COVARIANT DEFORMED OSCILLATOR ALGEBRAS

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Abstract

The general form and associativity conditions of deformed oscillator algebras are reviewed. It is shown how the latter can be fulfilled in terms of a solution of the Yang-Baxter equation when this solution has three distinct eigenvalues and satisfies a Birman-Wenzl-Murakami condition. As an example, an $SU_q(n) \times SU_q(m)$-covariant q-bosonic algebra is discussed in some details.

1 Introduction

Since the advent of quantum groups and q-algebras (see e.g. [1] and references quoted therein), much attention has been paid to deformations of the algebras of bosonic and fermionic creation and annihilation operators [2]–[6]. Different deformations of the latter arise depending on which property of the undeformed operators is preserved.

In the simple case of the $su(2)$ Lie algebra, two pairs of bosonic creation and annihilation operators $a_i^\dagger$, a_i, $i = 1, 2$, give rise to the Jordan-Schwinger realization

$$J_+ = a_1^\dagger a_2, \qquad J_- = a_2^\dagger a_1, \qquad J_0 = \tfrac{1}{2}(N_1 - N_2), \tag{1}$$

where $N_i = a_i^\dagger a_i$, $i = 1, 2$, are number operators. In addition, the creation operators $a_1^\dagger$, $a_2^\dagger$ (as well as the modified annihilation operators $\tilde{a}_1 = a_2$, $\tilde{a}_2 = -a_1$) are the components $+1/2$ and $-1/2$ of an $su(2)$ spinor, respectively. When extending these two properties to the corresponding q-algebra $su_q(2)$ (where q is real and positive), one gets two different sets of q-bosonic operators.

On the one hand, those first considered by Biedenharn [2], Macfarlane [3], Sun and Fu [4], give rise to a Jordan-Schwinger realization of $su_q(2)$ of the same type as (1), where $a_i^\dagger$, a_i, $i = 1, 2$, now satisfy the relations

$$a_i a_i^\dagger - q^{\pm 1} a_i^\dagger a_i = q^{\mp N_i}, \tag{2}$$

while operators with different indices do still commute, and $a_i^\dagger a_i = [N_i]_q \equiv (q^{N_i} - q^{-N_i})/(q - q^{-1})$. However, the operators $a_1^\dagger$, $a_2^\dagger$ do not transform any more under a definite representation of the algebra.

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On the other hand, the operators $A_i^\dagger$, A_i, $i = 1, 2$, introduced by Pusz and Woronowicz [5], satisfy different relations

$$
\begin{aligned}
A_i^\dagger A_j^\dagger - q^{-1} A_j^\dagger A_i^\dagger &= A_i A_j - q A_j A_i = 0, \qquad i < j, \\
A_i A_j^\dagger - q A_j^\dagger A_i &= 0, \qquad i \neq j, \\
A_i A_i^\dagger - q^2 A_i^\dagger A_i &= I + (q^2 - 1) \sum_{j=1}^{i-1} A_j^\dagger A_j,
\end{aligned}
\tag{3}
$$

where the two modes are not independent any more. As a result of this coupling, the operators $A_1^\dagger$, $A_2^\dagger$ (as well as $\tilde{A}_1 = q^{1/2} A_2$, $\tilde{A}_2 = -q^{-1/2} A_1$) are the components $+1/2$ and $-1/2$ of an $su_q(2)$ spinor respectively, but yield an $su_q(2)$ realization that is substantially more complicated than (1). The algebra (3) has also important covariance properties under the quantum group $SU_q(2)$, dual to $su_q(2)$.

The present communication is concerned with the construction of covariant deformed oscillator algebras that generalize the Pusz-Woronowicz algebra for other quantum groups than $SU_q(2)$ (or more generally $SU_q(n)$). The method used will be based on an R-matrix approach similar to that applied in noncommutative differential geometry [7,8]. In Sec. 2, after reviewing the general form and associativity conditions of deformed oscillator algebras, we will show how to fulfil the latter in terms of a solution of the Yang-Baxter equation with three distinct eigenvalues. The example of an $SU_q(n) \times SU_q(m)$-covariant q-bosonic algebra $\mathcal{A}_q(n,m)$ will be treated in some details in Sec. 3. Finally, in Sec. 4, an alternative derivation of the same algebra, based upon the q-algebra $u_q(n) + u_q(m)$ will be presented.

