

1995/16550

N95-22967

169

HARMONIC OSCILLATOR REPRESENTATION IN THE THEORY OF SCATTERING AND NUCLEAR REACTIONS

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Abstract

The following questions, concerning to the application of the harmonic oscillator representation (HOR) in the theory of scattering and reactions, are discussed: the formulation of the scattering theory in HOR; exact solutions of the free motion Schroedinger equation in HOR; separable expansion of the short range potentials and the calculation of the phase shifts; "isolated states" as generalization of the Wigner-von Neumann bound states embedded in continuum; a nuclear coupled channel problem in HOR; the description of true three body scattering in HOR. As an illustration the soft dipole mode in the ^{11}Li nucleus is considered in a frame of the $^9\text{Li}+n+n$ cluster model with taking into account of three body continuum effects.

1 Introduction

Usually harmonic oscillator wave functions are used for the description of bound states of quantum systems that belong to the discrete spectrum [1]. In this talk the application of the harmonic oscillator (HO) basis to the solution of the scattering problem, i.e. in *continuum*, will be discussed.

This line of investigations was begun in Refs. [2] (see also the papers cited there) and independently in the papers of Kiev [3] and Moscow [4] groups. The similar approach, also connected with an application of the HO basis to the scattering problem, was developed by the Hungarian group [5].

In order to illustrate the essence of the approach to the scattering problem in the harmonic oscillator basis, we shall consider at first the simplest problem of the scattering of a single particle by the central potential $V(r)$ [4]. Thus, we come to the Schroedinger equation

$$\left(\frac{p^2}{2m} + V(r) \right) \psi_{lm}(\mathbf{r}) = \varepsilon \psi_{lm}(\mathbf{r}) . \quad (1)$$

Its solution $\psi_{lm}(\mathbf{r}) = R_l(r) Y_{lm}(\Omega)$ will be sought in the form of an expansion in the eigenfunctions of the harmonic oscillator

$$R_l(r) = \sum_{n=0}^{\infty} C_{nl} R_{nl}(r) \quad (2)$$

where

$$R_e(r) = (-1)^n \left(\frac{2n!}{\Gamma(n+l+\frac{3}{2})} \right)^{1/2} r^l L_n^{l+1/2}(r^2) e^{-r^2/2} \quad (3)$$

is the radial wave function of three-dimensional harmonic oscillator. This wave function corresponds to the eigenvalues of the oscillator energy $E_n^{osc} = (2n+l+3/2)\hbar\omega$ [1]. The value $r_o = (\hbar/m\omega)^{1/2}$ is selected as a length scale in relations (1) and (2). Here ω is the oscillator frequency; the energy $\varepsilon = q^2/2$ is measured in units $\hbar\omega$; the wave vector k is expressed in units r_o^{-1} , $q = kr_o$ is the dimensionless momentum. Substituting expansion (2) in (1) and multiplying (1) scalarly by $R_{nl}Y_{lm}$, we obtain the following equation determining the wave function ψ_{lm} in the harmonic oscillator representation (HOR):

$$\sum_{n'} (H_{nn'} - \delta_{nn'}\varepsilon) C_{n'l} = 0, \quad n = 0, 1, 2, \dots \quad (4)$$

Here, $H = T + V$ and only the following matrix elements of the kinetic energy operator $T = P^2/2$ are nonvanishing:

$$\begin{aligned} T_{nn-1} &= -\frac{1}{2} \left[n \left(n + l + \frac{1}{2} \right) \right]^{1/2}, \\ T_{nn} &= \frac{1}{2} \left(2n + l + \frac{3}{2} \right), \\ T_{nn+1} &= -\frac{1}{2} \left[(n+1) \left(n + l + \frac{3}{2} \right) \right]^{1/2}. \end{aligned} \quad (5)$$

As to behaviour of the coefficient C_{nl} for $n \geq N$, their asymptotics are similar to the asymptotic of the wave function in the coordinate representation [3] if r is substituted by $2n^{1/2}r_o$:

$$C_{nl} \sim 2n^{1/4} \psi_{lm}(2\sqrt{n}r_o), \quad n \rightarrow \infty. \quad (6)$$

this result can be obtained if the WKB expression for the oscillator function $R_{nl}(r)$ is substituted in the expression for the coefficients

$$C_{nl} = \langle \psi_{nlm}(\mathbf{r}) | \psi_{lm}(\mathbf{r}) \rangle \quad (7)$$

and the integral (7) is calculated by the stationary phase method. The result (6) follows also from the fact that the finite-difference equation

$$\begin{aligned} & - \left[n \left(n + l + \frac{1}{2} \right) \right]^{1/2} C_{n-1l} + \left(2n + l + \frac{3}{2} - q^2 \right) C_{nl} \\ & - \left[(n+1) \left(n + l + \frac{3}{2} \right) \right]^{1/2} C_{n+1l} + 2 \sum_{n'} \langle nl | V | n'l \rangle C_{n'l} = 0 \end{aligned} \quad (8)$$

in the limit $n \gg \nu = l/2 + 3/4$ can be replaced by the following second-order differential equation [4]:

$$X_l'' - \frac{l(l+1)}{x^2} X_l - \int_0^\infty V(x, x') \sqrt{xx'} X_l(x') dx + q^2 X_l = 0. \quad (9)$$

Here $x = 2(n + \nu)^{1/2}$, $X_l(x) = x^{1/2} C_{nl}$. The Eq. (9) should be solved at the boundary condition

$$X_l(2\sqrt{\nu - 1}) = 0. \quad (10)$$

Thus, in the asymptotic limit for large n , the wave function of our system X_l for the partial wave with angular momentum l in the HOR obeys the conventional Schroedinger equation with nonlocal potential

$$V(x, x') \sqrt{xx'} \simeq 2 \langle nl | V | n'l \rangle [(n + \nu)(n' + \nu)]^{1/4} \quad (11)$$

where the value $2(n + \nu)^{1/2} r_0$ plays the role of "coordinate". In actual calculations, the potential matrix has to be cut off by the condition $V_{nn'} = 0$, if n or (and) $n' > N$. Then a set of equations (4) can be splitted in two parts:

$$a) \ n \leq N, \quad \sum_{n'=0}^N (H_{nn'} - \varepsilon \delta_{nn'}) C_{n'l} = -\delta_{nN} T_{NN+1} C_{N+1l}, \quad (12a)$$

$$b) \ n \geq N + 1, \quad T_{nn-1} C_{n-1l} + (T_{nn} - \varepsilon) C_{nl} + T_{nn+1} C_{n+1l} = 0. \quad (12b)$$

