

Boson Mapping Techniques applied to Constant Gauge Fields in QCD

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Abstract

Pairs of coordinates and derivatives of the constant gluon modes are mapped to new gluon-pair fields and their derivatives. Applying this mapping to the Hamiltonian of constant gluon fields results for large coupling constants into an effective Hamiltonian which separates into a one describing a scalar field and another one for a field with spin two. The ground state is dominated by pairs of gluons coupled to color and spin zero with slight admixtures of color zero and spin two pairs. As color group we used SU(2).

1 Introduction

In this contribution we report on a possible non-perturbative treatment of Quantum-Chromodynamics (QCD). As the color group we use SU(2). We further restrict to gluons only because due to their larger color charge, compared to quarks and anti-quarks, they will dominate at low energy, e.g., in the vacuum state. As has been indicated by several previous contributions [1, 2] the coupling to color and spin zero pairs are dominating the low energy structure of QCD, at least in perturbative calculations. This leads to assume that pair correlations play an important role in the lowest energy state (the vacuum) and that boson mapping techniques may help to make more transparent the physical structure. Combined with many body techniques of nuclear physics this can represent a possibility to solve non-perturbatively QCD. The method presented in this contribution can, e.g., be applied to the Hamiltonian as proposed in ref. [3]. There the complete Hilbert space in a finite universe (radius of several fm) is mapped to a model space of constant modes only. The non-constant modes are taken perturbatively into account, leading to renormalized interaction constants.

In section 2 we discuss the boson mapping after having introduced the Hamiltonian of constant modes. Furthermore, we give the result of the mapped effective Hamiltonian in the limit of large coupling constant g . Finally in section 3 conclusions are given and future applications are mentioned.

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2 A Boson Mapping of Pair Fields

Confinement properties of QCD are considered to be related to the infrared limit (large wave lengths) of the QCD. Therefore, in order to get a first idea one may just restrict to constant modes of gluons, i.e. the vector fields A_{ia} are approximated by constant fields denoted by c_{ia} . Here i is the space and a the color index, both ranging from 1 to 3 (in SU(2)- color the gluons are in the color T=1 representation). With this the Hamiltonian of gluonic QCD acquires the form[4]

$$\hat{H} = -\frac{1}{2} \sum_{ia} \frac{\partial^2}{\partial C_{ia} \partial C_{ia}} + \frac{1}{4} g^2 [(\sum_{ia} C_{ia} C_{ia})^2 - \sum_{ij} (\sum_a C_{ia} C_{ja}) (\sum_b C_{ib} C_{jb})] \quad (1)$$

where g is the coupling constant. If the non constant modes are included perturbatively higher terms will appear but the general pair structure, pairwise coupling to color zero (contraction over the indices a or b), remains. In equ. (1), having contracted over the color index, only spin zero and two pairs appear and therefore suggests to apply a boson mapping to the paired expressions. One often redefines $C_{ia} \rightarrow g^{-\frac{1}{2}} C_{ia}$ which as a result produces an overall factor $g^{\frac{3}{2}}$ in front of the Hamiltonian.

Normally boson mappings are related to boson creation and annihilation operators. For an excellent review see ref.[5]. One distinguishes between two types of boson mappings: (i) the Dyson (D) and the (ii) Holstein- Primakoff mapping (HP). The first one results into a non- hermitian Hamiltonian and the latter into a hermitian one. Both are equivalent and the problem is well defined but of course the HP gives a more pleasant hermitian structure of the Hamiltonian. Instead of using boson creation and annihilation operators we will use coordinates C_{ia} and derivatives $P^{ia} = \frac{\partial}{\partial C_{ia}}$ (for convenience we will use cartesian components, i.e. $P^{ia} = P_{ia}$). The reason for this is the more simpler and transparent structure of the Hamiltonian which would be very complicate in terms of the creation and annihilation operators. First we will give the Dyson mapping which is completely analog to the one using creation and annihilation operators. Then we go from there to the HP mapping which will be very different to the one in terms of creation and annihilation operators!