2 Deformed Oscillator Algebras

Let us consider the complex algebras generated by I, $A_i^\dagger$ $A_i = (A_i^\dagger)^\dagger$, $i = 1, \ldots, N$, subject to the relations [9,10]

$$
\begin{aligned}
A_i^\dagger A_j^\dagger &= X_{ij,kl} A_k^\dagger A_l^\dagger, \\
A_i A_j &= X_{ji,lk}^* A_k A_l, \\
A_i A_j^\dagger &= \delta_{ij} + Z_{jl,ik} A_k^\dagger A_l,
\end{aligned}
\tag{4}
$$

where X and Z are some complex $N^2 \times N^2$ matrices, and there are summations over dummy indices. As a consequence of the Hermiticity properties of the generators, X^* is the complex conjugate of X, and Z is a Hermitian matrix.

For these algebras to be associative, it is sufficient to require the braid transposition schemes for triples of generators. The braid scheme for $A_i^\dagger A_j^\dagger A_k^\dagger$ yields the condition

$$X_{ij,ab} X_{bk,cz} X_{ac,xy} = X_{jk,ab} X_{ia,xc} X_{cb,yz}, \tag{5}$$

i.e., in compact tensor notation, the Yang-Baxter equation for X (in the "braid" version)

$$X_{12} X_{23} X_{12} = X_{23} X_{12} X_{23}. \tag{6}$$

Similarly, for $A_i A_j^\dagger A_k^\dagger$, one gets the two conditions

$$\delta_{ji}\delta_{kx} - X_{jk,ix} + Z_{jk,ix} - X_{jk,ab}Z_{ab,ix} = 0, \tag{7}$$

and

$$Z_{kz,ac}Z_{ja,ib}X_{bc,xy} = X_{jk,ab}Z_{bz,cy}Z_{ac,ix}, \tag{8}$$

which may be written in compact form as

$$(I_{12} - X_{12})(I_{12} + Z_{12}) = 0, \tag{9}$$

and

$$Z_{23}Z_{12}X_{23} = X_{12}Z_{23}Z_{12}. \tag{10}$$

From the Hermiticity properties of the generators, it follows that the remaining two triple products $A_i A_j A_k$ and $A_i A_j A_k^\dagger$ will be associative if $A_i^\dagger A_j^\dagger A_k^\dagger$ and $A_i A_j^\dagger A_k^\dagger$ are so. Hence, eqs. (6), (9), and (10) are the only associativity conditions of algebra (4).

Let now R be any $N^2 \times N^2$ solution of the Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{11}$$

Then the corresponding braid matrix $\hat{R} = \tau R$, where τ is the twist operator (i.e., $\tau_{ij,kl} = \delta_{il}\delta_{jk}$), satisfies an equation similar to (6).

If $\hat{R}$ has three distinct eigenvalues λ_α, $\alpha = 1, 2, 3$, and satisfies a Birman-Wenzl-Murakami (BWM) condition[2]

$$(\hat{R} - \lambda_1 I)(\hat{R} - \lambda_2 I)(\hat{R} - \lambda_3 I) = 0, \tag{12}$$

then with each eigenspace of $\hat{R}$, one can associate two solutions of the set of associativity conditions (6), (9), and (10). In terms of the projector

$$\mathcal{P}_\alpha = \prod_{\beta \neq \alpha} \frac{(\hat{R} - \lambda_\beta I)}{(\lambda_\alpha - \lambda_\beta)} \tag{13}$$

onto the eigenspace corresponding to the eigenvalue λ_α, these two solutions can be written as

$$I - X \simeq \mathcal{P}_\alpha \qquad \text{and} \qquad Z = -\lambda_\alpha^{-1}\hat{R} \quad \text{or} \quad Z = -\lambda_\alpha \hat{R}^{-1}. \tag{14}$$