Thus, the coefficients C_{nl} with $n > N$ obey the equation of free motion (12b) or, in the asymptotic limit of continuous n , the Schroedinger equation of free motion

$$X_l'' - \frac{l(l+1)}{x^2} X_l + q^2 X_l = 0.$$

It means that the condition

$$C_{nl} \sim 2n^{1/4} e^{-2\sqrt{n}k} \quad (13a)$$

(where $\varepsilon = -k^2/2$ is the binding energy) must be satisfied for the bound states. The coefficients C_{nl} for the scattering problem have the following asymptotic behaviour:

$$C_{nl} \sim 2n^{1/4} \sin(2q\sqrt{n} - l\pi/2 + \delta_l) \quad (13b)$$

where $\varepsilon = q^2/2$. According to Eq.(6) the phase shift δ_l in Eq. (13b) coincides with the standard phase shift of the wave function in coordinate space. For the decaying resonance states, we get (see in [3]):

$$C_{nl} \sim 2n^{1/4} e^{2iq\sqrt{n}}. \quad (13c)$$

If the calculations are made up to sufficiently high values of $N \gg 1$ it is possible to use the asymptotic expressions (13) [3]. At modest N it is necessary to use the exact, rather than approximate, solution for the equation of free motion (12b) which was found in Refs. [2, 4] in order to calculate the binding energy, the scattering phases etc. Before considering the solution for the equation of free motion, we shall note that the solution for the set (12) is equivalent to the

solution for the Schroedinger equation with Hamiltonian $H = T + V^N$ containing the many-term separable potential

$$V^N = \sum_{n,n'}^N \langle nl|V|n'l \rangle |nl \rangle \langle n'l| \quad (14)$$

with harmonic oscillator formactors. The technique of solving such an equation in the frame of the momentum representation was described in [5, 6]. Here we shall describe an alternative method for solving the same problem in HOR.

2 Solution for the equation of free motion in the harmonic oscillator representation

Consider first the case of positive energy $\varepsilon = q^2/2 > 0$. The Schroedinger equation of free motion in the coordinate space has two linear-independent solutions (regular and irregular) [4]:

$$\begin{aligned} R_l^{reg} &= j_l(kr) \sim \frac{1}{kr} \sin(kr - \frac{l\pi}{2}) , \\ R_l^{irreg} &= n_l(kr) \sim \frac{1}{kr} \cos(kr - \frac{l\pi}{2}) . \end{aligned} \quad (15)$$

In accordance with this, the finite difference equation of free motion (12b) will have also two fundamental solutions in the HOR [4] namely the regular solution

$$\begin{aligned} C_{nl}^{reg}(q) &= \left(\frac{2\Gamma(n+l+\frac{3}{2})}{\Gamma(n+1)} \right)^{1/2} \frac{q^l}{\Gamma(l+\frac{3}{2})} e^{-q^2/2} M(-n, l+\frac{3}{2}; q^2) = \\ &= (-1)^n R_{nl}(q) \sim \frac{2^{3/2}}{\pi^{1/2}} n^{1/4} j_l(2\sqrt{n}q) \end{aligned} \quad (16)$$

satisfying the boundary condition (6) $C_{-1l}^{reg} = 0$, and the irregular solution

$$\begin{aligned} C_{nl}^{irreg}(q) &= \left(\frac{2\Gamma(n+1)}{\Gamma(n+l+\frac{3}{2})} \right)^{1/2} \frac{(-1)^l q^{-l-1}}{\Gamma(-l+\frac{1}{2})} e^{-q^2/2} M(-n-l-\frac{1}{2}, -l+\frac{1}{2}; q^2) = \\ &\sim \frac{2^{3/2}}{\pi^{1/2}} n^{1/4} n_l(2\sqrt{n}q) \end{aligned} \quad (17)$$

which is singular at the point $n = -1$.

The Casorati determinant K_{nl} for these two solutions which plays the same role for the difference equations as the Wronskian for the differential equations [7] is of the form:

$$K_{nl} = T_{n+1n} \begin{vmatrix} C_{nl}^{reg} & C_{nl}^{irreg} \\ C_{n+1l}^{reg} & C_{n+1l}^{irreg} \end{vmatrix} = \frac{-1}{\pi q} . \quad (18)$$

Since $K_{nl} \neq 0$ for any values of n and l , the expressions (16) and (17) constitute the fundamental set of solutions for equation (12b). An arbitrary solution for (12b) may be presented as a linear

combination of fundamental solutions. In particular, the solution for the set (12) for $n \geq N$ must be of the form

$$C_{nl}(q) = \cos \delta_l C_{nl}^{reg}(q) + \sin \delta_l C_{nl}^{irreg}(q) \quad (19)$$

whence it follows that

$$\tan \delta_l = -\frac{C_{nl}C_{n+1l}^{reg} - C_{n+1l}C_{nl}^{reg}}{C_{nl}C_{n+1l}^{irreg} - C_{n+1l}C_{nl}^{irreg}}. \quad (20)$$

The equivalent pair of fundamental solutions for the free motion equation has the asymptotic form of the type of Hankel functions

$$C_{nl}^{\pm}(q) = C_{nl}^{irreg}(q) \pm iC_{nl}^{reg}(q). \quad (21)$$

These solutions are useful for the calculation of the S -matrix and analyzing the decaying Gamov states. If we are interested in bound states ($\varepsilon = -k^2/2, q = ik$) the solution for the equation of free motion with a corresponding asymptotic

$$C_{nl}^{bound}(k) = i^l [C_{nl}^{irreg}(ik) + iC_{nl}^{reg}(ik)] \quad (22)$$

must be used. The numerical values of solutions (16), (17) can be obtained by using the book [8], where the function $M(a, b; z)$ is tabulated. Similarly to the regular and irregular solutions of the free motion Schroedinger equation the functions C_{nl}^{reg} and C_{nl}^{irreg} are oscillating functions of n and the period of oscillations decreases with increasing energy ε .

