators, C_{ij},

$$C_{ij} = \sum_\alpha a^\dagger_{\alpha i} a_{\alpha j}, \qquad i,j = 1,\ldots,4 \tag{24}$$

where i, j give the spin, isospin quantum numbers and α stands for all additional (orbital) quantum numbers needed to specify the single nucleon creation and annihilation operators. In terms of the C_{ij} the generators are

$$S_0 = \tfrac{1}{2}(C_{12} + C_{21} + C_{34} + C_{43})$$

$$
\begin{aligned}
S_+ &= \tfrac{1}{2}(-C_{13} - C_{23} + C_{14} + C_{24} - C_{31} + C_{32} - C_{41} + C_{42}) \\
S_- &= \tfrac{1}{2}(-C_{31} - C_{32} + C_{41} + C_{42} - C_{13} + C_{23} - C_{14} + C_{24}) \\
T_0 &= \tfrac{1}{2}(C_{12} + C_{21} - C_{34} - C_{43}) \\
T_+ &= \tfrac{1}{2}(C_{13} + C_{23} + C_{14} + C_{24} + C_{31} - C_{32} - C_{41} + C_{42}) \\
T_- &= \tfrac{1}{2}(C_{31} + C_{32} + C_{41} + C_{42} + C_{13} - C_{23} - C_{14} + C_{24}) \\
E_{00} &= \tfrac{1}{2}(C_{11} + C_{22} - C_{33} - C_{44}) \\
E_{10} &= \tfrac{1}{2\sqrt{2}}(C_{13} + C_{23} - C_{14} - C_{24} - C_{31} + C_{32} - C_{41} + C_{42}), \\
E_{-10} &= \tfrac{1}{2\sqrt{2}}(-C_{31} - C_{32} + C_{41} + C_{42} + C_{13} - C_{23} + C_{14} - C_{24}), \\
E_{01} &= \tfrac{1}{2\sqrt{2}}(-C_{13} - C_{23} - C_{14} - C_{24} + C_{31} - C_{32} + C_{42} - C_{41}), \\
E_{0-1} &= \tfrac{1}{2\sqrt{2}}(C_{31} + C_{32} + C_{41} + C_{42} - C_{13} + C_{23} - C_{24} + C_{14}), \\
E_{11} &= \tfrac{1}{2}(-C_{11} + C_{22} + C_{12} - C_{21}), \\
E_{-1-1} &= \tfrac{1}{2}(-C_{11} + C_{22} - C_{12} + C_{21}), \\
E_{1-1} &= \tfrac{1}{2}(C_{33} - C_{44} - C_{34} + C_{43}), \\
E_{-11} &= \tfrac{1}{2}(C_{33} - C_{44} + C_{34} - C_{43}).
\end{aligned} \tag{25}
$$

The SU(4) irreducible representations are labelled by 4-rowed Young tableaux partition labels $[f_1,f_2,f_3,f_4]$, by the SU(4) labels $\{\lambda_1,\ \lambda_2,\ \lambda_3\}$, or by the Wigner supermultiplet (or standard Cartan SO(6) labels (P, P′, P″)), with

$$
\begin{aligned}
\lambda_1 &= f_1 - f_2, \qquad \lambda_2 = f_2 - f_3, \qquad \lambda_3 = f_3 - f_4, \\
P &= \tfrac{1}{2}(\lambda_1 + 2\lambda_2 + \lambda_3), \quad P' = \tfrac{1}{2}(\lambda_1 + \lambda_3), \quad P'' = \tfrac{1}{2}(\lambda_1 - \lambda_3).
\end{aligned} \tag{26}
$$

These characterize the highest weight state $|\phi\rangle$ with

$$
\begin{aligned}
& C_{ij}|\phi\rangle = 0 \quad \text{for} \quad i < j \\
& C_{11}|\phi\rangle = (\lambda_1 + \lambda_2 + \lambda_3)|\phi >, \qquad C_{22}|\phi\rangle = (\lambda_2 + \lambda_3)|\phi >, \\
& C_{33}|\phi\rangle = \lambda_3|\phi\rangle, \qquad C_{44}|\phi >= 0.
\end{aligned} \tag{27}
$$

