AN ALGEBRAIC CLUSTER MODEL BASED ON THE HARMONIC OSCILLATOR BASIS

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Abstract

We discuss the semimicroscopic algebraic cluster model introduced recently, in which the internal structure of the nuclear clusters is described by the harmonic oscillator shell model, while their relative motion is accounted for by the Vibron model. The algebraic formulation of the model makes extensive use of techniques associated with harmonic oscillators and their symmetry group, $SU(3)$. The model is applied to some cluster systems and is found to reproduce important characteristics of nuclei in the *sd*-shell region. An approximate $SU(3)$ dynamical symmetry is also found to hold for the $^{12}C + {}^{12}C$ system.

1 Introduction

The harmonic oscillator and the $SU(3)$ group have proven to be invaluable tools of nuclear physics. (See, e.g. Ref. [1].) These concepts can be used to describe complex physical systems in a relatively straightforward way by utilizing the advantages of the group theoretical description. The harmonic oscillator picture has been found to be a suitable approach to various nuclear excitations, which sometimes could also be related to each other in terms of it.

Clustering can be considered a special collective excitation of certain nuclei. The structure of these (mainly light) nuclei can be interpreted in terms of a picture based on the relative motion of two (or more) nuclear clusters. In order to describe these nuclear systems cluster models have to take into account the relative motion, as well as the internal structure of the clusters. These models generally differ in their basic model assumptions, mathematical formulation and, consequently, also in the range of their applicability. Microscopic cluster models apply effective two-nucleon forces and rigorously take into account the effect of the Pauli principle by using antisymmetrized wavefunctions. However, fully microscopic calculations may turn out to be prohibitively difficult for a large number of realistic cluster systems. Phenomenologic cluster models, which are based on less strict model assumptions may have a wider range of applicability, and are generally used to describe a large amount of experimental data in a systematic way. Semimicroscopic cluster models utilize the advantageous sides of microscopic and phenomenologic models by combining the microscopic (antisymmetrized) basis with phenomenologic cluster–cluster interactions. This allows calculations in a wider range of nuclei without forgetting about the fermionic nature of the nucleons, i.e. without abandoning the Pauli principle.

From the technical point of view, most of the cluster models apply the geometric description, i.e. they use (nucleon–nucleon or cluster–cluster) potentials and work in the geometric space, while

some others prefer the algebraic description in terms of creation and annihilation operators and the second quantized formalism. Harmonic oscillators appear in a natural way in both approaches and offer a convenient way of interrelating them.

Recently we have intoduced a semimicroscopic algebraic cluster model [2,3] which makes extensive use of the harmonic oscillator picture in describing the relative motion as well as the internal structure of the clusters. In the first applications of the model we tested its ability of reproducing certain features of realistic cluster systems and tried to estimate the validity of the harmonic oscillator picture it is based on.

2 The Semimicroscopic Algebraic Cluster Model

Our earlier attempts of describing various nuclear cluster systems in terms of a pheonomenologic cluster model, the Vibron model [4] and its extensions have revealed [5,6] that these models can not distinguish between Pauli-forbidden and allowed states: complete forbidden shells can be excluded by a simple rule, the Wildermuth condition, but no such distinction can be made within allowed shells. These studies, however, have also pointed out the importance of the $SU(3)$ group as a possible of tool combining the relative motion and the internal structure of the clusters. This group appears in a special limit of the Vibron model accounting for the relative motion sector, and it can also be used to describe the internal excitations of the individual clusters. These preliminaries have paved the way to the introduction of the semimicroscopic algebraic cluster model [2,3].