3 The solution of the scattering problem in HOR

Consider now the solution for set (12). It follows from equations (12) that the coefficient C_{nl} for $n \geq N + 1$ obey the equation of free motion with an appropriate asymptotic, i.e. $C_{nl} = C_{nl}^0$, where C_{nl}^0 is the solution for the equation of free motion with asymptotic (19), (21) or (22). The coefficients $C_{nl}(n \geq N + 1)$ form the "external" part of the wave function in HOR. The coefficients $C_{nl}(n \leq N)$ belong to the "internal" part of this function. The equation

$$C_{Nl}^{intern} = C_{Nl}^{extern} \quad (23)$$

plays a role of "fitting" condition of "internal" and "external" parts of the wave function. The r.h.s. of this equation has one of the form (19), (21) or (22). Into the left hand side of Eq. (23) the solution of the set (12a) must be substituted. The last one can be found in the following manner [2]. At first we shall diagonalize the truncated Hamiltonian matrix $||H_{nn'}||$ using the unitary transformation Γ , i.e. turn from C_{nl} to the new coefficients

$$C'_{\lambda l} = \sum_{n=0}^N \Gamma_{\lambda n} C_{nl}, \quad \lambda = 0, 1, \dots, N. \quad (24)$$

As a result of this transformation, equation (12a) takes the form

$$(E_{\lambda} - \varepsilon)C'_{\lambda l} = -\Gamma_{\lambda N} T_{NN+1} C_{N+1l}, \quad \lambda = 0, 1, \dots, N$$

i.e.

$$C'_{\lambda l} = -\frac{\Gamma_{\lambda N} T_{NN+1}}{E_{\lambda} - \varepsilon} C_{N+1l} \quad (25a)$$

and

$$C_{nl} = -\sum_{\lambda=0}^N \frac{\Gamma_{\lambda n}^* \Gamma_{\lambda N}}{E_{\lambda} - \varepsilon} T_{NN+1} C_{N+1l} \quad (25b)$$

where E_{λ} - is the eigenvalues of the matrix $\|H_{nn'}\|$ ($n, n' \leq N$).

Substituting the "internal" solution (25b) at $n = N$ into Eq. (23) we obtain

$$P_{NN+1} C_{N+1l}^0 = -C_{Nl}^0, \quad P_{nN+1} = \sum_{\lambda=0}^N \frac{\Gamma_{\lambda n}^* \Gamma_{\lambda N}}{E_{\lambda} - \varepsilon} T_{NN+1}. \quad (26)$$

If we deal with bound states of Gamov resonances, C_{nl}^0 and C_{N+1l}^0 are the known functions of energy (see (21) and (22)). In such cases the condition (26) is the transcendent equation which may be used to find the energies ε_i of the bound or resonant states. For the scattering problem, we get in accordance with (19):

$$\begin{aligned} C_{Nl}^0 &= C_{Nl}^{reg} + \tan \delta_l C_{Nl}^{irreg}, \\ C_{N+1l}^0 &= C_{N+1l}^{reg} + \tan \delta_l C_{N+1l}^{irreg}. \end{aligned} \quad (27)$$

Substituting these expressions in (26), we find in accordance with Refs. [2]:

$$\tan \delta_l = -\frac{C_{Nl}^{reg} + P_{NN+1} C_{N+1l}^{reg}}{C_{Nl}^{irreg} + P_{NN+1} C_{N+1l}^{irreg}}. \quad (28)$$

It can be seen now that the scattering phase *at an arbitrary energy* ε can be obtained by diagonalizing the Hamiltonian matrix $\|H_{nn'}\|$ ($n, n' \leq N$) but *one time*.

In agreement with the Ritz variational principle, the negative eigenvalues $E_{\lambda} < 0$ of the Hamiltonian matrix $\|H_{nn'}\|$ ($n, n' \leq N$) may be treated as approximate values of the energies of discrete levels of a particle in the studied potential. In this case the approximation accuracy improves with increasing the size of the matrix N . The question arises, what is the sense of the matrix positive eigenvalues and of the respective wave functions? The question was answered in works [2, 4, 9] as follows. In the limit $\varepsilon \rightarrow E_{\lambda}$ expression (28) takes on the form

$$\tan \delta_l (E_{\lambda}) = -\frac{C_{N+1l}^{reg}}{C_{N+1l}^{irreg}}. \quad (29)$$

By comparing this result with formula (19) we get the coefficient $C_{N+1l}^{reg}(E_{\lambda}) = 0$ for E_{λ} . Thus, by diagonalizing the Hamiltonian matrix $\|H_{nn'}\|$ ($n, n' \leq N$), we find the solutions for equations (12) in the region of the continuum at such discrete energies $E_{\lambda} > 0$ which correspond to the vanishing of the HOR wave function $C_{nl}(E_{\lambda})$ at the point $n = N + 1$. The scattering phase can be calculated at such energies using simple formula (29). In the asymptotic limit of high N , the diagonalization of the Hamiltonian matrix on the cutoff basis $n \leq N$ means the solution for the Schroedinger equation (9) with the additional condition $X_l(b) = 0$, where $b = 2(N + \nu)^{1/2}$, i.e. when the system is placed within a rigid box of radius b . In this case the energy spectrum

for equation (9) gets discrete and the energy of any level becomes a function of the position of the boundary point $b = 2[(N + \nu)\hbar/m\omega]^{1/2}$. The condition $X_l(b) = 0$ is nothing other than the equation for the P -matrix poles in the system of radius b described by the Schrodinger equation (9) [10]. Thus the eigenvalues E_λ of the Hamiltonian matrix $\|H_{nn'}\|$ are poles of the discrete analogue of the P -matrix. The important point is a convergency of this approach. The practical calculations [3, 4] show that for rather smooth potentials $V(r)$ it is sufficient to use a number of terms $N \sim 20 - 30$ in the expansion (2) in order to calculate phase shifts and other scattering characteristics in reasonable accuracy. Therefore the using of HOR, or J -matrix approach [2], is a rather effective and practicable method for the study of continuum problems. Some additional example of application of this method will be discussed in section 6.