Considering for instance $Z = -\lambda_\alpha^{-1}\hat{R}$ leads to the following deformed oscillator algebra (written in compact tensor form)

$$A_2^\dagger A_1^\dagger = S A_1^\dagger A_2^\dagger, \qquad A_1 A_2 = S^* A_2 A_1, \qquad A_1 A_2^\dagger = I_{12} - \lambda_\alpha^{-1} R^{t_1} A_2^\dagger A_1, \tag{15}$$

where $S = \tau X$ is found from (13) and (14), and t_1 means transposition with respect to the first space in the tensor product.

Several examples of such deformed oscillator algebras have been worked out so far [9]–[11]. In all cases, the solution of the Yang-Baxter equation that has been considered is the fundamental R-matrix of some classical quantum group. In such circumstances, the deformed oscillator algebras

[2]The $SU_q(n)$-covariant algebra constructed by Pusz and Woronowicz [5] corresponds to the simpler case where $\hat{R}$ has only two distinct eigenvalues, and satisfies a Hecke condition (see Sec. 3).

are left invariant under the transformations induced by the quantum group. The construction presented here is not restricted however to such a choice, and any solution of (11) and (12) might actually be used. In a similar way, deformed oscillator algebras differing from that of Pusz-Woronowicz have been built by considering non-standard solutions of the Yang-Baxter equation and the Hecke condition [12].

The algebras constructed in refs. [9]–[11] include both standard and non-standard ones. The former [9,10] are either of q-bosonic or q-fermionic type, meaning that whenever $q \rightarrow 1$, they go over smoothly into ordinary bosonic or fermionic algebras, respectively. The latter [11], on the contrary, do not have such a smooth behaviour, but share instead some features with the quon algebra [13]. In the next section, we shall consider in more details a covariant q-bosonic algebra generalizing that of Pusz-Woronowicz.

3 An $SU_q(n) \times SU_q(m)$-Covariant q-Bosonic Algebra

The $SU_q(n)$ quantum group [1] is a complex associative algebra generated by I and the noncommutative elements T_{ij}, i, $j = 1, \ldots, n$ of an $n \times n$ matrix T, subject to the relations

$$RT_1T_2 = T_2T_1R, \qquad \det_q T = 1, \tag{16}$$

and the *-involution condition

$$T^* = (T^{-1})^t, \tag{17}$$

with q real. In (16), $\det_q$ denotes the quantum determinant, and R is the fundamental R-matrix associated with the A_{n-1} series of Lie algebras,

$$R = q\sum_{i=1}^{n} e_{ii} \otimes e_{ii} + \sum_{\substack{i,j=1\\ i\neq j}}^{n} e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{\substack{i,j=1\\ i<j}}^{n} e_{ij} \otimes e_{ji}, \tag{18}$$

where $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$. The coproduct, counit and antipode are defined by

$$\Delta(T) = T_1\dot{\otimes}T_2, \qquad \epsilon(T) = 1, \qquad S(T) = T^{-1}, \tag{19}$$

where $\Delta(T_{ij}) = T_{ik} \otimes T_{kj}$.

The braid matrix $\hat{R}$, corresponding to (18), is a real symmetric matrix with two distinct eigenvalues, q and $-q^{-1}$. Their respective multiplicities are $\frac{1}{2}n(n+1)$ and $\frac{1}{2}n(n-1)$, i.e., the dimensions of the symmetric and antisymmetric irreps $[2\dot{0}]_n$ and $[1^2\dot{0}]_n$ of $SU_q(n)$. The $\hat{R}$-matrix satisfies the Hecke condition

$$(\hat{R} - qI)(\hat{R} + q^{-1}I) = 0. \tag{20}$$

Similar relations are valid for $SU_q(m)$. Its generators and fundamental R-matrix will be denoted by $\mathcal{T}_{st}$, s, $t = 1, \ldots, m$, and $\mathcal{R}$, respectively, to distinguish them from the corresponding quantities for $SU_q(n)$. Note that T_{ij} and $\mathcal{T}_{st}$ are assumed to commute with one another.