The double rotor expansion uses the double rotation operator $R(\Omega) \equiv R(\Omega_S)R(\Omega_T)$, with Euler angles $\alpha_S, \beta_S, \gamma_S \equiv \Omega_S$ and $\alpha_T, \beta_T, \gamma_T \equiv \Omega_T$ in the spin and isospin space. Draayer [15] has shown that the set of states, $\{R\ (\Omega)\ |\phi\rangle\}$, obtained by rotation of the highest weight state through all possible angles $\alpha_S, \ldots, \gamma_T$ span the full SU(4) space. Arbitrary state vectors $|\Psi\rangle$ in this space are now tranformed into their coherent state realizations with coherent state wave function

$$
\Psi(\Omega) = \langle\phi|R(\Omega)|\Psi\rangle. \tag{28}
$$

A state $|\alpha S M_S T M_T\rangle$ with definite spin and isospin quantum numbers is represented by

$$
\begin{aligned}
& \Psi_{\alpha S M_S T M_T}(\Omega) = \langle\phi|R(\Omega)|\alpha S M_S T M_T\rangle = \\
& \sum_{K_S, K_T} \langle\phi|\alpha S K_S T K_T\rangle D^S_{K_S M_S}(\Omega_S) D^T_{K_T M_T}(\Omega_T).
\end{aligned} \tag{29}
$$

Draayer [15] has shown that the SU(4) irreducible representations $[f_1, f_2, f_3, f_4] \equiv \{\lambda_1\lambda_2\lambda_3\}$ are spanned by the double rotor wave functions with K_S, K_T-values restricted by

$$\begin{aligned}(K_S + K_T) &= \pm\lambda_1, \pm(\lambda_1 - 2), \pm(\lambda_1 - 4), \ldots, 0(or \pm 1),\\ (K_S - K_T) &= \pm\lambda_3, \pm(\lambda_3 - 2), \pm(\lambda_3 - 4), \ldots, 0(or \pm 1).\end{aligned} \tag{30}$$

Again, it will be useful to introduce symmetrized combinations of the D-functions. The double rotor coherent state wave functions are then spanned by the symmetrized (normalized) double rotor functions

$$\begin{aligned}&\frac{1}{8\pi^2}\left[\frac{(2S+1)(2T+1)}{2(1+\delta_{K_S0}\delta_{K_T0})}\right]^{\frac{1}{2}}\\ &\times\{D^S_{K_SM_S}(\Omega_S)D^T_{K_TM_T}(\Omega_T) + (-1)^{\lambda_2+\lambda_3+S+T}D^S_{-K_SM_S}(\Omega_S)D^T_{-K_TM_T}(\Omega_T)\},\end{aligned} \tag{31}$$