4 Multichannel case

Let us consider the case of two open (binary, spinless) channel for simplicity. The wave function has the form of a column

$$\psi(r) = \begin{pmatrix} \psi_1(r) \\ \psi_2(r) \end{pmatrix} \quad (30)$$

and the Hamiltonian is the operator matrix of a size 2×2 :

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}. \quad (31)$$

Let us assume that the wave function of the entrance channel $\psi_1(r)$ is characterized by the following asymptotic behaviour

$$\psi_1(r) \sim (e^{-ik_1 r} - S_{11} e^{ik_1 r})/r \quad (32a)$$

while in the second channel only the outgoing wave presents

$$\psi_2(r) \sim -((v_1/v_2)^{1/2} S_{21} e^{ik_2 r})/r. \quad (32b)$$

The transition into n -representation consists in the expansion of both channel wave functions

$$\begin{aligned} \psi_1(r) &= \sum_n C_{1n} |n, r_{01}\rangle, \\ \psi_2(r) &= \sum_m C_{2m} |m, r_{02}\rangle, \end{aligned} \quad (33)$$

in terms of harmonic oscillator wave functions $|n, r_{01}\rangle$, $|m, r_{02}\rangle$ with a unique frequency $\hbar\omega$ while the linear scale parameters $r_{0i} = (\hbar/\mu_i\omega)^{1/2}$ can be different for the channels 1 and 2 if the reduced masses μ_1 and μ_2 of two fragments in these channels are different. Assuming that it is possible to restrict ourselves to a truncated matrix of the potential energy

$$\begin{aligned} V_{1n,1n'} \quad (0 \leq n, n' \leq N_1), \quad V_{2m,2m'} \quad (0 \leq m, m' \leq N_2), \\ V_{1n,2m}, \quad V_{2m,1n} \quad (0 \leq n \leq N_1, 0 \leq m \leq N_2) \end{aligned}$$

(generally speaking $N_1 \neq N_2$) we obtain the following set of equations for C_{1n} , C_{2m} coefficients instead of Eqs., (12), (23), (27)

$$\begin{aligned}
(H - E)C &= -TC^0, \\
C_{1N_1} &= C_{1N_1}^0, \quad C_{2N_2} = C_{2N_2}^0, \\
C_{1n}^0 &= C_{1n}^- - S_{11}C_{1n}^+, \quad n \geq N_1, \\
C_{2m}^0 &= -(q_2/q_1)^{1/2} S_{21}C_{2m}^+, \quad m \geq N_2.
\end{aligned} \tag{34}$$

Here C is a column of $N_1 + N_2 + 2$ coefficients $C_{10}C_{11}\dots C_{1N_1}C_{20}\dots C_{2N_2}$, H is the matrix of Hamiltonian in a truncated basis $|n, r_{01}\rangle$, $|m, r_{02}\rangle$ ($n \leq N_1$, $m \leq N_2$). The column TC contains only two nonvanishing elements, namely $T_{N_1, N_1+1}C_{1N_1+1}^0$ in the $N_1 + 1$ -th row and $T_{N_2, N_2+1}C_{2N_2+1}^0$ in the last row. The functions $C_{ik}^\pm = C_{ik}^{reg} \pm iC_{ik}^{irreg}$ are the same ones as in Eq. (21). As it was shown in Ref. [2, 11] the asymptotic of the function $\psi^\pm = \sum_{n=0}^\infty C_n^\pm |n\rangle$ is of the form $k \exp(\pm ikr)$ for $r \rightarrow \infty$. This fact and the difference of r_{0i} in various channels are the origin of the factor $(q_2/q_1)^{1/2}$ in Eq. (34) instead of the usual velocities ratio $(v_2/v_1)^{1/2}$ in Eq. (32). Solving the Eq. (34) similarly to Eq. (12) we obtain the following results instead of Eq. (26)

$$\begin{aligned}
C_{1N_1} &= C_{1N_1}^- - S_{11}C_{1N_1}^+ = P_{11}(C_{1N_1+1}^- - S_{11}C_{1N_1+1}^+) + \\
&\quad + P_{12} \left(-(q_2/q_1)^{1/2} S_{21}C_{2N_2+1}^+ \right), \\
C_{2N_2} &= -S_{21}C_{2N_2}^+ (q_2/q_1)^{1/2} = P_{21} \left(C_{1N_1+1}^- - S_{11}C_{1N_1+1}^+ \right) + \\
&\quad + P_{22} \left(-(q_2/q_1)^{1/2} S_{21}C_{2N_2+1}^+ \right)
\end{aligned} \tag{35}$$

where

$$P_{ij} = \sum_\lambda \frac{\Gamma_{\lambda N_1} \Gamma_{\lambda N_2}}{E_\lambda - E} T_{N_j, N_j+1}.$$

E_λ is the eigenvalue of the truncated matrix H , $(\Gamma_{\lambda 0} \dots \Gamma_{\lambda N_1} \dots \Gamma_{\lambda N_2})$ is the corresponding eigenvector of this matrix.