For the product $SU_q(n) \times SU_q(m)$, one can introduce a "large" R-matrix, $\boldsymbol{R} = q^{-1}R\mathcal{R}$, of dimension $(nm)^2 \times (nm)^2$ [10]. Its matrix elements are defined by

$$\boldsymbol{R}_{(is)(jt),(ku)(lv)} = q^{-1}R_{ij,kl}\mathcal{R}_{st,uv}. \tag{21}$$

From the properties of the two "small" braid matrices $\hat{R}$ and $\hat{\mathcal{R}}$, it follows that $\hat{\boldsymbol{R}} = q^{-1}\hat{R}\hat{\mathcal{R}}$ has three distinct eigenvalues q, $-q^{-1}$, and q^{-3}, with respective multiplicities corresponding to the dimensions of the representations $[2\dot{0}]_n[2\dot{0}]_m$, $[2\dot{0}]_n[1^2\dot{0}]_m + [1^2\dot{0}]_n[2\dot{0}]_m$, and $[1^2\dot{0}]_n[1^2\dot{0}]_m$ of $SU_q(n) \times SU_q(m)$, and satisfies the BWM condition (12).

By applying the results of the previous section to the antisymmetric (reducible) eigenspace of $\hat{\boldsymbol{R}}$ associated with the eigenvalue $-q^{-1}$, one gets a deformed oscillator algebra of type (15), which will be denoted by $\mathcal{A}_q(n,m)$, and whose defining relations are [10]

$$\boldsymbol{A}_2^\dagger \boldsymbol{A}_1^\dagger = \boldsymbol{S}\boldsymbol{A}_1^\dagger \boldsymbol{A}_2^\dagger, \qquad \boldsymbol{A}_2\boldsymbol{A}_1 = \boldsymbol{A}_1\boldsymbol{A}_2\boldsymbol{S}, \qquad \boldsymbol{A}_2\boldsymbol{A}_1^\dagger = \boldsymbol{I}_{21} + q\boldsymbol{R}^{t_1}\boldsymbol{A}_1^\dagger \boldsymbol{A}_2, \tag{22}$$

where

$$\boldsymbol{S} = \tau(\boldsymbol{I} - (q+q^{-1})\boldsymbol{\mathcal{P}}_A), \qquad \boldsymbol{\mathcal{P}}_A = \frac{(\hat{\boldsymbol{R}} - q\boldsymbol{I})(\hat{\boldsymbol{R}} - q^{-3}\boldsymbol{I})}{(q+q^{-1})(q^{-1}+q^{-3})}, \tag{23}$$

and the creation and annihilation operators $\boldsymbol{A}_{is}^\dagger$, $\boldsymbol{A}_{is}$ now have two indices, $i = 1, 2, \ldots, n$, and $s = 1, 2, \ldots, m$. Whenever $q \to 1$, $\boldsymbol{R}$ and $\boldsymbol{S}$ go over into $\boldsymbol{I}$, so that (22) becomes an ordinary bosonic algebra.

The defining relations (22) of the q-bosonic algebra $\mathcal{A}_q(n,m)$ may be rewritten in terms of the two "small" R-matrices as

$$R\boldsymbol{A}_1^\dagger \boldsymbol{A}_2^\dagger = \boldsymbol{A}_2^\dagger \boldsymbol{A}_1^\dagger \mathcal{R}, \qquad R\boldsymbol{A}_2\boldsymbol{A}_1 = \boldsymbol{A}_1\boldsymbol{A}_2\mathcal{R}, \qquad \boldsymbol{A}_2\boldsymbol{A}_1^\dagger = I_{21}\mathcal{I}_{21} + R^{t_1}\mathcal{R}^{t_1}\boldsymbol{A}_1^\dagger \boldsymbol{A}_2, \tag{24}$$

