SHAPES AND STABILITY OF ALGEBRAIC NUCLEAR MODELS [1]

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Abstract

A generalization of the procedure to study shapes and stability of algebraic nuclear models introduced by Gilmore is presented. One calculates the expectation value of the hamiltonian with respect to the coherent states of the algebraic structure of the system. Then equilibrium configurations of the resulting energy surface, which depends in general on state variables and a set of parameters, are classified through the Catastrophe theory. For one and two-body interactions in the hamiltonian of the interacting boson model-1, the critical points are organized through the Cusp catastrophe. As an example, we apply this *Separatrix* to describe the energy surfaces associated to the Rutenium and Samarium isotopes.

1 Introduction

The geometry of algebraic nuclear models can be studied by means of the time-dependent variational principle [1], [2]. This formalism provides us with a classical limit of the nuclear model, in particular we are mainly concerned with the static properties of the hamiltonian function (energy surface) associated to the considered algebraic nuclear model. In general these hamiltonian functions depend on state variables and a set of parameters, then the appropriate mathematical tool to determine the most general behaviour of their equilibrium configurations is the Catastrophe formalism [3].

A connection between the interacting boson model-1(IBM-1) [4] and the geometrical approach of Bohr-Mottelson [5] was done by expressing the IBM-1 hamiltonian in terms of shape variables. This can be achieved by means of the intrinsic boson states defined by [6] or by the corresponding coherent states [2]. Analysis of shape and phase transitions in this model have been done by [7], [8]. In this work we apply the procedure introduced in Ref. [2] to the interacting boson model, but for the general hamiltonian of one and two-body central interactions involving s and d bosons [4] and determining its associated *Separatrix*. We show that the equilibrium configurations can be

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classified through the Cusp catastrophe, this means that two parameters are enough to describe the most general energy surface. Therefore this analysis generalize those presented previously [4], in which only transitions between pairs of exact $SU(5)$, $O(6)$ and $SU(3)$ symmetries are considered. In the last decade, effective hamiltonians of the IBM-1 have been used to describe energy spectra and transition probabilities of chains of isotopes and isotones [9], [10]. In particular, the effective hamiltonians for Ru [11] and Sm [12] isotopes were determined, *i.e.*, the best choice of the parameters of the general IBM-1 hamiltonian that reproduced the corresponding experimental data. Using these effective hamiltonians we construct their energy surfaces and show that their critical points follow a curve in the parameters space organized by the Cusp *Separatrix*. This let us to know: i) How many equilibrium configurations yield the system and ii) If the behaviour of the model around the critical points may or may not be approximated by an harmonic oscillator.

In the Second Section we review how the energy surfaces can be determined considering a hamiltonian constructed in terms of the generators of a Heisenberg-Weyl algebra. In the Third Section a brief summary of the IBM-1 is presented. In the Fourth Section, an analysis of the shape and stability of the most general energy surfaces of the IBM-1 is made, also the curves associated to the Ru and Sm isotopes are plotted in the parameters space. Finally some remarks are indicated in the last section.

2 Energy Surfaces of Algebraic Models

The energy surfaces (ES) of algebraic models can be determined by means of the coherent states of the associated algebraic structure of the hamiltonian. As an example, a hamiltonian written in terms of the generators of a Heisenberg-Weyl algebra is considered, *i.e.*,

$$H = c_0\, b^\dagger b + c_2\, (b^{\dagger 2} + b^2) + c_1\, (b^\dagger + b)\ , \tag{1}$$

where the operators $b^\dagger$ and b satisfy standard creation and annihilation commutation relations. Although this hamiltonian can be solved analytically by means of a Bogoliubov transformation, we use it to illustrate the procedure to construct the coherent states and the ES of an algebraic model.

The coherent state is defined by the action of the raising generator on the vacuum state [1]

$$|\alpha\rangle = \exp(\alpha^*\, b^\dagger)|0\rangle\ . \tag{2}$$

The Baker-Campbell-Hausdorff formulas can be used to calculate the overlap of two coherent states and the coherent state representation of the creation and annihilation boson operators

$$\langle \alpha' | \alpha \rangle = \exp\left(\alpha' \alpha^*\right)\ , \tag{3}$$

$$\langle \alpha | b = \frac{\partial}{\partial \alpha} \langle \alpha |\ , \qquad \langle \alpha | b^\dagger = \alpha \langle \alpha |\ . \tag{4}$$

Then the energy surface is given by

$$E(\alpha) = \lim_{\alpha' \to \alpha} \frac{\langle \alpha' | H | \alpha \rangle}{\langle \alpha' | \alpha \rangle}\ . \tag{5}$$

Substituting the expressions (1), (3) and (4) into the previous one, one gets the result

$$E(\rho) = (c_0 + 2c_2)\rho^2 + 2c_1\rho \ , \tag{6}$$

where the reality condition $\alpha = \alpha^* = \rho$ was also used.

