GALACTIC OSCILLATOR SYMMETRY

G. Rosensteel

Physics Department, Tulane University
New Orleans, Louisiana 70118 USA

Abstract

Riemann ellipsoids model rotating galaxies when the galactic velocity field is a linear function of the Cartesian coordinates of the galactic masses. In nuclear physics, the kinetic energy in the linear velocity field approximation is known as the collective kinetic energy. But, the linear approximation neglects intrinsic degrees of freedom associated with nonlinear velocity fields. To remove this limitation, the theory of symplectic dynamical symmetry is developed for classical systems. A classical phase space for a self-gravitating symplectic system is a co-adjoint orbit of the noncompact group Sp(3,R). The degenerate co-adjoint orbit is the 12 dimensional homogeneous space Sp(3,R)/U(3), where the maximal compact subgroup U(3) is the symmetry group of the harmonic oscillator. The Hamiltonian equations of motion on each orbit form a Lax system $\dot{X} = [X, F]$, where X and F are elements of the symplectic Lie algebra. The elements of the matrix X are the generators of the symplectic Lie algebra, viz., the one-body collective quadratic functions of the positions and momenta of the galactic masses. The matrix F is composed from the self-gravitating potential energy, the angular velocity, and the hydostatic pressure. Solutions to the Hamiltonian dynamical system on Sp(3,R)/U(3) are given by symplectic isospectral deformations. The Casimirs of Sp(3,R), equal to the traces of powers of X, are conserved quantities.

1 Riemann Ellipsoids

A remarkably unified picture of rotating systems is attained by adopting an algebraic perspective. Classical rotating bodies such as galaxies (period=10^{15}s), stars (10^6s), and fluid droplets (1s), and quantum rotating nuclei (10^{-20}s) may be described in terms of a single subgroup GCM(3) (for general collective motion in 3 dimensions) of the noncompact symplectic Lie group Sp(3,R). In classical physics, the GCM(3) theory is identical to the Riemann ellipsoidal model [1, 2, 3].

A Riemann ellipsoid is a uniform density fluid with an ellipsoidal boundary whose velocity field is a linear function of the inertial frame Cartesian position coordinates $\vec{X}$. The isodensity surfaces of elliptical galaxies are very nearly ellipsoidal [4]. Linear velocity fields $\vec{U}^L$ (the superscript L indicates a laboratory inertial frame quantity) span the dynamical continuum from rigid rotation, $\vec{U}^L(\vec{X}) = \vec{\omega}^L \times \vec{X}$, to irrotational flow, $\vec{\nabla} \times \vec{U}^L = 0$. Thus, Riemann ellipsoids can model a wide class of rotating systems.

The principal aim of this paper is to present the classical symplectic model with particular emphasis upon its relationship with the Riemann ellipsoidal model [5]. But first the Riemann model and its equivalence to the algebraic GCM(3) theory will be reviewed. To describe a linear velocity field, the dynamical group GCM(3) contains the general linear group GL(3,R) as a subgroup. In

addition, to characterize the size, deformation, and orientation of an ellipsoid, the GCM(3) Lie algebra includes the inertia tensor.

There are several advantages to adopting the powerful dynamical group method. First, the Euler fluid equations of motion for a Riemann ellipsoid can be proven to form a Hamiltonian dynamical system [3, 6]. A Riemann ellipsoid phase space is a co-adjoint orbit of GCM(3), and its Poisson bracket is inherited from the Lie algebra structure of GCM(3). Moreover, this Hamiltonian system is a special Lax pair system [7]. Second, the group method is not restricted to continuum fluids. GCM(3) dynamical symmetry applies equally well to discrete systems of particles. Third, GCM(3) dynamical symmetry also applies to some quantum rotating bodies. For example, the Bohr-Mottelson irrotational surface wave model of collective rotational and vibrational states forms an irreducible unitary representation of GCM(3) [8, 9, 10]. Finally, GCM(3) symmetry suggests a natural extension to symplectic Sp(3,R) dynamical symmetry [11]. The latter replaces the collective kinetic energy of the GCM(3) theory by its exact microscopic expression.

The hydrodynamic Riemann ellipsoidal model provides a physical interpretation to the abstract GCM(3) theory: The length C of the Kelvin circulation vector, a constant of the motion for a frictionless, homoentropic fluid flow, is the Casimir invariant for GCM(3) [6].