or, in a more explicit form, as

$$\begin{aligned}
R_{ij,kl}\boldsymbol{A}_{ks}^\dagger \boldsymbol{A}_{lt}^\dagger &= \boldsymbol{A}_{jv}^\dagger \boldsymbol{A}_{iu}^\dagger \mathcal{R}_{uv,st}, \\
R_{ij,kl}\boldsymbol{A}_{lt}\boldsymbol{A}_{ks} &= \boldsymbol{A}_{iu}\boldsymbol{A}_{jv}\mathcal{R}_{uv,st}, \\
\boldsymbol{A}_{is}\boldsymbol{A}_{jt}^\dagger &= \delta_{ij}\delta_{st} + R_{ki,jl}\mathcal{R}_{us,tv}\boldsymbol{A}_{ku}^\dagger \boldsymbol{A}_{lv}.
\end{aligned} \tag{25}$$

Let us consider the map $\varphi : \mathcal{A}_q(n,m) \to \mathcal{A}_q(n,m) \otimes (SU_q(n) \times SU_q(m))$, defined by

$$\begin{aligned}
\boldsymbol{A}_{is}'^\dagger &= \varphi(\boldsymbol{A}_{is}^\dagger) = \boldsymbol{A}_{jt}^\dagger T_{ji}\mathcal{T}_{ts}, \\
\boldsymbol{A}_{is}' &= \varphi(\boldsymbol{A}_{is}) = \boldsymbol{A}_{jt}T_{ji}^*\mathcal{T}_{ts}^* = T_{ij}^{-1}\mathcal{T}_{st}^{-1}\boldsymbol{A}_{jt},
\end{aligned} \tag{26}$$

where the elements T_{ij} and $\mathcal{T}_{st}$ of $SU_q(n) \times SU_q(m)$ are assumed to commute with $\boldsymbol{A}_{is}^\dagger$ and $\boldsymbol{A}_{is}$. As a consequence of (16) and its counterpart for $SU_q(m)$, this map leaves the defining relations (25) of $\mathcal{A}_q(n,m)$ invariant. Hence, the latter is an $SU_q(n) \times SU_q(m)$-covariant algebra.

In the next section, an important part will be played by the modified annihilation operators

$$\tilde{\boldsymbol{A}}_{is} = \boldsymbol{A}_{jt}C_{ji}\mathcal{C}_{ts}, \qquad C_{ji} = (-1)^{n-j}q^{-(n-2j+1)/2}\delta_{j,i'}, \qquad \mathcal{C}_{ts} = (-1)^{m-t}q^{-(m-2t+1)/2}\delta_{t,s'}, \tag{27}$$

where $i' \equiv n+1-i$, $s' \equiv m+1-s$. Eq. (24) can be rewritten in terms of $\boldsymbol{A}_{is}^\dagger$, $\tilde{\boldsymbol{A}}_{is}$ as

$$R\boldsymbol{A}_1^\dagger \boldsymbol{A}_2^\dagger = \boldsymbol{A}_2^\dagger \boldsymbol{A}_1^\dagger \mathcal{R}, \qquad R\tilde{\boldsymbol{A}}_1\tilde{\boldsymbol{A}}_2 = \tilde{\boldsymbol{A}}_2\tilde{\boldsymbol{A}}_1\mathcal{R}, \qquad \tilde{\boldsymbol{A}}_2\boldsymbol{A}_1^\dagger = C_{12}\mathcal{C}_{12} + q^2\boldsymbol{A}_1^\dagger\tilde{\boldsymbol{A}}_2\tilde{R}^{-1}\tilde{\mathcal{R}}^{-1}, \tag{28}$$

where $\tilde{R}$ is defined by

$$\tilde{R} = \sum_{i=1}^{n} e_{ii} \otimes e_{i'i'} + q \sum_{\substack{i,j=1 \\ i \neq j'}}^{n} e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{\substack{i,j=1 \\ i<j}}^{n} (-q)^{i-j+1} e_{ij} \otimes e_{i'j'}, \tag{29}$$