and it will therefore be sufficient to choose $K_S \geq 0$, and for $K_S = 0 : K_T \geq 0$. The requirement $S \geq |K_S|, T \geq |K_T|$ together with the structure of the $\mathcal{K}\mathcal{K}^\dagger$-matrices will determine the multiplicity of a given S, T value. For states with low values of $S + T$, for which the eigenvalues of $\mathcal{K}\mathcal{K}^\dagger$ are all nonzero (no redundant states), the number of occurrences of a given S, T will be determined by the number of possible K_S, K_T combinations. States with the maximum possible value of $S + T = \lambda_1 + \lambda_2 + \lambda_3 = f_1 - f_4$, and with S(or T)$\geq \frac{1}{2}(\lambda_1 + \lambda_3)$, always have an occurrence of 1. For these S, T-values the $\mathcal{K}\mathcal{K}^\dagger$ matrix always has only a single nonzero eigenvalue giving only a single nonredundant or physically allowed state. In general, the states with $S + T \geq \lambda_2 + 2$ will have $\mathcal{K}\mathcal{K}^\dagger$-matrices with some zero eigenvalues and hence some physically forbidden states. Table 1 gives a specific example, the possible S, T-values for the irreducible representation [8620] with $\{\lambda_1\lambda_2\lambda_3\} = \{242\}$. In this case there are five possible symmetrized states of the type of eq. (31), with $K_SK_T = 20, 11, 1-1, 02$, and 00. Note that states with $K_SK_T = 00$ must have $S + T =$ even since $\lambda_2 + \lambda_3 =$ even. States with both S and $T \geq 2$ can thus have a 5-fold occurrence for $S + T =$ even and a 4-fold occurrence for $S + T =$ odd. The maximum $S + T$-value is 8 in this case. States with $S + T = 8$, S(or T)≥ 2, are all single as indicated in the first column of the table. The $\mathcal{K}\mathcal{K}^\dagger$-matrix for this case has four eigenvalues of 0. In addition, it can be shown that the $\mathcal{K}\mathcal{K}^\dagger$-matrices for states with $S + T = 7$ have two eigenvalues of 0, thus reducing the possible number of physical states by two, while states with $S + T = 6$ lead to $\mathcal{K}\mathcal{K}^\dagger$-matrices with one eigenvalue of 0 reducing the possible number of physical states by one.

In the VCS rotor expansion operators, $\mathbf{O}$, are transformed into their VCS realizations, $\Gamma(\mathbf{O})$, through $\mathbf{O}|\Psi\rangle \rightarrow \Gamma(\mathbf{O})\Psi(\Omega)$, with (cf. eq. (6)),

$$\Gamma(\mathbf{O})\Psi(\Omega) = \langle\phi|R(\Omega)\mathbf{O}|\Psi\rangle. \tag{32}$$

The SU(4) generators, $\mathbf{O} = \mathbf{S}, \mathbf{T}, \mathbf{E}$ are again of greatest interest. Again, both the left and right realizations of $\mathbf{S}$ and $\mathbf{T}$ can be expressed in terms of the Euler angles α_S, β_S, γ_S and α_T, β_T, γ_T as in eqs. (8) and (11). Now $\Gamma(S_0)$ has the simple eigenvalue M_S whereas $\bar{\Gamma}(S_0)$ has eigenvalue K_S; while $\Gamma(S_+)$, ($\Gamma(S_-)$ are M_S-raising, (lowering) operators, whereas $\bar{\Gamma}(S_+)$, ($\bar{\Gamma}(S_-)$) are K_S-lowering, (raising) operators; with similar properties for the $\Gamma(\mathbf{T})$ and $\bar{\Gamma}(\mathbf{T})$.

The generators $\mathbf{E}$ can be transformed into left-action operators via

$$\begin{aligned} & \Gamma(E_{m_S m_T})\Psi(\Omega) = \langle\phi|R(\Omega)E_{m_S m_T}|\Psi\rangle \\ = & \langle\phi|(R(\Omega)E_{m_S m_T}R^{-1}(\Omega))R(\Omega)|\Psi\rangle \\ = & \sum_{k_S k_T}\langle\phi|E_{k_S k_T}R(\Omega)|\Psi\rangle D^1_{k_S m_S}(\Omega_S)D^1_{k_T m_T}(\Omega_T). \end{aligned} \tag{33}$$

Using the properties of the highest weight state, eq. (27), and the specific expressions of the generators, eqs. (25), it can be seen that

$$\begin{aligned} \langle\phi|E_{00} &= \tfrac{1}{2}(\lambda_1+2\lambda_2+\lambda_3)\langle\phi|, \\ \langle\phi|E_{\pm10} &= -\tfrac{1}{\sqrt{2}}\langle\phi|S_\pm, \qquad \langle\phi|E_{0\pm1} = -\tfrac{1}{\sqrt{2}}\langle\phi|T_\pm, \\ \langle\phi|E_{\pm1\pm1} &= \tfrac{1}{2}\langle\phi|(-\lambda_1 \pm S_0 \pm T_0), \qquad \langle\phi|E_{\pm1\mp1} = \tfrac{1}{2}\langle\phi|(\lambda_3 \mp S_0 \pm T_0)). \end{aligned} \tag{34}$$