3 The Interacting Boson Model

In 1975 this model [4] was introduced to describe collective properties of even-even nuclei far from closed shells through the interactions between two kinds of bosons, one with angular momentum $L = 0$ (the s-boson) and another with angular momentum $L = 2$ (the d-boson). The six possible boson states give rise to a $U(6)$ group structure. The bosons represent pairs of fermions, the s-boson reflects the strong pairing attraction of identical nucleons whereas the d-boson is a result of the weaker $J = 2^+$ attraction [4]. Therefore nuclei are pictured as systems of s and d bosons, whose number is equal to half the number of the valence nucleons, the core being considered inert. When a shell is more than half full, hole-pairs are counted instead of particle-pairs.

The most general one and two body hamiltonian that conserves the total number of bosons is

$$\begin{aligned} H_{IBM} &= \varepsilon_s N_s + \varepsilon_d N_d + \sum_{L=0,2,4} \frac{c_L}{2}\sqrt{2L+1}\,[[\mathrm{d}^\dagger \times \mathrm{d}^\dagger]^{[L]} \times [\tilde{\mathrm{d}} \times \tilde{\mathrm{d}}]^{[L]}]^{[0]} \\ &+ \frac{v_0}{\sqrt{2}}\,([[\mathrm{d}^\dagger \times \mathrm{d}^\dagger]^{[2]} \times \tilde{\mathrm{d}}]^{[0]}\, s + s^\dagger\,[\mathrm{d}^\dagger \times [\tilde{\mathrm{d}} \times \tilde{\mathrm{d}}]^{[2]}]^{[0]}) \\ &+ \frac{v_2}{2}\,([[\mathrm{d}^\dagger \times \mathrm{d}^\dagger]^{[0]}\, s^2 + s^{\dagger 2}\,[\tilde{\mathrm{d}} \times \tilde{\mathrm{d}}]^{[0]}) \\ &+ \sqrt{5}u_0\, s^\dagger s\,[\mathrm{d}^\dagger \times \tilde{\mathrm{d}}]^{[0]} + u_2\, s^{\dagger 2} s^2 \ , \end{aligned} \tag{7}$$

where the sets of boson operators s, $s^\dagger$ and $d^\dagger_\mu$, d_μ satisfy the following, different from zero, commutators

$$\left[s,\, s^\dagger\right] = 1 \ , \qquad \left[d_\mu,\, d^\dagger_{\mu'}\right] = \delta_{\mu,\mu'} \ . \tag{8}$$

Now, we construct the coherent state of a six dimensional harmonic oscillator, following the procedure indicated in the previous section. However in this case the associated group is compact and then we restrict the exponential to only one term of the Taylor series expansion,

$$|N,\, \vec{\alpha}\rangle = A_N(s^\dagger + \sum_\mu \alpha_\mu d^\dagger_\mu)^N|0\rangle \ , \tag{9}$$

where A_N is the normalization constant. Evaluating the corresponding Eq.(5) one arrives to the formulae for the energy surface of the model [8], *i.e.*,

$$E(\beta,\, \gamma) = N\,\epsilon\,\frac{\beta^2}{(1+\beta^2)} + \frac{N\,(N-1)}{(1+\beta^2)^2}\,(a_1\,\beta^4 + a_2\,\beta^3\cos 3\gamma + a_3\,\beta^2 + u_2) \ , \tag{10}$$

where it was used that the laboratory variables α_μ can be expressed in terms of two intrinsic parameters β and γ plus three Euler angles. Besides as the energy surface is a rotational invariant

all the dependence in the Euler angles disappear. The parameters a_1, a_2, a_3, and ϵ of the Eq.(10) are combinations of those that appear in the IBM-1 hamiltonian (7)

$$a_1 = \frac{c_0}{10} + \frac{c_2}{7} + \frac{9c_4}{35} , \tag{11}$$

$$a_2 = -\sqrt{\frac{8}{35}} v_0 , \tag{12}$$

$$a_3 = -\frac{2}{\sqrt{5}} v_2 + u_0 , \tag{13}$$

$$\epsilon = \epsilon_d - \epsilon_s . \tag{14}$$

4 Shapes and Stability of Energy Surfaces

The energy surfaces define functions of state variables and a set of parameters, and the Catastrophe theory is used to analize their equilibrium configurations. This formalism let us organize all the possible shapes of the ES into well defined separated regions of the parameters space.