and a similar definition holds for $\tilde{\mathcal{R}}$. Under map φ of eq. (26), $\tilde{\boldsymbol{A}}_{is}$ is transformed into

$$\tilde{\boldsymbol{A}}'_{is} = \varphi(\tilde{\boldsymbol{A}}_{is}) = \tilde{\boldsymbol{A}}_{jt}\tilde{T}_{ji}\tilde{\mathcal{T}}_{ts}, \qquad \tilde{T} \equiv C^{-1}(T^{-1})^t C, \qquad \tilde{\mathcal{T}} \equiv \mathcal{C}^{-1}(\mathcal{T}^{-1})^t\mathcal{C}. \tag{30}$$

Finally, combining eqs. (18) and (25) yields the detailed form of the $\mathcal{A}_q(n,m)$ defining relations

$$\begin{aligned}
\boldsymbol{A}^\dagger_{is}\boldsymbol{A}^\dagger_{it} - q^{-1}\boldsymbol{A}^\dagger_{it}\boldsymbol{A}^\dagger_{is} &= 0, \qquad s<t,\\
\boldsymbol{A}^\dagger_{is}\boldsymbol{A}^\dagger_{js} - q^{-1}\boldsymbol{A}^\dagger_{js}\boldsymbol{A}^\dagger_{is} &= 0, \qquad i<j,\\
\boldsymbol{A}^\dagger_{is}\boldsymbol{A}^\dagger_{jt} - \boldsymbol{A}^\dagger_{jt}\boldsymbol{A}^\dagger_{is} &= 0, \qquad i>j, \qquad s<t,\\
\boldsymbol{A}^\dagger_{is}\boldsymbol{A}^\dagger_{jt} - \boldsymbol{A}^\dagger_{jt}\boldsymbol{A}^\dagger_{is} &= -(q-q^{-1})\boldsymbol{A}^\dagger_{js}\boldsymbol{A}^\dagger_{it}, \qquad i<j, \qquad s<t,
\end{aligned} \tag{31}$$

and

$$\begin{aligned}
\boldsymbol{A}_{is}\boldsymbol{A}^\dagger_{jt} - \boldsymbol{A}^\dagger_{jt}\boldsymbol{A}_{is} &= 0, \qquad i\neq j, \qquad s\neq t,\\
\boldsymbol{A}_{is}\boldsymbol{A}^\dagger_{js} - q\boldsymbol{A}^\dagger_{js}\boldsymbol{A}_{is} &= (q-q^{-1})\sum_{t=1}^{s-1}\boldsymbol{A}^\dagger_{jt}\boldsymbol{A}_{it}, \qquad i\neq j,\\
\boldsymbol{A}_{is}\boldsymbol{A}^\dagger_{it} - q\boldsymbol{A}^\dagger_{it}\boldsymbol{A}_{is} &= (q-q^{-1})\sum_{j=1}^{i-1}\boldsymbol{A}^\dagger_{jt}\boldsymbol{A}_{js}, \qquad s\neq t,\\
\boldsymbol{A}_{is}\boldsymbol{A}^\dagger_{is} - q^2\boldsymbol{A}^\dagger_{is}\boldsymbol{A}_{is} &= I + (q^2-1)\Bigg(\sum_{j=1}^{i-1}\boldsymbol{A}^\dagger_{js}\boldsymbol{A}_{js} + \sum_{t=1}^{s-1}\boldsymbol{A}^\dagger_{it}\boldsymbol{A}_{it}\\
&\quad - (q^{-2}-1)\sum_{j=1}^{i-1}\sum_{t=1}^{s-1}\boldsymbol{A}^\dagger_{jt}\boldsymbol{A}_{jt}\Bigg),
\end{aligned} \tag{32}$$

together with the Hermitian conjugates of (31). Whenever $m=1$, substituting $A^\dagger_i$, A_i for $\boldsymbol{A}^\dagger_{i1}$, $\boldsymbol{A}_{i1}$ in (31) and (32) yields the Pusz-Woronowicz relations (3) for arbitrary n values. Hence, the covariant q-bosonic algebra $\mathcal{A}_q(n,m)$ is a generalization of that of Pusz-Woronowicz for values of m greater than 1.