In this model the internal structure of a cluster is described in terms of the $SU(3)$ (harmonic oscillator) shell model [7], therefore its wavefunction is characterized by the $U_C^{ST}(4) \otimes U_C(3)$ symmetry, where C refers to cluster, and $U^{ST}(4)$ is Wigner's spin-isospin group [8]. The relative motion of the clusters is accounted for by the vibron model with $U_R(4)$ group structure [4]. The representation labels of the group chain

$$\begin{aligned} & U_{C_1}^{ST}(4) \otimes U_{C_1}(3) \otimes U_{C_2}^{ST}(4) \otimes U_{C_2}(3) \otimes U_R(4) \\ & \supset U_C^{ST}(4) \otimes U_C(3) \otimes U_R(3) \supset U_C^S(2) \otimes U(3) \\ & \supset U_C^S(2) \otimes O(3) \supset U(2) \supset O(2) \end{aligned} \tag{1}$$

provide us with the quantum numbers for the basis states of a two-cluster system. From this set we have to skip those states, which are Pauli forbidden, or which correspond to spurious excitations of the center of mass. A simple recipe for eliminating these states is applying a matching requirement between the quantum numbers of the shell model basis of the whole nucleus and its cluster model basis [2,3]. This recipe is based on the connection between the harmonic oscillator shell model and harmonic oscillator cluster model [10]. This procedure corresponds to a special truncation of the extensive shell model basis in the sense, that only those states survive, which are Pauli-allowed, and are relevant to the cluster structure under study.

When the internal structure of each cluster is described by a single $U_C^{ST}(4) \otimes U_C(3)$ representation, then the physical operators of the system can be obtained in terms of the generators of the $U_{C_1}^{ST}(4) \otimes U_{C_1}(3) \otimes U_{C_2}^{ST}(4) \otimes U_{C_2}(3) \otimes U_R(4)$ group. In such a case the description is algebraically closed, i.e. the matrix elements can be deduced by means of group theoretical techniques. In the

limiting case when the Hamiltonian is given by the invariant operators of (1), then the eigenvalue problem has an analytical solution, and a $U(3)$ dynamical symmetry is said to hold.

The problem can be simplified further if one or both of the clusters are even–even nuclei (i.e. they consist of even number of protons and neutrons). In this case the clusters are said to be $U_C^{ST}(4)$ scalars, furthermore, if the clusters are closed shell nuclei, then they are also $U_C(3)$ scalars. In this case these groups and the quantum numbers associated with them do not appear explicitly in the formulas. In Ref. [3] the formalism is presented in detail for the $U_C(3) \otimes U_R(4)$ and $U_{C_1}(3) \otimes U_{C_2}(3) \otimes U_R(4)$ models, as well as for the restricted $U_C^{ST}(4) \otimes U_C(3) \otimes U_R(4)$ model. In this latter case the restriction implies that only spin and isospin free interactions and a single $U_C^{ST}(4)$ representation are considered. If both of the clusters are $U_C^{ST}(4)$ and $U_C(3)$ scalars, the model reduces to that of the simple vibron model with a basis truncation corresponding to the Wildermuth condition [5].

Here we give a brief account of the $U_C(3) \otimes U_R(4)$ model, which is able to describe two–cluster systems in which one of the clusters is a closed–shell nucleus (e.g. ^4He, ^{16}O, or ^{40}Ca), while the other one is an even–even nucleus. In this simple case the basis states can be labeled without explicit reference to the $U^{ST}(4)$ group, (unless some higher excitations of the non–closed–shell nucleus are also considered), and the cluster model basis states are characterized by the representation labels of the group chain:

$$\begin{array}{l} U_C(3) \otimes U_R(4) \supset U_C(3) \otimes U_R(3) \supset SU_C(3) \otimes SU_R(3) \supset SU(3) \supset O(3) \supset O(2) \\ |[n_1^C, n_2^C, n_3^C], [N,0,0,0], \quad [n_\pi,0,0,], \ (\lambda_C,\mu_C) \ , \ (n_\pi,0) \ , \ (\lambda,\mu), K_L, L \quad , \ M \ \rangle. \end{array} \tag{2}$$

The irreducible representations (λ,μ) of $SU(3)$ are obtained by taking the outer product of $(\lambda_C,\mu_C) \otimes (n_\pi,0)$. N stands for the maximal number of the excitation quanta assigned to the relative motion, and it determines the size of the model space. The angular momentum content of a (λ,μ) representation is given by the usual relations of the Elliott model [7]. For technical reasons, however, it is more convenient to use the orthonormal $SU(3)$ basis of Draayer and Akiyama [11], rather than the Elliott basis, which is not orthogonal. The parity of the basis states is determined by the parity assigned to the relative motion: $P_R = (-1)^{n_\pi}$. (The internal states of the non–$U(3)$–scalar cluster carry positive parity $P_C = (-1)^{n_1^C+n_2^C+n_3^C}$, unless major shell excitations of the clusters are also considered.)