To illustrate how this is done, we consider the potential energy surfaces (PES) of the simplest version of the Generalized Collective Model [5], *i.e.*,

$$V(\beta, \gamma; C_2, C_3) = \beta^4 - C_3 \beta^3 cos3\gamma + C_2 \beta^2 . \tag{15}$$

The equilibrium or critical points are determined by solving the equation $\nabla V(\beta, \gamma) = 0$. The results are given by $(0, 0)$ and (β_0, γ_0), with

$$\beta_0 = \frac{3C_3 \pm \sqrt{9C_3^2 - 32C_2}}{8} , \quad \gamma_0 = 0, \pi/3 . \tag{16}$$

The set of degenerated critical points defines a locus in the parameters space which is called the *Separatrix*. This can be obtained through the determinant of the Hessian matrix or by other procedures, in this case it is immediate that the critical points are double degenerated if the parameters satisfy the expressions

$$9\,C_3^2 - 32\,C_2 = 0 , \tag{17}$$

$$C_2 = 0 , \quad C_3 \neq 0 . \tag{18}$$

For the expression (17) the critical points are localized in $\beta_0 = 3C_3/8$ while for (18) in $\beta_0 = 0$. Besides, it is straight to prove that if $C_2 = 0$ and $C_3 = 0$ the critical point is triple degenerated and localized at $\beta_0 = 0$. The *Separatrix* of the system is shown in Fig.1. It divides the space in regions each characterizing a typical shape yield by the model. By means of the transformation $\beta = y + C_3/4$ the *Separatrix* is taken to the canonical form of the Cusp catastrophe.

Now we study the equilibrium configurations of the energy surface associated to the IBM-1, which is given in Eq.(10). Then we calculate the critical points by taking the derivatives with respect to β and γ variables. It is straightforward to see that the critical points correspond only to $\gamma = 0$ (prolate case) or $\gamma = \pi/3$ (oblate case). Therefore we can restrict to the prolate case,

without losing generality. The energy surface (10), with $\gamma = 0$, can be re-written in terms of the following parameters

$$r_1 = -\frac{2u_2(N-1) - (N-1)a_3 - \epsilon}{2a_1(N-1) - (N-1)a_3 + \epsilon}\,, \tag{19}$$

$$r_2 = -\frac{2a_2(N-1)}{2a_1(N-1) - (N-1)a_3 + \epsilon}\,, \tag{20}$$

and it takes the form

$$E(\beta) = \frac{1}{(1+\beta^2)^2}\left\{\beta^4 + r_1\beta^2(\beta^2+2) - r_2\beta^3\right\}\,, \tag{21}$$

where $E(\beta) = E(\beta, \gamma = 0)/N - \epsilon_s - u_2(N-1)$. One has to notice that the oblate case can be regained by interchanging r_2 by $-r_2$ or equivalently a_2 by $-a_2$.

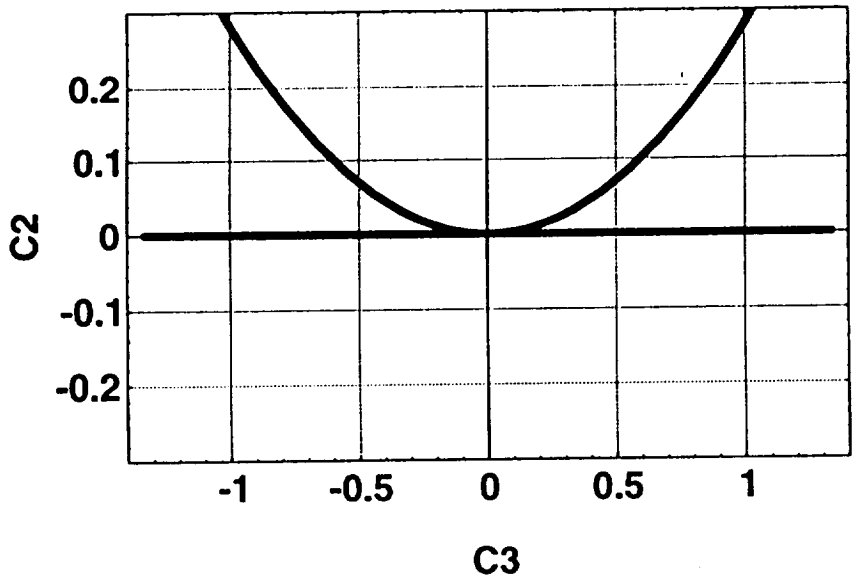

FIG. 1. Separatrix for the Bohr Mottelson Hamiltonian

To find the extrema in β of the Eq.(21) one needs to solve the equation

$$\beta\,(r_2\,\beta^3 + 4\,\beta^2 - 3\,r_2\,\beta + 4\,r_1) = 0\,. \tag{22}$$

From this expression, one determines the locus in the parameters space $(r_2,\, r_1)$ of degenerated critical points. Then the *Separatrix* of the model is defined by the curves