4 Alternative Derivation in Terms of $u_q(n) + u_q(m)$

An alternative approach to the construction of covariant deformed oscillator algebras, based upon q-algebras, has been developed elsewhere [14,15]. In the case of the algebra $\mathcal{A}_q(n,m)$ introduced in the previous section, one considers the q-algebra $u_q(n)+u_q(m)$. The Cartan-Chevalley generators of $u_q(n)$ are denoted by $E_{ii} = (E_{ii})^\dagger$, $i=1, 2, \ldots, n$, $E_{i,i+1}$, $E_{i+1,i} = (E_{i,i+1})^\dagger$, $i=1, 2, \ldots, n-1$, and satisfy the commutation relations

$$\begin{aligned}
[E_{ii}, E_{jj}] &= 0, \qquad [E_{ii}, E_{j,j+1}] = (\delta_{ij} - \delta_{i,j+1})E_{j,j+1},\\
[E_{ii}, E_{j+1,j}] &= (\delta_{i,j+1} - \delta_{ij})E_{j+1,j}, \qquad [E_{i,i+1}, E_{j+1,j}] = \delta_{ij}[H_i]_q,
\end{aligned} \tag{33}$$

together with the quadratic and cubic q-Serre relations. In (33), $H_i \equiv E_{ii} - E_{i+1,i+1}$. The algebra is endowed with a Hopf algebra structure with coproduct Δ, counit ϵ, and antipode S, defined by

$$\Delta(E_{ii}) = E_{ii}\otimes I + I\otimes E_{ii}, \qquad \Delta(E_{i,i+1}) = E_{i,i+1}\otimes q^{H_i/2} + q^{-H_i/2}\otimes E_{i,i+1},$$

$$\Delta(E_{i+1,i}) = E_{i+1,i} \otimes q^{H_i/2} + q^{-H_i/2} \otimes E_{i+1,i}, \quad (34)$$
$$\epsilon(E_{ii}) = \epsilon(E_{i,i+1}) = \epsilon(E_{i+1,i}) = 0, \quad (35)$$
$$S(E_{ii}) = -E_{ii}, \qquad S(E_{i,i+1}) = -qE_{i,i+1}, \qquad S(E_{i+1,i}) = -q^{-1}E_{i+1,i}. \quad (36)$$

The Cartan-Chevalley generators of $u_q(m)$ are denoted by $\mathcal{E}_{ss}$, $s = 1, 2, \ldots, m$, $\mathcal{E}_{s,s+1}$, $\mathcal{E}_{s+1,s}$, $s = 1, 2, \ldots, m-1$, and satisfy relations similar to (33)-(36), while commuting with the generators of $u_q(n)$.

In the approach based upon $u_q(n)+u_q(m)$, the q-bosonic creation operators $\boldsymbol{A}^\dagger_{is}$, $i = 1, 2, \ldots, n$, $s = 1, 2, \ldots, m$, belonging to $\mathcal{A}_q(n,m)$, are defined as the components of a double irreducible tensor $T^{[1\dot{0}]_n[1\dot{0}]_m}$ with respect to this q-algebra. This means that they fulfil the relations

$$E_{jj}(\boldsymbol{A}^\dagger_{is}) = \delta_{ji}\boldsymbol{A}^\dagger_{is}, \qquad E_{j,j+1}(\boldsymbol{A}^\dagger_{is}) = \delta_{j,i-1}\boldsymbol{A}^\dagger_{i-1,s}, \qquad E_{j+1,j}(\boldsymbol{A}^\dagger_{is}) = \delta_{ji}\boldsymbol{A}^\dagger_{i+1,s}, \quad (37)$$
$$\mathcal{E}_{tt}(\boldsymbol{A}^\dagger_{is}) = \delta_{ts}\boldsymbol{A}^\dagger_{is}, \qquad \mathcal{E}_{t,t+1}(\boldsymbol{A}^\dagger_{is}) = \delta_{t,s-1}\boldsymbol{A}^\dagger_{i,s-1}, \qquad \mathcal{E}_{t+1,t}(\boldsymbol{A}^\dagger_{is}) = \delta_{ts}\boldsymbol{A}^\dagger_{i,s+1}, \quad (38)$$