The coupled wavefunction can be expressed in terms of $SU(3) \supset O(3)$ Wigner coefficients:

$$\begin{aligned} &|(\lambda_C,\mu_C), N(n_\pi,0); (\lambda,\mu)\chi LM\rangle \\ &= \sum_{\chi_C L_C M_C} \sum_{L_R M_R} \langle (\lambda_C,\mu_C)\chi_C L_C M_C; N(n_\pi,0) L_R M_R | (\lambda,\mu)\chi LM\rangle \\ &\times |(\lambda_C,\mu_C)\chi_C L_C M_C\rangle |N(n_\pi,0) L_R M_R\rangle. \end{aligned} \tag{3}$$

The physical operators can be constructed from the generators of the groups present in group chain (2). In particular, the most general form of the Hamiltonian can be obtained in terms of a series expansion of these generators. In the simplest case, however, when we use the $SU(3)$ dynamical symmetry approximation, and consider only one $U_C(3)$ representation to describe the structure of the non–closed–shell even–even cluster, the energy eigenvalues can be obtained in a closed form:

$$E \ = \ \epsilon + \gamma n_\pi + \delta n_\pi^2 + \eta C_2(\lambda,\mu) + \beta L(L+1). \tag{4}$$

In this approximation the energy levels can easily be assigned to rotational bands labeled by the quantum numbers $n_\pi(\lambda,\mu)\chi$. (See Eqs. (3) and (4).) Bands following an approximate rotational pattern usually appear in the energy spectrum of nuclear cluster systems.

The electromagnetic transition operators are also constructed from the group generators, which automatically implies selection rules in the dynamical symmetry approximation. The electric quadrupole transition operator, for example, is written as the sum of the rank–2 generators of the $U_C(3)$ and the $U_R(3)$ groups:

$$T^{(E2)} = q_R Q_R^{(2)} + q_C Q_C^{(2)}. \tag{5}$$

The matrix elements of the operators with the basis states (3) are calculated using tensor algebraic techniques [12].

The formulation of the $U_{C_1}(3) \otimes U_{C_2}(3) \otimes U_R(4)$ and $U_C^{ST}(4) \otimes U_C(3) \otimes U_R(4)$ models can be done via a straightforward generalization of the results presented here. These models can also be used away from the $SU(3)$ dynamical symmetry limit: in this case the diagonalization of the Hamiltonian becomes necessary. Although the interactions applied in this approach are phenomenological ones, they can be related to the effective two-nucleon forces, due to the use of the microscopic $SU(3)$ cluster model basis. See Ref. [13] for the details.

3 Applications

The applications of the semimicroscopic algebraic cluster model have been carried out so far within the $SU(3)$ dynamical symmetry approximation. This approximation allows exact analytical expression of the energy eigenvalues and electromagnetic transition rates in terms of reduced matrix elements, Clebsch–Gordan coefficients, etc. obtained from the algebraic description. Its validity, and also that of the underlying oscillator picture can be estimated from the comparison of the results with the corresponding experimental data.

As an illustrative example we present here the description of the $T = 0$ states of the ^{24}Mg nucleus in terms of a $^{12}C + {}^{12}C$ cluster model [14]. The structure of this nucleus has been studied carefully via various reactions both in the ground–state region and in the ragion of molecular resonances observed in $^{12}C + {}^{12}C$ heavy ion collisions. These experiments have resulted a large amount of experimental information on the structure of the ^{24}Mg nucleus. Most of the theoretical investigations have focused only on one of the two regions mentioned above, and relatively little effort has been put into their simultaneous investigation.