At this stage the usefulness of the transformation (22) can be appreciated. Although it seemingly complicates the relations of the group generators in terms of the C_{ij}, it can now be seen that the transformation (22) makes it possible to express the operators $E_{k_S k_T}$ in their left actions on the single highest weight state into equivalent left actions of components of $\mathbf{S}$ or $\mathbf{T}$ or the Cartan generators C_{ii}. The relations (34) lead to

$$\begin{aligned} & \Gamma(E_{m_S m_T})\Psi(\Omega) = \{\tfrac{1}{2}(\lambda_1+2\lambda_2+\lambda_3)D^1_{0m_S}(\Omega_S)D^1_{0m_T}(\Omega_T) \\ & -\tfrac{1}{\sqrt{2}}[D^1_{1m_S}(\Omega_S)\bar{\Gamma}(S_+) + D^1_{-1m_S}(\Omega_S)\bar{\Gamma}(S_-)]D^1_{0m_T}(\Omega_T) \\ & -\tfrac{1}{\sqrt{2}}D^1_{0m_S}(\Omega_S)[D^1_{1m_T}(\Omega_T)\bar{\Gamma}(T_+) + D^1_{-1m_T}(\Omega_T)\bar{\Gamma}(T_-)] \\ & +\tfrac{1}{2}D^1_{1m_S}(\Omega_S)D^1_{1m_T}(\Omega_T)(-\lambda_1+\bar{\Gamma}(S_0)+\bar{\Gamma}(T_0)) \\ & +\tfrac{1}{2}D^1_{-1m_S}(\Omega_S)D^1_{-1m_T}(\Omega_T)(-\lambda_1-\bar{\Gamma}(S_0)-\bar{\Gamma}(T_0)) \\ & +\tfrac{1}{2}D^1_{1m_S}(\Omega_S)D^1_{-1m_T}(\Omega_T)(\lambda_3-\bar{\Gamma}(S_0)+\bar{\Gamma}(T_0)) \\ & +\tfrac{1}{2}D^1_{-1m_S}(\Omega_S)D^1_{1m_T}(\Omega_T)(\lambda_3+\bar{\Gamma}(S_0)-\bar{\Gamma}(T_0))\}\langle\phi|R(\Omega)|\Psi\rangle. \end{aligned} \tag{35}$$

Finally, using the identity

$$[\mathbf{S}^2, D^1_{0m_S}(\Omega_S)] = \sqrt{2}(D^1_{1m_S}(\Omega_S)\bar{\Gamma}(S_+) + D^1_{-1m_S}(\Omega_S)\bar{\Gamma}(S_-)) + 2D^1_{0m_S}(\Omega_S), \tag{36}$$

and the similar relation for the isospin operators, we obtain

$$\begin{aligned} \Gamma(E_{m_S m_T}) &= \{\tfrac{1}{2}(\lambda_1+2\lambda_2+\lambda_3)+2\}D^1_{0m_S}(\Omega_S)D^1_{0m_T}(\Omega_T) \\ &- \tfrac{1}{2}\{[\mathbf{S}^2, D^1_{0m_S}(\Omega_S)]D^1_{0m_T}(\Omega_T) + D^1_{0m_S}(\Omega_S)[\mathbf{T}^2, D^1_{0m_T}(\Omega_T)]\} \\ &+ \tfrac{1}{2}D^1_{1m_S}(\Omega_S)D^1_{1m_T}(\Omega_T)(-\lambda_1+\bar{\Gamma}(S_0)+\bar{\Gamma}(T_0)) \\ &+ \tfrac{1}{2}D^1_{-1m_S}(\Omega_S)D^1_{-1m_T}(\Omega_T)(-\lambda_1-\bar{\Gamma}(S_0)-\bar{\Gamma}(T_0)) \\ &+ \tfrac{1}{2}D^1_{1m_S}(\Omega_S)D^1_{-1m_T}(\Omega_T)(\lambda_3-\bar{\Gamma}(S_0)+\bar{\Gamma}(T_0)) \\ &+ \tfrac{1}{2}D^1_{-1m_S}(\Omega_S)D^1_{1m_T}(\Omega_T)(\lambda_3+\bar{\Gamma}(S_0)-\bar{\Gamma}(T_0)). \end{aligned} \tag{37}$$