$$r_1 = \pm\frac{(16+9r_2^2)^{3/2}}{54r_2^2} - \frac{32}{27r_2^2} - 1\,, \tag{23}$$

$$r_1 = 0\,. \tag{24}$$

This *Separatrix* is shown in Fig.2 and it corresponds to the Cusp catastrophe although it does not has the canonical form.

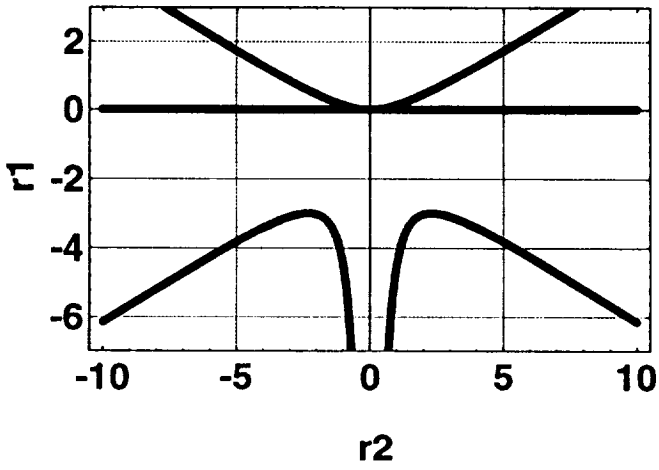

FIG. 2. Separatrix for the IBM-1 Hamiltonian

Now we applied the results to find the shapes and stability of the Ru and Sm isotopes. For the Ru case, one has the number of valence protons pairs, $N_\pi = 3$. As we consider isotopes with a mass number varying from $A = 98 - 110$, the corresponding number of neutron bosons runs from $N_\nu = 2$ to $N_\nu = 8$. Thus the size of the space in the IBM-1 is determined by the total number of bosons, which is the sum of the numbers of proton and neutron bosons. In the Sm case, one has the number of proton bosons, $N_\pi = 6$, and as we take into account $A = 148 - 160$, the number of neutron bosons runs from $N_\nu = 2$ to $N_\nu = 8$. An analysis of their energy spectra and electromagnetic transitions using the hamiltonian (7) is made in [11] and [12]. For both isotope chains, the parameters used are presented in the first eight columns of Table I. Substituting these parameters into the equations (11) to (13) we get the values of a_1 , a_2 and a_3. These are given in the last three columns of Table I, by means of which one can easily construct the corresponding energy surface of each nucleus.

To find the region of the *Separatrix*, Fig. 2, where the different isotopes are localized one calculates the parameters r_1 and r_2 through the equations (19) and (20), as functions of the total number of bosons. For the Ru, one gets the expressions

$$r_1 = \frac{990.2 + 146.2(N-1)}{990.2 + 40.2(N-1)} , \qquad r_2 = 0 , \tag{25}$$

while for the Sm isotopes the parameters are

$$r_1 = \frac{2171.2 + 258.3\,(N-1)}{2171.2 + 151.2\,(N-1)} , \qquad r_2 = \frac{86\,(N-1)}{2175 - 151\,(N-1)} . \tag{26}$$

The localization of the points (25) and (26) are shown in Fig. 3 and Fig 4, respectively.

TABLE I. Parameters, in KeV units, used to describe the Ru and Sm isotope chains

	ϵ	c_0	c_2	c_4	u_0	u_2	v_0	v_2	a_1	a_2	a_3
Ru	990.2	-185.4	-77.4	-.0.4	-53.0	23.3	0	-52.1	-29.7	0	-52.1
Sm	2170.6	-613.7	-318.8	-377.6	227.4	0.4	89.9	-33.0	-204.1	-43.0	-256.9