Our aim was to give a unified description of these two domains in terms of the $U_{C_1}(3) \otimes U_{C_2}(3) \otimes U_R(4)$ model. In this description the internal structure of the ^{12}C clusters is accounted for by the $(\lambda_C,\mu_C) = (0,4)$ $U_C(3)$ representation, which corresponds to an oblate deformation in the geometric picture. We have analyzed about 150 experimental levels in the energy range of 0 to 40 MeV (see Fig. 1.), and nearly 100 electric quadrupole transition probability data in our study, which is a more complete account of the energy spectrum and $E2$ transitions of the ^{24}Mg nucleus than any previous model calculation. We have displayed the $B(E2)$ values for the in–band transitions in Table I. Our results for interband $E2$ transitions are also satisfactory. The fact that most of the transitions forbidden by the selection rules due to the $SU(3)$ dynamical symmetry have very weak experimental counterparts seems to indicate that the $SU(3)$ dynamical symmetry approach is a realistic approximation of the actual physical situation here. The model

was able to describe the general features of the molecular resonance spectrum as well. $E2$ transition probabilities calculated for in-band transitions within this region were significantly smaller than most of the corresponding results of other models. The example of the $^{12}C + {}^{12}C$ system demonstrated that a large number of experimental data, including the ground-state region as well as the highly excited molecular resonances can be reconciled in terms of relatively straightforward calculations, which is one of the major advantages of the semimicroscopic algebraic cluster model.

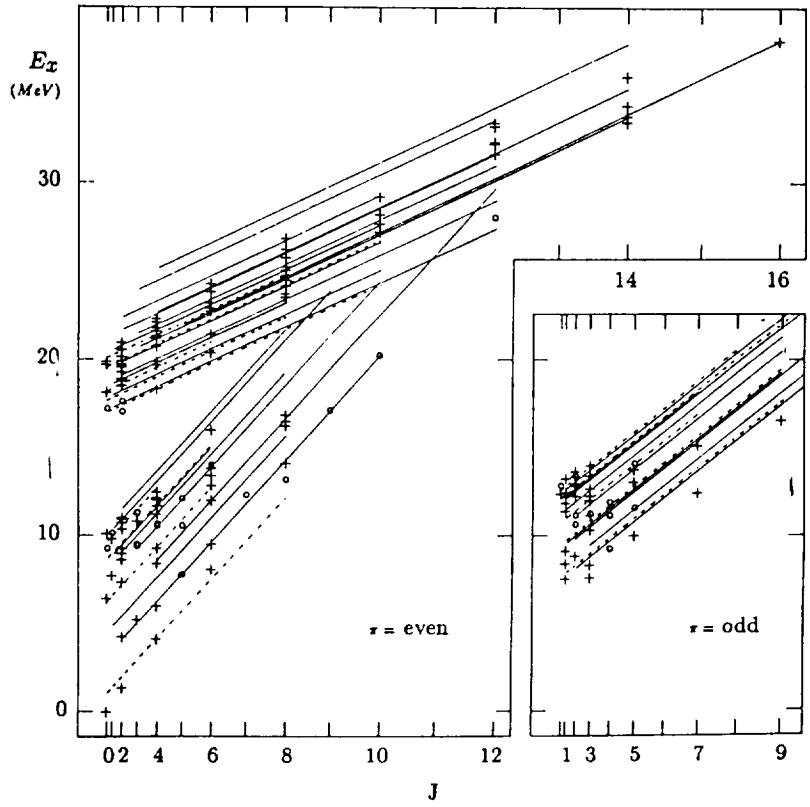

FIG. 1. Positive- and negative-parity $T = 0$ energy levels of the ^{24}Mg nucleus displayed separately in rotational diagram form [14]. Circles (o) stand for states with uncertain J^π assignment. The lines denote the position of the calculated model bands. (Dashed lines indicate bands with $\chi = 0$, which contain only every second possible J value.)

Similar conclusions have been drawn from another application of the model to the $^{14}C + \alpha$ system in terms of the restricted $U_C^{ST}(3) \otimes U_C(3) \otimes U_R(4)$ model, describing the $T = 1$ states of the ^{18}O nucleus [15]. Being a considerably less complex nuclear system than $^{12}C + {}^{12}C$, this example also allowed comparison of our results with those of microscopic calculations. We have found strong correlation between these two data sets, which seems to indicate, that the semimicroscopic algebraic cluster model approximates certain microscopic features reasonably well.