The relations (35) should be considered as the equations for elements of the S -matrix. The solutions of these equations are of the form

$$\begin{aligned}
S_{11} &= \frac{1}{D} \left[(C_{1N_1}^- + P_{11}C_{1N_1+1}^-)(C_{2N_2}^- + P_{22}C_{2N_2+1}^-) - P_{21}P_{12}C_{1N_1+1}^- C_{2N_2+1}^- \right] \\
S_{21} &= \frac{1}{D} \frac{2iP_{12}P_{21}}{\sqrt{\pi q_1 q_2}},
\end{aligned} \tag{36}$$

$$D = \left[(C_{1N_1}^+ + P_{11}C_{1N_1+1}^+)(C_{2N_2}^+ + P_{22}C_{2N_2+1}^+) - P_{12}P_{21}C_{1N_1+1}^+ C_{2N_2+1}^+ \right].$$

Here the property of the Casorati determinant

$$T_{NN+1} \begin{vmatrix} C_N^- & C_N^+ \\ C_{N+1}^- & C_{N+1}^+ \end{vmatrix} = \frac{2i}{\pi q}$$

was used. The elements S_{22} , S_{12} of the S -matrix can be obtained from (36) by substitution of indices 1 and 2. Obviously the S -matrix is symmetrical in accordance with the time reflection

symmetry of the Hamiltonian. As in the previous section, the eigenvalues E_λ are the poles of the discrete analogue of the P -matrix.

The eigenfunctions $\psi_\lambda = \sum_n \Gamma_{\lambda n} |n\rangle$ are discrete analogs of "primitives" (in terms of the paper [10]).

The expressions (36) allow us to find the numerical values of the S -matrix elements and then to calculate the cross-sections of elastic scattering and reactions

$$\sigma_{el} = \frac{\pi}{k_1^2} |S_{11} - 1|^2, \quad (37)$$

$$\sigma_r = \frac{\pi}{k_1^2} |S_{21}|^2, \quad (38)$$

the differential cross-sections, various polarization characteristics (taking into account the spin degree of freedom) etc. If we want to describe the reaction with three-four fragments in the final states it is necessary to extend the above developed formalism, which is valid only for two body (binary) channels, to three, four body collisions.

5 The description of "true" many body scattering in a hyperspherical HOR

We restrict ourselves by the case of the so called "true" many body scattering (TMBS) when the wave function of an A body system is in the asymptotic region of the form

$$\psi_K(\rho, \Omega) = \delta_{K\gamma, K_0\gamma_0} e^{-ik\rho} Y_{K_0\gamma_0}(\Omega) - \sum_{K'\gamma'} S_{K'\gamma', K_0\gamma_0} e^{ik\rho} Y_{K'\gamma'}(\Omega), \quad \rho \rightarrow \infty. \quad (39)$$

$\rho^2 = \sum_{i=1}^A (\mathbf{r}_i - \mathbf{R})^2$ is a global radius in $3(A-1)$ dimensional space, the angles Ω are hyperspherical coordinates in this space. \mathbf{R} is the center-of-mass of the system, $Y_{K\gamma}(\Omega)$ is a hyperspherical harmonic with a global momentum K , γ substitutes all the rest quantum numbers labelling this harmonic. The approximation taking into account only the contribution of TMBS is valid if there is a "democracy" in the A body system i.e. there is no pair of particles with dominating interaction between them in comparison with the rest of the interactions. The TMBS - approximation is applicable to a lot of processes of three, four body decay of light nuclei and hypernuclei [17] (for example disintegration $^{12}\text{C} \rightarrow 3\alpha$ etc.).

For the description of TMBS we shall use the expansion of the A -body wave function $\psi(\vec{r}_1 \dots \vec{r}_A)$ in terms of $A-1$ body oscillator wave function (the center of mass variable \mathbf{R} is excluded)

$$|nK\gamma\rangle = R_{nK}(\rho) Y_{K\gamma}(\Omega) \quad (40)$$

depending on hyperspherical coordinates ρ, Ω :

$$\psi = \sum_{nK\gamma} \langle nK\gamma | \psi \rangle |nK\gamma\rangle. \quad (41)$$

Further consideration is totally parallel to sections 1-4 and we represent the result in very short form. Instead of Eq. (3) we have for the many body case

$$R_{NK}(\rho) = \rho^{-(3A-4)/2} \phi_n^\ell(\rho), \quad (42)$$

$$\phi_n^\mathcal{L}(\rho) = (-1)^n \sqrt{\frac{2n!}{\Gamma(n + \mathcal{L} + \frac{3}{2})}} \rho^{\mathcal{L}+1} e^{-\rho^2/2} L_n^{\mathcal{L}+1/2}(\rho^2) \quad (43)$$

$\mathcal{L} = K + (3A - 6)/2$, ρ is taken in units of r_0 . The Eq. (4) takes on the form

$$\sum_{n'K'\gamma'} \langle nK\gamma | H - E | n'K'\gamma' \rangle \langle n'K'\gamma' | \psi \rangle = 0. \quad (44)$$

The kinetic term T in the Hamiltonian $H = T + V$ is diagonal in the quantum numbers K and γ . As for the main quantum number, n , the matrix is three diagonal with respect to n and its matrix elements coincide with Eq. (5) except for substitution of l by \mathcal{L} . We also truncate the matrix of potential energy $V = \sum_{i < j}^A V_{ij}$ to $n \leq N$, $K \leq K_{max}$. Then for $n \geq N$ the expansion coefficients $\langle nK\gamma | \psi \rangle \equiv C_{n\mathcal{L}}$ obey the three-term recurrent relation similar to Eq. (12b)

$$\begin{aligned} \sqrt{n(n + \mathcal{L} + \frac{1}{2})} \langle n - 1K\gamma | \psi \rangle - \left(2n + \mathcal{L} + \frac{3}{2} - q^2\right) \langle nK\gamma | \psi \rangle + \\ \sqrt{(n + 1)(n + \mathcal{L} + \frac{1}{2})} \langle n + 1K\gamma | \psi \rangle = 0, \quad q = \sqrt{2E}. \end{aligned} \quad (45)$$