This is the analogue of eq. (12). Using the symmetrized (normalized) rotor basis states of eq. (31) the standard S and T-space rotational measure, and a standard definition of a spin, isospin reduced double-barred matrix element, together with the well-known triple D-function integrals, we obtain (for the general K_SK_T case with $K_S + K_T > 1$) the result

$$\begin{aligned}
&\langle K_SK_T; S'T'||\Gamma(E)||K_SK_T; ST\rangle \\
&= [(2S+1)(2T+1)]^{\frac{1}{2}}\langle SK_S10|S';K_S\rangle\langle TK_T10|T'K_T\rangle \\
&\times\{\tfrac{1}{2}(\lambda_1+2\lambda_2+\lambda_3)+2-\tfrac{1}{2}S'(S'+1)+\tfrac{1}{2}S(S+1)-\tfrac{1}{2}T'(T'+1)+\tfrac{1}{2}T(T+1)\}, \\
&\langle (K_S\pm 1)(K_T\pm 1); S'T'||\Gamma(E)||K_SK_T; ST\rangle \\
&= \tfrac{1}{2}[(2S+1)(2T+1)]^{\frac{1}{2}}\langle SK_S1\pm 1|S'(K_S\pm 1)\rangle\langle TK_T1\pm 1|T'(K_T\pm 1)\rangle(-\lambda_1\pm K_S\pm K_T), \\
&\langle (K_S\pm 1)(K_T\mp 1); S'T'||\Gamma(E)||K_SK_T; ST\rangle \\
&= \tfrac{1}{2}[(2S+1)(2T+1)]^{\frac{1}{2}}\langle SK_S1\pm 1|S'(K_S\pm 1)\rangle\langle TK_T1\mp 1|T'(K_T\mp 1)\rangle(\lambda_3\mp K_S\pm K_T). \quad (38)
\end{aligned}$$

The special cases $K_SK_T = \frac{1}{2}\,\frac{1}{2};\ \frac{1}{2},\text{-}\frac{1}{2}$; 10 and 01 will again require additional terms, (the analogues of the special case K = 1 for eq. (13)). The details can be found in ref. [16]. As for the SU(3) case, the reduced matrix elements of the $\Gamma(\mathbf{E})$ are given by very simple expressions involving ordinary spin (S) and isopin (T) Wigner coefficients with projection labels K_S and K_T. Since the $\Gamma(\mathbf{E})$ are nonunitary realizations of the generators $\mathbf{E}$ these first have to be translated to unitary form via the $\mathcal{K}$-operators through the analogs of eq. (15) and (16). The $(\mathcal{K}\mathcal{K}^\dagger)$-matrix elements are now calculated most easily through recursion relations such as

$$\begin{aligned}
&\sum_{K'_{S_2}K'_{T_2}} (\mathcal{K}\mathcal{K}^\dagger(S',T'))_{K'_{S_1}K'_{T_1};K'_{S_2}K'_{T_2}}\langle K_{S_2}K_{T_2}; ST||\Gamma(E)||K'_{S_2}K'_{T_2}; S'T'\rangle(-1)^{S+T-S'-T'} \\
&= \sum_{K_{S_1}K_{T_1}} \langle K'_{S_1}K'_{T_1}; S'T'||\Gamma(E)||K_{S_1}K_{T_1}; ST\rangle(\mathcal{K}\mathcal{K}^\dagger(S,T))_{K_{S_1}K_{T_1};K_{S_2}K_{T_2}} \quad (39)
\end{aligned}$$