The velocity fields of rigid rotors and irrotational droplets have very different Kelvin circulation vectors $\vec{C}$. Suppose the rotating system has an ellipsoidal boundary with semi-axes lengths a_k. The inertial frame Kelvin circulation vector, projected onto the kth body-fixed axis, is defined as the line integral of the velocity field $\vec{U}$ around the boundary of the ellipse D_k in the $i - j$ principal plane for i, j, k cyclic. According to Stoke's theorem, these line integrals equal the surface integrals of the curl of the velocity field,

$$ C_k = \frac{M}{5\pi} \oint_{\partial D_k} \vec{U} \cdot d\vec{l} = \frac{M}{5\pi} \iint_{D_k} \vec{\nabla} \times \vec{U} \cdot d\vec{S}, \tag{1} $$

where $\vec{U}$ denotes the projection of the inertial frame velocity field onto the body-fixed axes, and M is the fluid's mass.

By definition, the curl of the velocity field of an irrotational droplet is zero, and, hence, the Kelvin circulation of an irrotational fluid vanishes, $\vec{C} = 0$. For a rigid rotor velocity field, $\vec{\nabla} \times \vec{U} = 2\vec{\omega}$. Because $\pi a_i a_j$ is the area of the ellipse D_k, the rigid rotor circulation components equal $C_k = (2M/5)a_i a_j \omega_k$. For a general linear velocity field, the curl is a constant vectorfield $\vec{\nabla} \times \vec{U} = \vec{\zeta} + 2\vec{\omega}$, where $\vec{\zeta}$ is called the uniform vorticity. As the uniform vorticity ranges continuously from zero to the negative of twice the angular velocity, the complete Riemann sequence from rigid rotation to irrotational flow is traversed.

2 GCM(3) Dynamical Symmetry

The symplectic algebra Sp(3,R) consists of the inertia, virial momentum, and kinetic tensors [11]:

$$ \begin{aligned} Q_{ij}^{\mathsf{L}} &= \sum m_\alpha X_{\alpha i} X_{\alpha j}, \\ N_{ij}^{\mathsf{L}} &= \sum X_{\alpha i} P_{\alpha j}, \\ T_{ij}^{\mathsf{L}} &= \sum m_\alpha^{-1} P_{\alpha i} P_{\alpha j}, \end{aligned} \tag{2} $$

where the sums are carried over the particle index $\alpha = 1, \ldots, A$, m_α denotes the mass of particle α, and $\vec{X}_\alpha$, $\vec{P}_\alpha$ are the inertial frame vector Cartesian position and momentum of particle α. In fluid dynamics, the sums over particles are replaced by integrals over the density distribution, e.g., $Q_{ij}^{\mathrm{L}} = \int \rho(X) X_i X_j d^3 X$. The Poisson brackets close to form the symplectic algebra:

$$
\begin{aligned}
\left\{ N_{ij}^{\mathrm{L}}, N_{kl}^{\mathrm{L}} \right\} &= \delta_{il} N_{kj}^{\mathrm{L}} - \delta_{jk} N_{il}^{\mathrm{L}}, \\
\left\{ Q_{ij}^{\mathrm{L}}, Q_{kl}^{\mathrm{L}} \right\} &= 0, \\
\left\{ Q_{ij}^{\mathrm{L}}, N_{kl}^{\mathrm{L}} \right\} &= \delta_{il} Q_{jk}^{\mathrm{L}} + \delta_{jl} Q_{ik}^{\mathrm{L}} \\
\left\{ T_{ij}^{\mathrm{L}}, T_{kl}^{\mathrm{L}} \right\} &= 0, \\
\left\{ N_{ij}^{\mathrm{L}}, T_{kl}^{\mathrm{L}} \right\} &= \delta_{il} T_{jk}^{\mathrm{L}} + \delta_{ik} T_{jl}^{\mathrm{L}} \\
\left\{ Q_{ij}^{\mathrm{L}}, T_{kl}^{\mathrm{L}} \right\} &= \delta_{ik} N_{jl}^{\mathrm{L}} + \delta_{il} N_{jk}^{\mathrm{L}} + \delta_{jk} N_{il}^{\mathrm{L}} + \delta_{jl} N_{ik}^{\mathrm{L}}.
\end{aligned}
$$

The general collective motion GCM(3) subalgebra includes only the inertia tensor Q_{ij}^{L} and the virial momentum tensor N_{ij}^{L}. The rotational ROT(3) subalgebra is spanned by just the inertia tensor and the antisymmetric part of the viral momentum tensor, viz., the angular momentum $L_k^{\mathrm{L}} = \epsilon_{ijk} N_{ij}^{\mathrm{L}}$. The Lie algebra GL(3,R) of the general linear group is generated by the virial momentum tensor, and the Lie algebra SO(3) of the rotation group is generated by the angular momentum. The inertia tensor generates a 6 dimensional R^6 abelian Lie algebra. GCM(3) and ROT(3) are semidirect sum Lie algebras of the abelian ideal R^6 with GL(3,R) and SO(3), respectively.