This difference equation has two fundamental solutions

$$C_{n\mathcal{L}}^{reg} = \sqrt{\frac{2n!}{\Gamma(n + \mathcal{L} + \frac{3}{2})}} q^{\mathcal{L}+1} e^{-q^2/2} L_n^{\mathcal{L}+1/2}(q^2) \quad (46)$$

and

$$C_{n\mathcal{L}}^{irreg} = -\frac{2q}{\pi C_{0\mathcal{L}}^{reg}(q)} v.p. \int_0^\infty \frac{C_{0\mathcal{L}}^{reg}(q') C_{n\mathcal{L}}^{reg}(q')}{q^2 - q'^2} dq' \quad (47)$$

or the equivalent pair of solutions

$$C_{n\mathcal{L}}^\pm = -\frac{2q}{\pi C_{0\mathcal{L}}^{reg}(q)} \int_0^\infty \frac{C_{0\mathcal{L}}^{reg}(q') C_{n\mathcal{L}}^{reg}(q')}{q^2 - q'^2 \pm i0} dq'. \quad (48)$$

The problem of TMBS is similar to the multichannel problem described in section 4. Thus the wave function (39) with ingoing wave in some channel $K_0\gamma_0$ and outgoing waves in each channels $K'\gamma'$ under consideration takes on the form at $n > N$ (in principle the truncation boundary N may be different in the various channels $K\gamma$):

$$\langle nK\gamma | \psi \rangle = \delta_{K\gamma, K_0\gamma_0} C_{n\mathcal{L}}^-(q) - \sum_{K'\gamma'} S_{K\gamma, K'\gamma'} C_{n\mathcal{L}}^+(q). \quad (49)$$

In analogy with Eq. (36) we can obtain

$$S = A^{-1}B \quad (50)$$

where

$$\begin{aligned} (A)_{K'\gamma', K\gamma} &= P_{K'\gamma', K\gamma} C_{N+1\mathcal{L}}^+(q) - \delta_{K\gamma, K'\gamma'} C_{N\mathcal{L}}^+(q), \\ (B)_{K'\gamma', K\gamma} &= P_{K'\gamma', K\gamma} C_{N+1\mathcal{L}}^-(q) - \delta_{K\gamma, K'\gamma'} C_{N\mathcal{L}}^-(q), \end{aligned} \quad (51)$$

$$P_{K\gamma, K'\gamma'} = \sum_{\lambda} \frac{\langle NK\gamma|\lambda \rangle \langle \lambda|NK'\gamma' \rangle}{E - E_{\lambda}} T_{NK', N+1K'} .$$

E_{λ} and $\langle NK\gamma|\lambda \rangle$ are eigenvalues and eigenvector components of the truncated Hamiltonian matrix $\langle nK\gamma|H|n'K'\gamma' \rangle$ ($n, n' \leq N$). The poles of the S -matrix (i.e. bound states and Gamov resonance states) can be found from the equation

$$\det A = 0 . \quad (52)$$

Thus we have all expressions that are necessary for the construction of the wave function for few body states belonging to continuum or discrete spectrum in the frame of TMBS approximation.

6 Soft dipole mode in ^{11}Li and three body continuum

In order to demonstrate the effectiveness of the HOR in an analysis of concrete nuclear processes the calculations of the properties of the low energy EI excitations in the ^{11}Li nucleus were done. The remarkable feature of this nucleus is a presence of a large neutron halo formed by two neutrons weakly bound with the ^9Li core. In this connection the following model was used for the description of this nucleus.

6.1 The model

It was assumed that the ^{11}Li ground and continuum states can be interpreted in the framework of the three-body cluster structure $^9\text{Li}+n+n$.

1) The cluster ^9Li is supposed to be structureless and the excitations of its internal degrees of freedom are not considered.

2) We don't account for non-central components of the interaction between two valence neutrons and between valence neutron and the cluster ^9Li . Therefore, the wave function can be characterized by the three-body orbital angular momentum L , spin $S = 3/2$, total angular momentum J and its projection M .

3) The states with the total spin of the valence neutron pair $S = 0$ are only considered, and the ground state three-body orbital angular momentum is supposed to be equal to zero: $L = 0$.

4) n - ^9Li interaction is described by the shallow potential of Johansen et al [13]. NN -interaction is described by the Gaussian potential [13].

5) Only democratic decay channels are allowed for.

The wave function of the system $^9\text{Li}+n+n$, $\psi_{JM}(\mathbf{x}, \mathbf{y})$, is expanded in three-body hyperspherical functions, $\Phi_K^{l_x l_y JM}(\hat{\Omega})$ (including the internal wave function of ^9Li with a spin $S = 3/2$)

$$\psi_{JM}(\mathbf{x}, \mathbf{y}) = \sum_{K l_x l_y} \psi_K^{(J) l_x l_y}(\rho) \Phi_K^{l_x l_y JM}(\hat{\rho}) , \quad (53)$$

where K is hypermomentum, l_x and l_y are the angular momenta corresponding to the Jacobi coordinates

$$\mathbf{x} = \sqrt{\frac{m\omega}{2\hbar}}(\mathbf{r}_1 - \mathbf{r}_2) , \quad \mathbf{y} = \sqrt{\frac{18 m\omega}{11 \hbar}}\left(\frac{\mathbf{r}_1 + \mathbf{r}_2}{2} - \mathbf{r}_3\right) , \quad (54)$$

respectively, m is the neutron mass, \mathbf{r}_i are coordinates of the valence neutrons ($i = 1, 2$) and the cluster ${}^9\text{Li}$ ($i = 3$), $\rho = (\mathbf{x}^2 + \mathbf{y}^2)^{1/2}$ is a three-body hyperradius.