If the quantum numbers $(\lambda_1 + \lambda_3)$ - are not too large, the dimensions of the $(\mathcal{K}\mathcal{K}^\dagger)$ matrices will be of manageable size so that analytic expressions can be given for the matrix elements as functions of S, T, and the SU(4) quantum numbers. As a special example, the irreducible representation $[f_1f_2f_3f_4] = [y+2, y, 0, 0] = \{\lambda_1\lambda_2\lambda_3\} = \{2y0\}$ has the simple $(\mathcal{K}\mathcal{K}^\dagger)$-matrix elements

$$\begin{aligned}
(\mathcal{K}\mathcal{K}^\dagger(S,T))_{11,11} &= \tfrac{1}{2}[(y+3)(y+4) - S(S+1) - T(T+1)]\mathrm{CF}, \\
(\mathcal{K}\mathcal{K}^\dagger(S,T))_{00,00} &= [(y+3)(y+2) - S(S+1) - T(T+1)]\mathrm{CF}, \\
(\mathcal{K}\mathcal{K}^\dagger(S,T))_{11,00} &= -[2S(S+1)T(T+1)]^{\frac{1}{2}}\mathrm{CF}, \quad (40)
\end{aligned}$$

with common factor given by

$$\mathrm{CF} = \frac{\mathrm{Num}}{(y+4+S+T)!!(y+2-S-T)!!(y+3+S-T)!!(y+3-S+T)!!}, \quad (41)$$

with Num given by

$$\begin{aligned}
\mathrm{Num} &= (y+4)!!y!!(y+3)!!(y+1)!! \qquad \text{for } y\text{=even}, \\
&= (y+5)!!(y-1)!!(y+2)!!(y+2)!! \qquad \text{for } y\text{=odd}. \quad (42)
\end{aligned}$$

Similar $(\mathcal{K}\mathcal{K}^\dagger)$-matrix elements are given in ref. [16] for most of the important SU(4) irreducible representations.

For the Wigner supermultiplet therefore, as for the SU(3) $\supset$ SO(3) scheme, the matrix elements of double (spin and isospin-space) spherical tensor operators, (not necessarily group generators), can be evaluated by a simple two-step process. By mapping the SU(4) states of good spin and isospin onto the symmetrized orthonormal basis states of the double rotor coherent state expansion, very simple expressions are gained for the reduced matrix elements of the $\Gamma(O^{r_S r_T})$. By converting the nonunitary $\Gamma(O^{r_S r_T})$ to unitary form via the $\mathcal{K}$-matrix technique, these can then be converted to standard Hilbert space matrix elements in which the labels $K_S K_T$ are replaced with the quantum numbers ν which enumerate the nonzero eigenvalues Λ_ν, of the $(\mathcal{K}\mathcal{K}^\dagger)$-matrix. These Λ_ν again give an author-independent meaning to the quantum numbers, ν, where now

$$\langle \nu'; S'T'||O^{r_S r_T}||\nu, ST\rangle = \sum_{K_S K_T}\sum_{K'_S K'_T} (\mathcal{K}^{-1})_{\nu', K'_S K'_T} \langle K'_S K'_T; S'T'||\Gamma(O^{r_S r_T})||K_S K_T; ST\rangle (\mathcal{K})_{K_S K_T, \nu} \tag{43}$$

The $\mathcal{K}$-matrix thus effectively elevates the Draayer (Elliott-type) projection labels K_S, K_T to the status of good quantum numbers.