The model has been applied in other areas of nuclear physics as well, where the cluster picture may be relevant. In particular, the link between superdeformed and cluster states of α-like ($N = Z = even$) nuclei has been discussed [16]; the allowed and forbidden binary fission modes

of ground–state–like configurations in *sd*–shell nuclei have been studied [17]; and the possibility of describing exotic cluster radioactivity has been pointed out via the example of $^{210}Pb + {}^{14}C$ clusterization of the ^{224}Ra nucleus [18]. In this latter case the model has to be adapted to heavy nuclei by introducing the pseudo–$SU(3)$ scheme.

TABLE I. In–band transitions for the ^{24}Mg nucleus. See Ref. [14] for the sources of the experimental data. The quantun numbers $n_\pi(\lambda,\mu)\chi$ assigned to the bands are also displayed.

$J_i^\pi(E_{xi})$	$J_f^\pi(E_xf)$	$B(E2)_{Exp}$	$B(E2)_{Th}$	$n_\pi(\lambda,\mu)\chi$
$2^+(1.37)$	$0^+(0.0)$	21.0 ± 0.4[a]	21.0[a]	12(8,4)0
$4^+(4.12)$	$2^+(1.37)$	37.8 ± 3.0	28.0	
$6^+(8.11)$	$4^+(4.12)$	38 ± 13	27.1	
$8^+(13.21)$	$6^+(8.11)$	30 ± 14	23.1	
$3^+(5.24)$	$2^+(4.24)$	38.0 ± 5.5	37.5	12(8,4)2
$4^+(6.01)$	$2^+(4.24)$	18.7 ± 2.4	11.4	
$5^+(7.81)$	$3^+(5.24)$	35.0 ± 4.9	17.5	
$5^+(7.81)$	$4^+(6.01)$	24 ± 10	19.5	
$6^+(9.53)$	$4^+(6.01)$	18 ± 8	18.0	
$7^+(12.35)$	$5^+(7.81)$	21 ± 14	19.7	
$8^+(14.15)$	$6^+(9.53)$	9.1 ± 2.4	13.7	
$2^+(8.65)$	$0^+(6.43)$	14.0 ± 4.3	12.4	12(6,2)0
$6^+(12.86)$	$4^+(9.30)$	11.2 ± 2.1	12.2	
$5^-(10.03)$	$3^-(8.36)$	20^{+8}_{-5}	34.7	13(9,4)0
$7^-(12.44)$	$5^-(10.03)$	51 ± 10	32.3	
$5^-(13.06)$	$3^-(10.33)$	22 ± 4	28.1	13(8,3)1
$4^-(9.30)$	$3^-(7.62)$	29 ± 6	35.1	13(8,3)3
$5^-(11.60)$	$3^-(7.62)$	4.6 ± 1.4	7.3	
$5^-(11.60)$	$4^-(9.30)$	37 ± 11	31.8	

[a] Used to fit model parameters.

4 Summary and Outlook

We have discussed the new semimicroscopic algebraic cluster model, in which a harmonic oscillator picture is used to account for the internal structure and the relative motion of nuclear clusters. The model combines a microscopic harmonic oscillator basis with phenomenologic interactions formulated in algebraic terms. Its first applications to realistic nuclear systems have shown, that it is able to describe a large amount of experimental data in a coherent way, and also seems to reproduce certain microscopic effects reasonably well. The $SU(3)$ dynamical symmetry limit of the model was found to be a realistic approximation for several *sd*–shell nuclei.

The model can be developed further along several lines. First, the treatment of cluster systems with arbitrary open-shell structure can be considered by introducing spin and isospin degrees of freedom. The formalism of the model can also be extended to incorporate several internal configurations, including major shell excitations. Furthermore, by considering symmetry breaking terms in the Hamiltonian a more realistic description of nuclei can be given, relaxing, for example the selection rules imposed by the $SU(3)$ dynamical symmetry.

Acknowledgments

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