In the principal axis frame, the inertia tensor Q is, by definition, diagonal, and its eigenvalues are proportional to the squared axis lengths a_i^2 of the inertia ellipsoid.

Although the exact kinetic tensor T_{ij} is not an element of GCM(3), its linear velocity field value, the collective kinetic tensor, is a function of the algebra generators, [12] $t = {}^t N \cdot Q^{-1} \cdot N$. The Kelvin circulation of a linear velocity field may be expressed in terms of the GCM(3) generators as $C_k = \epsilon_{kij} (Q^{-1/2} \cdot N \cdot Q^{1/2})_{ij}$.

Time evolution in the classical collective models based upon ROT(3), GCM(3), and Sp(3,R) is governed by Hamiltonian dynamics of a special type known as a Lax system. Consider first the simple case of ROT(3) for which the dynamics corresponds to Euler rigid body rotation. If the inertia ellipsoid is rotating with an angular velocity $\Omega_{ij} = \epsilon_{ijk} \omega_k$ and $L_{ij} = N_{ij} - N_{ji}$ is the angular momentum tensor, then Hamiltonian dynamics is given by

$$
\dot{L} = [\Omega, L]. \tag{3}
$$

In terms of vectors, this equation is the familiar law $\dot{\vec{L}} = -\vec{\omega} \times \vec{L}$ that determines the precession of the angular momentum vector in the body-fixed frame.

A matrix equation of the form $\dot{X} = [F, X]$ is called a Lax equation and $X - F$ are referred to as a Lax pair [13, 14]. A useful property of any Lax equation is that the trace of any power of X is conserved. Let I_p denote the trace of the pth power of the matrix X.

$$
I_p = \frac{1}{p} \mathrm{Tr}(X)^p. \tag{4}
$$

For any Lax equation, it is evident that every I_p is a constant of the motion,

$$\dot{I}_p = \text{Tr}\left(X^{p-1} \cdot \dot{X}\right) = \text{Tr}\left(X^{p-1} \cdot [F, X]\right) = \text{Tr}\left(X^{p-1}FX - X^pF\right) = 0. \tag{5}$$

In the case of the Euler equation, $I_2 = -\vec{L} \cdot \vec{L}$ is the negative of the squared length of the angular momentum vector. If p is odd, then I_p is zero. If $p > 2$ is even, then I_p is a function of the squared length of the angular momentum vector. Thus, there is only one independent invariant among the Lax invariants.

Suppose that $X(t)$ is a solution to the Lax equation, $\dot{X} = [F, X]$, corresponding to the initial condition $X = X_0$. If $g(t)$ is a smooth curve of invertible matrices satisfying the matrix differential equation $\dot{g} = F \cdot g$ with the initial condition $g = I$, then the solution to the Lax equation is just the isospectral deformation,

$$X(t) = g(t) \cdot X_0 \cdot g(t)^{-1}. \tag{6}$$

This is proven using the identity $dg^{-1}/dt = -g^{-1}\dot{g}g^{-1}$. If Ω is constant, the matrix differential equation $\dot{g} = \Omega \cdot g$ for the Euler equation has the unique solution $g(t) = \exp(\Omega t)$ for the initial condition $g(0) = I$. Thus, $g(t)$ is a curve in the rotation group $SO(3)$, and the isospectral deformation $L(t) = g(t)L_0 g(t)^{-1}$ describes explicitly the precession of the angular momentum in the body-fixed frame resulting from the rotation $g(t)$ of the intrinsic frame relative to the laboratory frame. Because of the choice of initial conditions for g, L_0 represents the constant angular momentum vector in the inertial laboratory frame.