4 A Rotor Coherent State Expansion for the SO(5) $\supset$ SO(3) Chain

Very recently, Rowe [18] has also given a vector coherent state rotor realization for the special irreducible representations [N0] and [NN] of SO(5). With a slight change of emphasis [19] this rotor construction can be put into exact parallel with that used for the SU(3) $\supset$ SO(3) and SU(4) $\supset$ SU(2) x SU(2) group chains. In the SO(5) basis of good orbital angular momentum, $|[N_1N_2],\ldots,\ldots, LM\rangle$, there are two missing quantum numbers, in contrast to the mathematically natural basis $|\ [N_1N_2]\ sm_s tm_t\rangle$ which exploits the local isomorphism between SO(5) and Sp(4) and labels the states with the quantum numbers of the SU(2) x SU(2) subgroup generated by two angular momentum generators **s** and **t** (not to be confused with the spin and isospin of the last section). For this reason it will be convenient to express the group generators in the Sp(4) notation in terms of the particle creation and annihilation operators for a family of spin-$\frac{3}{2}$ particles with states m = + $\frac{3}{2}$, + $\frac{1}{2}$, - $\frac{1}{2}$, - $\frac{3}{2}$ to be denoted by labels a, b, c, d, respectively. In order to generate the rotor states in terms of a single intrinsic (maximal weight) state, it will be convenient to make a rotation in the m = + $\frac{3}{2}$, - $\frac{3}{2}$ subspace, viz.

$$\begin{aligned} |a\rangle &= \tfrac{1}{\sqrt{2}}(|1\rangle + |4\rangle), \qquad |b\rangle = |2\rangle,, \\ |d\rangle &= \tfrac{1}{\sqrt{2}}(-|1\rangle + |4\rangle), \qquad |c\rangle = |3\rangle; \end{aligned} \tag{44}$$

where this will achieve the same purpose as the analagous eq. (23).

Since the totally symmetric SO(5) irreducible representations, [N0], are of greatest interest in nuclear physics applications, we will focus on this case. The rotor coherent state can now again be given in terms of a single intrinsic state $|\phi\rangle$ via $\langle\phi|R(\Omega|\Psi\rangle$. For the totally symmetric irreps,

[N0], $|\phi\rangle$ is now chosen such that

$$\left\{ \begin{array}{c} \frac{1}{2}(C_{11} - C_{44}) \\ \frac{1}{2}(C_{22} - C_{33}) \end{array} \right\} |\phi\rangle = \left\{ \begin{array}{c} +s_{max} \\ -s_{max} \end{array} \right\} |\phi\rangle = \left\{ \begin{array}{c} +\frac{1}{2}N \\ -\frac{1}{2}N \end{array} \right\} |\phi\rangle \tag{45}$$

with

$$\left\{ \begin{array}{c} C_{14} \\ C_{32} \end{array} \right\} |\phi\rangle = 0; \left\{ \begin{array}{c} (C_{12} + C_{34}) \\ (C_{13} - C_{24}) \\ (C_{31} - C_{42}) \end{array} \right\} |\phi\rangle = 0. \tag{46}$$

The group generators are now given by the orbital angular momentum vector $\mathbf{L}$, (a spherical tensor of rank 1), and the 7 components of a spherical tensor of rank 3, O^3_m. Eqs. (45) and (46) assure that the left action of these octupole generators can be replaced by operators $\bar{\Gamma}(L_k)$. In particular,

$$\begin{aligned}
\langle\phi|O_0 &= -\langle\phi|(\tfrac{L_0}{3} + \tfrac{5}{3}N), \\
\langle\phi|O_{-1} &= \tfrac{\sqrt{3}}{2}\langle\phi|L_-, \qquad \langle\phi|O_{+1} = \tfrac{1}{\sqrt{3}}\langle\phi|L_+, \\
\langle\phi|O_{+2} &= 0, \qquad \langle\phi|O_{-2} = -\sqrt{\tfrac{5}{6}}\langle\phi|L_+, \\
\langle\phi|O_{+3} &= \tfrac{\sqrt{5}}{3}\langle\phi|(L_0 - N), \langle\phi|O_{-3} = \tfrac{\sqrt{5}}{3}\langle\phi|(L_0 + 2N),
\end{aligned} \tag{47}$$