The boson- pair mapping is given by

$$\begin{aligned} (\sum_a C_{ia} C_{ja})_D &= q_{ij} \\ (\sum_a P_{ia} P_{ja})_D &= \frac{1}{2} \sum_{k,k'} (p_{k'i} q_{k'k} p_{kj} + p_{kj} q_{kk'} p_{k'i}) - p_{ij} \\ (\sum_a C_{ia} P_{ja} + \frac{3}{2} \delta_{ij})_D &= \sum_k q_{ik} p_{kj} + \frac{3}{2} \delta_{ij} \end{aligned} \quad (2)$$

with

$$[p_{ij}, q_{nm}] = \delta_{in} \delta_{jm} + \delta_{im} \delta_{jn} \quad (3)$$

In equation (2) the index D refers to "Dyson mapping". As can be seen the pair of derivatives does not preserve their hermitian structure under the D- mapping. Also the operator in the last line, which is introduced in order to obtain a closed algebra and is anti- hermitian in the original

space, does also not preserve the anti-hermitian property in the mapped space. The q_{ij} and p_{ij} are not yet normalized as can be seen from equ. (3).

That the hermitian properties are not preserved has to do with an additional assumption, namely that the volume element is of the simple form $d\mathbf{q}$ ($=\prod_{i \leq j} q_{ij}$). However, if one assumes a more complicate volume element $d\mathbf{q}K^2(\mathbf{q})$ (in the argument of K the notation \mathbf{q} refers to the dependence on all q_{ij}) we can choose then $K(\mathbf{q})$ such that all hermitian properties are conserved. In order to recover the simple volume element we have to redefine all operators of equ. (2) (denoted now collectively by \hat{O}) and the wave functions Ψ by

$$\begin{aligned}(\hat{O})_{HP} &= K(\mathbf{q})(\hat{O})_D K^{-1}(\mathbf{q}) \\ (\Psi)_{HP} &= K(\mathbf{q})(\psi)_D\end{aligned}\quad (4)$$

where the index HP now refers to the Holstein- Primakoff mapping.

The difference to the HP mapping using creation and annihilation operators becomes obvious when one remembers that in the latter the K is an operator depending on the Casimir operators of the unitary group $U(3)$ [5, 6] (the generators are given in the last line of equ. (2) when C_{ia} is substituted by a creation and P_{ia} by an annihilation operator) while in our proposal the K is a function in the coordinates q_{ij} only. The equivalent in the other case would be a function in pairs of creation operators. Besides this essential difference the HP mapping results always into a non-polynomial function in the operators, except this does not represent a difficulty when we deal with coordinates. Even if the function K is complicate we always can integrate numerically!

In order to determine the function K we require that the anti-hermitian property of $(\sum_a C_{ia} P_{ia} + \frac{3}{2}\delta_{ij})$ is preserved, i.e.

$$(K(\mathbf{q})(\sum_a C_{ia} P_{ja} + \frac{3}{2}\delta_{ij})_D K^{-1}(\mathbf{q}))^\dagger = -K(\mathbf{q})(\sum_a C_{ia} P_{ja} + \frac{3}{2}\delta_{ij})_D K^{-1}(\mathbf{q})\quad (5)$$

which results into the condition

$$\begin{aligned}\sum_{ik} (q_{ik} p_{ki} K(\mathbf{q})) &= -\frac{3}{2} K(\mathbf{q}) \\ \sum_k (q_{ik} p_{kj} K(\mathbf{q})) &= 0 \quad , \text{for } i \neq j\end{aligned}\quad (6)$$

This implies that $K(\mathbf{q})$ is a spin scalar and K^{-1} a sum of monomials of order 3 (note that $\sum_{ik} (q_{ik} p_{ki} q_{nn}) = 2q_{nn}$).