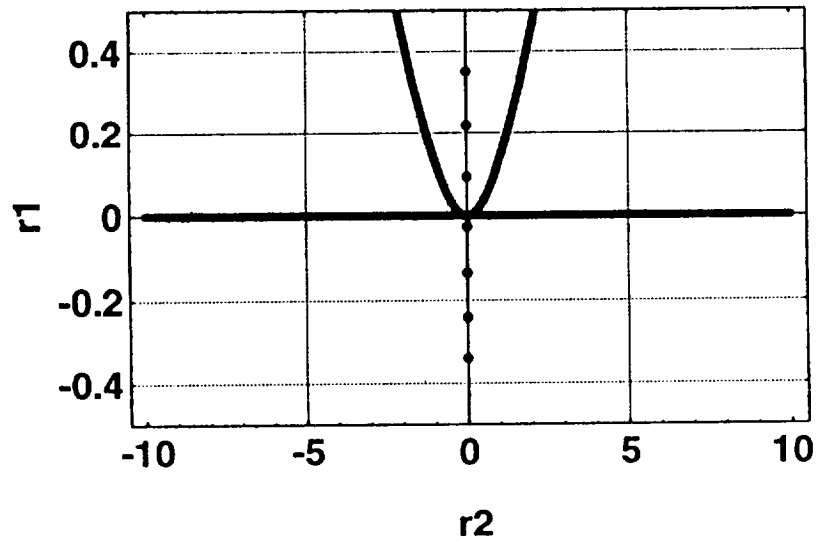

FIG. 3. Plot of the points (25) associated to the Ru isotopes chain.

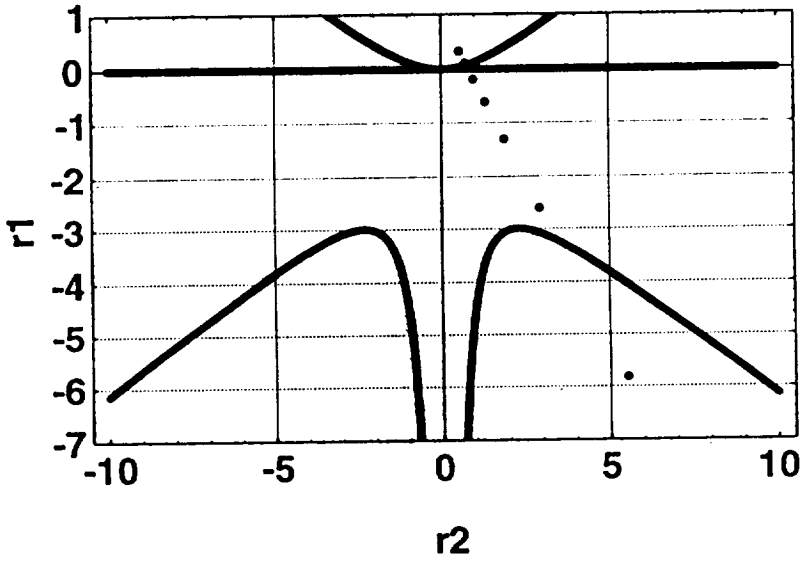

FIG. 4. Plot of the points (26) associated to the Sm isotopes chain.

5 Conclusions

For the Bohr-Mottelson model the yielded shapes and equilibrium configurations are classified by the *Separatrix* of Fig.1. One can identify four regions: (i) Above the parabola, the PES have one minimum at $\beta = 0$. ii) Between the parabola and the C_2 axis, in the PES appear additionally a second minumum at $\beta \neq 0$. iii) Below the C_3 axis the PES have two mimima at $\beta \neq 0$ and

the maximum occurs at $\beta = 0$. The PES around these critical points can be approximated by an harmonic oscillator. iv) On the locus of points that define the *Separatrix*, we have shape transitions and in the vecinity of the critical points (16) the PES cannot be approximated by quadratic functions. It is important to remark that the PES have a mirror symmetry along the C_3 axis, which physically represents transitions from prolate to oblate shapes.

For the IBM-1 hamiltonian one gets (see Fig. 2): (i) For positive values of r_1 one has two regions, above the curve the ES present one minimum at $\beta = 0$ and below the curve they have a second minumum at $\beta \neq 0$. ii) For negative values of r_1 one has again two regions, above the curve the ES are built with two minima at $\beta \neq 0$ and a maximum at $\beta = 0$ and below of it, the ES have one mimimum point at $\beta \neq 0$ and a maximum at $\beta = 0$. For the critical points mentioned above the energy surfaces can be approximated by an harmonic oscillator. iii) Finally on the *Separatrix*, there are shape transitions, and in the vecinity of their critical points the ES cannot be approximated by quadratic functions. It is important to remark that these ES have also a mirror symmetry along the r_2 axis, representing transitions from prolate to oblate shapes.

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