leads to an expression for $\Gamma(O_m)$ in terms of the $\bar{\Gamma}(L_k)$ and $D^3_{km}(\Omega)$. Analogs of eq. (36) lead to the simplest form for $\Gamma(O_m)$

$$\begin{aligned}
\Gamma(O_m) &= -D^3_{0m}(\Omega)(\tfrac{1}{3}\bar{\Gamma}(L_0) + \tfrac{5}{3}N + 2) + \tfrac{1}{6}[\mathbf{L}^2, D^3_{0m}] \\
&\quad + \tfrac{1}{2\sqrt{3}}D^3_{-1m}(\Omega)\bar{\Gamma}(L_-) + \tfrac{\sqrt{5}}{3}D^3_{+3m}(\Omega)(\bar{\Gamma}(L_0) - N) \\
&\quad + \sqrt{5}D^3_{-3m}(\Omega)(-\tfrac{2}{3}\bar{\Gamma}(L_0) + \tfrac{2}{3}N + 2) - \tfrac{\sqrt{5}}{6}[\mathbf{L}^2, D^3_{-3m}].
\end{aligned} \tag{48}$$

Note the parallels between this expression and the comparable eqs. (12) and (37) of sections 2 and 3; but also note that in this case it was now not possible to eliminate both $\bar{\Gamma}(L_-)$ and $\bar{\Gamma}(L_+)$. However, the K-raising matrix element of $\bar{\Gamma}(L_-)$ in combination with the K-lowering of the $D^3_{-1m}(\Omega)$ operator leads to a simple contribution to the matrix element diagonal in K in the rotor basis, $D^L_{KM}(\Omega)$. The $\Gamma(O_m)$ of eq. (48) thus lead to very simple matrix elements in the rotor basis with $K' =$ K, K + 3, and K - 3. Williams and Pursey [20] have shown that the allowed K sequences for the irreps [N0] are the following (with n = integer)

For N = 3n	K=	..., -6, -3, 0, +3, +6, +9, ...
For N = 3n+1	K=	..., -8, -5, -2, +1, +4, +7, ...
For N = 3n+2	K=	..., -7, -4, -1, +2, +5, +8, ...

Starting with the simplest state for [N0] = [10], with L = 2, with the normalized rotor state

$$\sqrt{\frac{5}{16\pi^2}}\{D^2_{+1M}(\Omega) + D^2_{-2,M}(\Omega)\}, \tag{49}$$

totally symmetric rotor states for N > 1 can be built up from simple products of D-functions.

In such a basis the reduced matrix elements of the $\Gamma(O^3_m)$ of eq. (48) are again given by very simple expressions involving ordinary Wigner coefficients with projection labels K. The $\mathcal{K}$-matrix technique of coherent state theory can again be used to convert these to the status of good quantum numbers, ν, through the eigenvalues Λ_ν of the $\mathcal{K}\mathcal{K}^\dagger$-matrix.

Table 1: The Possible ST-values for the Irrep. [8620]

		$(60)^1$					
	$(61)^1$		$(50)^1$				
$(62)^1$		$(51)^3$		$(40)^2$			
	$(52)^2$		$(41)^3$		$(30)^1$		
$(53)^1$		$(42)^4$		$(31)^4$		$(20)^2$	
	$(43)^2$		$(32)^4$		$(21)^3$		
$(44)^1$		$(33)^4$		$(22)^5$		$(11)^3$	$(00)^1$
	$(34)^2$		$(23)^4$		$(12)^3$		
$(35)^1$		$(24)^4$		$(13)^4$		$(02)^2$	
	$(25)^2$		$(14)^3$		$(03)^1$		
$(26)^1$		$(15)^3$		$(04)^2$			
	$(16)^1$		$(05)^1$				
		$(06)^1$					
$\uparrow$	$\uparrow$	$\uparrow$					
4	2	1					

The numbers below the arrows give the number of zeros of the $(\mathcal{K}\mathcal{K}^\dagger)$-matrices in the columns indicated by the arrows.

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