Because of lack of space we cannot go into details here but merely give a rough description of the results. The detailed analysis is given elsewhere[7]. The $K(\mathbf{q})$ is a function in the pair coordinates q_{ij} . Instead of using decoupled indices we can introduce coordinates of a given spin, i.e. $q_m^{[l]}$ with $l = 0, 2$. The exact dependence is obtained by using a linear combination of all possible monomials of order three with total spin zero. After that we made a change of variables by transforming $q_m^{[2]}$ to an intrinsic system very similar to what is done in the collective model of a nucleus where one transforms from the deformation quadrupole coordinate (which has also angular momentum 2) to a system where the quadrupole operator is diagonal[8]. Also here appear

some kind of "deformation" coordinates β and γ . The physical interpretation is that they describe the deformation (distribution) of the wave function in coordinate space. Also we have transformed the coordinate $q_0^{[0]}$ to $\sqrt{2}q_0^{[0]} = q + \sqrt{2}\beta\cos(\gamma + \frac{\pi}{3})$. With this we obtain the final expression of the exact mapping of the Hamiltonian. However, this expression appears complicate at first sight. It gets more transparent when one develops around the minimum values of the potential. One finds that in lowest order the Hamiltonian can be separated in a sum of a pure q and $q_m^{[2]}$ dependent part:

$$\begin{aligned}\hat{H}_q & \pm 2\sqrt{\frac{2}{3}}g^{\frac{2}{3}} \left\{ -\frac{d}{dq}q\frac{d}{dq} + \frac{9}{4q} + \frac{1}{2}\sqrt{\frac{3}{2}}q^2 \right\} \\ \hat{H}_\beta & = 2\sqrt{5}g^{\frac{2}{3}} \left\{ -\left[\frac{d}{dq^{[2]}} \times \frac{d}{dq^{[2]}}\right]_0^{[0]} + \frac{1}{2\sqrt{5}}\beta^2\left[4\cos^2(\gamma + \frac{\pi}{3}) + \frac{1}{2}\right] \right\}\end{aligned}\quad (7)$$

where the square bracket with the cross (\times) inbetween means standart angular momentum coupling. This result is only a good approximation when the coupling constant is large! Nevertheless we can construct a basis of functions with which we can also diagonalize the general expression. The interesting part of the above result is that we have a Hamiltonian in q which has a minimum in its potential for values of $q \neq 0$! This has as a consequence that the ground state will contain a q -condensate. The Hamiltonian in β is just an anharmonic oscillator, i.e., the ground state will contain small admixtures in the spin two pair. Within a rough approximation, and taking into account the relation of q with $q_0^{[0]}$ and β, γ , we can state that within the model of constant modes in QCD the vacuum state is dominated by a spin and color zero condensate.

3 Conclusions

We have applied a boson mapping technique to the model of constant modes of QCD. Instead of using creation and annihilation operators we used coordinates and derivatives. The non- hermitian Dyson mapping works very similar to the standart boson mapping[5, 6]. However, going from there to the Holstein- Primakoff mapping is quite different! The mapped Hamiltonian of the model of constant modes separates for large coupling constant into a part depending on q (essential the spin and color zero gluon pair) and the other depending on the color zero and spin two gluon pair. The spin zero part shows a minimum in the potential at values different from zero and thus produces a spin and color zero condensate for the vacuum state. The spin two part is an anharmonic oscillator and indicates slight admixtures of those bosons to the vacuum state. For large coupling constant the Hamiltonian separates into a sum of a pure (q, p) and a pure $(q_m^{[2]}, p_m^{[2]})$ depending part.

The model used is of course very simple. Nevertheless, using the more realistic Hamiltonian of ref.[3] the principal qualitative results will not change. This contribution has to be seen as a further step towards the non- perturbative description of QCD. The detailed analysis of the results presented here are given in ref.[7].

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