To present the time evolution for Riemann ellipsoids as a Lax system, suppose the potential energy in the body-fixed frame $V = V(a_1, a_2, a_3)$ is a smooth function of the axes lengths. For a star or galaxy, V is the attractive gravitational self-energy. For a nucleus, V may be approximated by the sum of the attractive surface energy and the repulsive Coulomb energy. Define the Chandrasekhar potential energy tensor W in the rotating frame to be the diagonal matrix,

$$W_{ij} = -\delta_{ij}a_i\frac{\partial V}{\partial a_i}, \tag{7}$$

and, to impose a constraint to constant volume, define the pressure tensor $\Pi = pv$ to be the product of the hydrostatic pressure p times the ellipsoid's volume $v = 4\pi a_1a_2a_3/3$. Hamiltonian dynamics for Riemann ellipsoids is given as follows [7]:

Theorem. If the inertia ellipsoid is rotating with an angular velocity $\Omega_{ij} = \epsilon_{ijk}\omega_k$, then the Riemann ellipsoid Hamiltonian dynamical system is equivalent to the Lax system, $\dot{X} = [F, X]$, where the 6 × 6 real matrices X and F in the body-fixed frame are given by

$$X = \begin{pmatrix} N & -Q \\ {}^t\! Q & -{}^t\! N \end{pmatrix}, \quad F = \begin{pmatrix} \Omega & I \\ (W + \Pi) \cdot Q^{-1} & \Omega \end{pmatrix}. \tag{8}$$

The quadratic Lax invariant equals the negative of the squared length of the Kelvin circulation vector, $I_2 = \text{Tr}(N^2 - {}^t\! Q \cdot Q) = -C'^2$. The higher order Lax invariants are either zero (odd powers) or are functions of the circulation vector's squared length.

The phase space for a Riemann ellipsoid obeying the Lax equation is a co-adjoint orbit of GCM(3):

Theorem. Each Riemann ellipsoid orbit is diffeomorphic to some coset space of GCM(3). The coset depends upon the value of the circulation C:

$$\mathcal{O}_C = \begin{cases} GCM(3)/SO(2) \cong R^{12} \times S_2, & C \neq 0, \quad \dim = 14 \\ GCM(3)/SO(3) \cong R^{12}, & C = 0, \quad \dim = 12 \end{cases} \tag{9}$$

The degenerate orbit is diffeomorphic to 12-dimensional Euclidean space. This irrotational flow phase space, coordinatized by $\alpha_{2\mu}, \pi_{2\mu}, \alpha_0, \pi_0$ for the quadrupole and monopole degrees of freedom, was quantized by A. Bohr. The generic orbits $C \neq 0$ were undiscovered for many years because the significant role of Lie groups in this problem was not appreciated by the Copenhagen school. The generic orbits are diffeomorphic to the Cartesian product $R^{12} \times S_2$ of Euclidean space with the two-dimensional sphere. The topology of the sphere forces the circulation to be quantized to integer multiples of $\hbar$ in a way parallel to the usual angular momentum quantization. Thus, the spectrum of the squared length of the quantum circulation operator is quantized to $C(C+1)\hbar^2$, where C is a nonnegative integer.

3 Sp(3,R) Dynamical Symmetry

Classical symplectic Sp(3,R) time evolution in the rotating frame is given by the Lax equation, $\dot{X} = [F, X]$, if, in the Lax matrix X, the linear approximation t to the kinetic energy is replaced by its exact expression T. In this way, the restriction to linear velocity fields of the Riemann GCM(3) model is removed in the symplectic Sp(3,R) theory.

The symplectic conservation laws are provided by the Lax invariants I_p. The quadratic Casmir invariant of the symplectic algebra is the quadratic Lax invariant, $C^{(2)} = \text{Tr}(N^2 - Q \cdot T)$. Note that for a linear velocity field, the quadratic symplectic invariant simplifies to the negative of the squared length of the Kelvin circulation vector. The odd order invariants vanish. The quartic symplectic Casimir invariant is the quartic Lax invariant,

$$C^{(4)} = \text{Tr}\left[(NQ - Q^t N)(TN - {}^t NT) \right] - 1/2 \, \text{Tr}\left[(N^2 - QT)^2 \right]. \tag{10}$$

There is only one more independent Casimir and Lax invariant $C^{(6)} = I_6$; the higher order invariants are functionally dependent upon the three independent Casimirs $C^{(p)} = I_p$ for $p = 2, 4, 6$.

Since the matrices X and F are elements of the symplectic Lie algebra, the following theorem may be proved:

Theorem. Every solution to the classical symplectic Lax system is given by a isospectral transformation $g(t) \in \text{Sp}(3,R)$ applied to the initial state

$$X(t) = g(t) \cdot X_0 \cdot g(t)^{-1}, \tag{11}$$

where X_0 and X are elements of the symplectic Lie algebra sp(3,R). The group element $g(t)$ is a solution to the matrix differential equation $\dot{g} = Fg$ with the initial condition $g = I$ if and only if X is a solution to the Lax equation with the initial condition $X = X_0$.