In the c. m. frame the Hamiltonian is of the form:

$$H = T + V_{12} + V_{13} + V_{23} , \quad (55)$$

where T is the three-body relative motion kinetic energy operator, and V_{ij} are the two-body potentials. For the radial wave functions $\psi_{K l_x l_y}^{(J)}(\rho)$ we have the usual set of the K -harmonic method coupled equations (see, e.g., [12]). The equations are solved by expanding the radial wave function.

$$\psi_{K l_x l_y}^{(J)}(\rho) = \sum_{n=0}^{\infty} D_{n K l_x l_y}^{(J)}(E) \varphi_{n K}(\rho) , \quad (56)$$

in the six-dimensional harmonic oscillator eigenfunctions. To calculate the bound state energy, i.e. to locate the corresponding S -matrix pole, one should solve the nonlinear equation [11]

$$\det A^{(+)} = 0 , \quad (57)$$

where the matrix A is given by the Eq. (51).

For the continuum spectrum states we calculate S -matrix for any positive energy E using (50)

The interactions of the valence neutrons with each other and with the cluster ${}^9\text{Li}$ are described by the potentials $V_{12}(r_{12})$ and $V_{13}(r_{13}) = V_{23}(r_{23})$, respectively. We use the following parametrization of the potentials [13]:

$$\begin{aligned} V_{ij}(r) &= V_{ij}^{(1)} \exp[-(r/b_{ij}^{(1)})^2] + V_{ij}^{(2)} \exp[-(r/b_{ij}^{(2)})^2] , \\ V_{12}^{(1)} &= -31 \text{ MeV}, \quad V_{12}^{(2)} = 0, \quad b_{12}^{(1)} = 1.8 \text{ fm}; \\ V_{13}^{(1)} &= -7 \text{ MeV}, \quad V_{13}^{(2)} = -1 \text{ MeV}, \quad b_{13}^{(1)} = 2.4 \text{ fm}, \quad b_{13}^{(2)} = 3.0 \text{ fm}. \end{aligned}$$

In the external asymptotic region $n \geq N$ we consequently allow for channels Γ characterized by $K = K_{min}, K_{min} + 2, \dots$ (K_{min} is the minimal possible value of K for a given J) until the convergence for all physical properties under consideration is achieved. The convergence is found to be very good, and the allowance for the decay channels with $K > K_{min} + 2$ do not yield any visual variation of the results. So, we consider in the external asymptotic region $n > N$ the channels with $K \leq K_{min} + 2$ only. Note, that components with all possible values of $K \leq N$ are accounted for in the calculation of the wave function in the inner region $n \leq N$.

The parameter $\hbar\omega$ is set to be equal to 7.1 MeV in our calculations. This value corresponds approximately to the minimum of ground state energy E_0 .

6.2 The ground state

The results for the ${}^{11}\text{Li}$ ground state for different values of the truncation parameter N are presented in the table 1. The variational ground state energies, $E_0^{(d)}$, obtained by the pure diagonalization of the truncated Hamiltonian matrix are listed in the second column, while the J -matrix results, E_0 , which are the solutions of the eq. (57), are listed in the third column. It is seen, that by locating the S -matrix pole using eq. (57) that is equivalent to the allowance for the

Table 1: ^{11}Li ground state properties (see text for details).

Truncation boundary N	Ground state energy, MeV		Neutron halo mean square radius $\langle r^2 \rangle_{11}^{1/2}$, fm	
	$E_0^{(d)}$	E_0	$\langle r^2 \rangle_{11}^{1/2 (d)}$	$\langle r^2 \rangle_{11}^{1/2}$
12	-0.012	-0.150	2.83	3.31
16	-0.116	-0.199	2.91	3.29
20	-0.171	-0.225	2.98	3.31
24	-0.202	-0.240	3.04	3.32
Experiment	-0.247±0.080		3.16±0.11	

long asymptotic tail of the wave function, we improve essentially the convergence for the binding energy.

The results presented in the table 1 have been obtained using Lanczos smoothing of the three-body potential energy matrix [5, 14].

The ^{11}Li r.m.s. radius, $\langle r^2 \rangle_{11}^{1/2}$, can be calculated by the following equation:

$$\langle r^2 \rangle_{11} = \frac{9}{11} \langle r^2 \rangle_9 + \frac{\hbar}{11 m \omega} \langle \rho^2 \rangle, \quad (58)$$

where $\langle r^2 \rangle_9^{1/2}$ is the ^9Li r.m.s. radius and the mean square value of the hyperradius, $\langle \rho^2 \rangle$, can be easily calculated using the ground state wave function. The values of $\langle r^2 \rangle_{11}^{1/2 (d)}$ and $\langle r^2 \rangle_{11}^{1/2}$ obtained by the pure diagonalization of the truncated Hamiltonian matrix and with the allowance for the asymptotic tail of the wave function, respectively, are presented in the 4-th and the 5-th columns of the table 1.

It is seen that in calculation of the ground state, the allowance for the wave function asymptotics is very important for a weakly-bound system like ^{11}Li . The terms of expansion (56) with the number of total oscillator quanta $N \simeq 100$ that cannot be obtained in the usual oscillator-basis variational calculations, play an essential role in the description of the transverse momentum distribution, r.m.s. radius, etc. The convergence of $\langle r^2 \rangle_{11}^{1/2}$, transverse momentum distribution and other properties of the wave function in the full J -matrix calculation is rather good. Nevertheless, it is seen that the r.m.s. radius converges to a value that is somewhat larger than the experimental one, and the calculated transverse momentum distribution appears narrower than the experimental one. These shortcomings can be overcome by the adjustment of n ^9Li potential. We have not aimed to fit the potential to the ^{11}Li properties, we have just take its parameters from ref. [13].

6.3 The soft dipole mode

The dipole transition operator in our model is of the form

$$\mathcal{M}(E1 \mu) = -\frac{N_v Z}{A} e y Y_{1\mu}(\hat{y}), \quad (59)$$

where e is the proton charge, $A = 11$, $Z = 3$ and the number of valence neutrons, $N_v = 2$. The operator (59) corresponds to the excitation of the three-body cluster modes only. The excitation energy of the first excited state of ${}^9\text{Li}$ is relatively high (~ 4 MeV). So, low-energy $E1$ -transitions correspond to the excitation of the cluster degrees of freedom only and should be described by the operator (59).

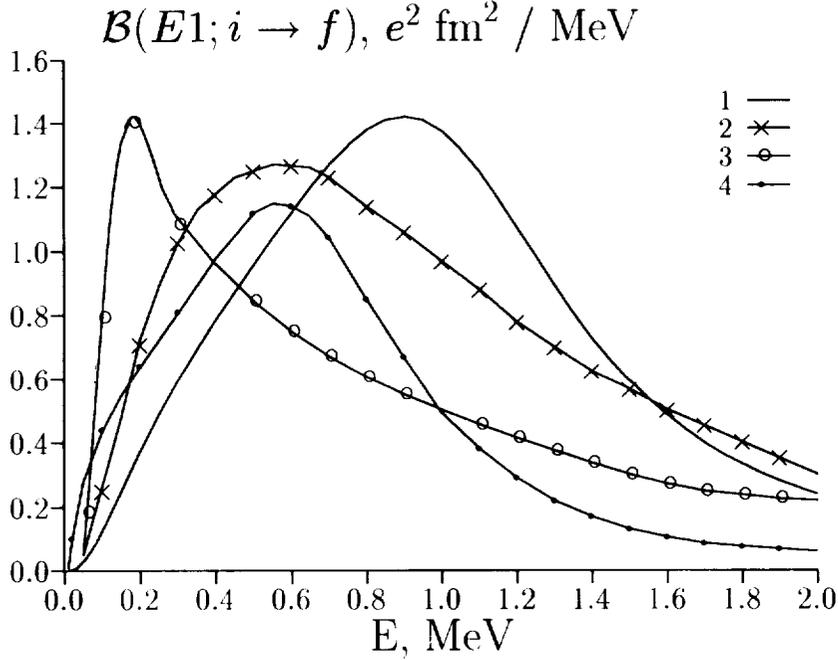


Figure 1: Comparison of our results for $\mathcal{B}(E1; g.s. \rightarrow \text{continuum})$ in ${}^{11}\text{Li}$ with results of other authors. 1 — this work (J -matrix method), 2 — ref. [15], 3 — ref. [16], 4 — experimental data parametrization of ref. [17].

The cluster reduced probability of the $E1$ -transition, $\mathcal{B}(E1; E_f - E_0)$, associated with the operator (59), is displayed on the figure 1.

This figure shows the comparison of the results of our calculations of cluster $\mathcal{B}(E1; E_f - E_0)$ with the parametrization of experimental data of ref. [17]. The agreement is reasonable. The form of the $\mathcal{B}(E1; E_f - E_0)$ peak is well reproduced, the discrepancy in the position of the $\mathcal{B}(E1; E_f - E_0)$ maximum is supposed to be eliminated by the adjustment of the potentials. The results of the $\mathcal{B}(E1; E_f - E_0)$ calculations of refs. [15, 16] are also depicted. All these calculations give a low energy peak which can be associated with the soft dipole mode.

The soft dipole mode exhausts about 90% of the cluster sum rule (EWSR) associated with the operator (59). The contribution from the soft dipole mode to the total EWSR is relatively small. In the vicinity of the sharp $\mathcal{B}(E1; E_f - E_0)$ maximum at the excitation energy $E \approx 1\text{-}2$ MeV only $\sim 8\%$ fraction of the total EWSR is exhausted. Nevertheless, the account for the soft dipole mode results in an essential increase of the electromagnetic dissociation cross section of 0.8 GeV/nucleon ${}^{11}\text{Li}$ beams on Pb and Cu targets. Using the sums of the ${}^{11}\text{Li}$ and target nucleus charge radii as impact parameter we obtain for the electromagnetic dissociation cross sections the values of 0.966 barn for the Pb target and 0.132 barn for the Cu target; the corresponding experimental values

are 0.890 ± 0.110 barn and 0.21 ± 0.04 barn, respectively [18]. $E0$ - and $E2$ -transitions give only 1.2% contribution in the cross sections.

Thus, it is shown, that cluster model ${}^9\text{Li}+n+n$ yields a good description of the ground state properties and $E1$ -transitions in the ${}^{11}\text{Li}$ nucleus. The HOR may be used successfully in the studies of weakly-bound systems with long-tailed wave functions, e.g., in the study of neutron halo properties. For both bound and continuum states the correct account of the wave function asymptotics in the framework of the oscillator representation of scattering theory is very important in such studies. Low-energy $E1$ -transitions in ${}^{11}\text{Li}$ are of the cluster nature. The widths and the position of resonant states calculated in the democratic decay approximation are in a reasonable agreement with experiment.

Appendix. Isolated States. The scattering problem with nonlocal separable potential V^N can display some peculiarities which we explain here using a simple example when the Hamiltonian H is approximated by the matrix of a size 2×2 (i.e. $N = 1$). In specific situation when $T_{01} = -V_{01}$, i.e. the nondiagonal matrix elements of the kinetic and potential energies cancel each other $H_{01} = 0$, we obtain that the Harmonic oscillator wave function $R_{00}(r)$ is an eigenfunction of this Hamiltonian corresponding to the eigenvalue $E_0 = T_{00} + V_{00}$. If $E_0 > 0$ we find an example of the bound state embedded in continuum [19]. It is clear that the eigenfunction $R_{00}(r)$ is not connected with the rest basis states $R_{n0}(r)$. Thus it is isolated from continuum states and can be called an *isolated states*. The phase shift $\delta_0(k)$ displays a narrow resonance near energy E_0 at small value of h_{01} . It transforms into the resonance of zero width when $H_{01} \rightarrow 0$.

Acknowledgments

We are thankful to Profs. J.Bang, B.Danilin, M.Moshinsky, I.Thompson and J.Vaagen for valuable discussions.

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