Consider the co-adjoint orbit of the symplectic group through the point X,

$$\mathcal{O}_X = \left\{ g \cdot X \cdot g^{-1} \mid g \in \mathrm{Sp}(3, \mathrm{R}) \right\}. \tag{12}$$

The co-adjoint orbit is regarded as a surface in the Euclidean symplectic dual space, $sp(3, R)^*$. A manifold that intersect each co-adjoint orbit exactly once is called a "transversal." A transversal $\mathcal{T}$ for the symplectic co-adjoint group action is provided by a three-dimensional surface [15, 16]

$$\mathcal{T} = \left\{ \hat{S} = \begin{pmatrix} 0 & -S \\ S & 0 \end{pmatrix} \in sp(3, R)^* \mid S = \mathrm{diag}(s_1, s_2, s_3) \right\}. \tag{13}$$

Transversal points correspond to elementary systems for which the virial momentum tensor vanishes, $N = 0$, and the inertia and kinetic tensors are equal and diagonal, $Q = T = S$. Since the inertia and kinetic tensors are positive-definite, the physically relevant transversal consists of only those points for which S is positive-definite, $s_i > 0$.

An orbit of the transversal point $\hat{S} \in \mathcal{T}$ is diffeomorphic to a coset space of the symplectic group modulo the isotropy subgroup. These isotropy subgroups may be proven to be subgroups of the unitary group,

$$U(3) \simeq \left\{ \begin{pmatrix} U & -V \\ V & U \end{pmatrix} \in Sp(3, R) \mid U + iV \in U(3) \right\}, \tag{14}$$

and, thereby, the coset spaces are given explicitly as follows [15, 16, 5]:

Theorem. The symplectic phase spaces are diffeomorphic to coset spaces of Sp(3,R):

$$\mathcal{O}_S = \begin{cases} Sp(3, R)/[U(1) \times U(1) \times U(1)], & s_i \text{ distinct}, & \dim = 18 \\ Sp(3, R)/[U(2) \times U(1)], & s_1 = s_2 \neq s_3, & \dim = 16 \\ Sp(3, R)/U(3), & s_1 = s_2 = s_3, & \dim = 12 \end{cases} \tag{15}$$

The degenerate orbit $Sp(3,R)/U(3)$ is diffeomorphic to the complex Siegel half-plane. In future work, the dynamical system on the Siegel half-plane will be reported.

Acknowledgments

References

[1] S. Chandrasekhar, *Ellipsoidal Figures of Equilibrium* (Yale Univ., New Haven, 1969).

[2] J. Binney and S. Tremaine, *Galactic Dynamics* (Princeton Univ., Princeton, 1987), Section 2.3.

[3] G. Rosensteel and Huy Q. Tran, Astrophys. J. **366**, 30 (1991).

[4] D. Mihalas and J. Binney, *Galactic Astronomy* (Freeman, San Francisco, 1981), Section 5.2.

[5] G. Rosensteel, Astrophys. J. **416**, 291 (1993).

[6] G. Rosensteel, Ann. Phys. (N.Y.) **186**, 230 (1988).

[7] G. Rosensteel, Appl. Math. Lett. **6**, 55 (1993).

[8] A. Bohr and B. Mottelson, *Nuclear Structure* (Benjamin, Reading, Mass., 1975), Vol. II.

[9] G. Rosensteel and D.J. Rowe, Ann. Phys. (N.Y.) **96**, 1 (1976).

[10] O.L. Weaver, R.Y. Cusson, and L.C. Biedenharn, Ann. Phys. (N.Y.) **102**, 493 (1976).

[11] D.J. Rowe, Rep. Prog. Phys. **48**, 1419 (1985).

[12] R.Y. Cusson, Nucl. Phys. **A114**, 289 (1968).

[13] P. Lax, Commun. Pure Appl. Math. **21**, 467 (1968).

[14] Ju. Moser, Adv. Math. **16**, 197 (1975).

[15] T.A. Springer and R. Steinberg, Lecture Notes in Mathematics **131** (Springer, Berlin, 1970).

[16] P. Kramer, Ann. Phys. (N.Y.) **141**, 254, 269 (1982)