where, for any $u_q(n) + u_q(m)$ generator X, $X(\boldsymbol{A}^\dagger_{is})$ denotes the quantum adjoint action $X(\boldsymbol{A}^\dagger_{is}) = \sum_r X^1_r \boldsymbol{A}^\dagger_{is} S(X^2_r)$, with $\Delta(X) = \sum_r X^1_r \otimes X^2_r$. The modified annihilation operators $\tilde{\boldsymbol{A}}_{is}$, $i = 1, 2, \ldots, n$, $s = 1, 2, \ldots, m$, of eq. (27), are similarly defined as the components of a double irreducible tensor $T^{[\dot{0}-1]_n[\dot{0}-1]_m}$ with respect to $u_q(n) + u_q(m)$, and satisfy the relations

$$E_{jj}(\tilde{\boldsymbol{A}}_{is}) = -\delta_{ji'}\tilde{\boldsymbol{A}}_{is}, \qquad E_{j,j+1}(\tilde{\boldsymbol{A}}_{is}) = \delta_{ji'}\tilde{\boldsymbol{A}}_{i-1,s}, \qquad E_{j+1,j}(\tilde{\boldsymbol{A}}_{is}) = \delta_{j,i'-1}\tilde{\boldsymbol{A}}_{i+1,s}, \quad (39)$$
$$\mathcal{E}_{tt}(\tilde{\boldsymbol{A}}_{is}) = -\delta_{ts'}\tilde{\boldsymbol{A}}_{is}, \qquad \mathcal{E}_{t,t+1}(\tilde{\boldsymbol{A}}_{is}) = \delta_{ts'}\tilde{\boldsymbol{A}}_{i,s-1}, \qquad \mathcal{E}_{t+1,t}(\tilde{\boldsymbol{A}}_{is}) = \delta_{t,s'-1}\tilde{\boldsymbol{A}}_{i,s+1}. \quad (40)$$

The operators $\boldsymbol{A}^\dagger_{is}$ and $\tilde{\boldsymbol{A}}_{is}$ can be explicitly written down in terms of m independent copies of the Pusz-Woronowicz operators [14]. By using such expressions and exploiting the tensorial character of the operators, it is straightforward to prove that their q-commutation relations are given in coupled form by

$$\begin{aligned}
[\boldsymbol{A}^\dagger, \boldsymbol{A}^\dagger]^{[2\dot{0}]_n[1^2\dot{0}]_m} &= [\boldsymbol{A}^\dagger, \boldsymbol{A}^\dagger]^{[1^2\dot{0}]_n[2\dot{0}]_m} = [\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{A}}]^{[\dot{0}-2]_n[\dot{0}(-1)^2]_m} = [\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{A}}]^{[\dot{0}(-1)^2]_n[\dot{0}-2]_m} = 0, \\
[\tilde{\boldsymbol{A}}, \boldsymbol{A}^\dagger]^{[1\dot{0}-1]_n[1\dot{0}-1]_m} &= [\tilde{\boldsymbol{A}}, \boldsymbol{A}^\dagger]^{[1\dot{0}-1]_n[\dot{0}]_m}_{q^m} = [\tilde{\boldsymbol{A}}, \boldsymbol{A}^\dagger]^{[\dot{0}]_n[1\dot{0}-1]_m}_{q^n} = 0, \qquad (41) \\
[\tilde{\boldsymbol{A}}, \boldsymbol{A}^\dagger]^{[\dot{0}]_n[\dot{0}]_m}_{q^{n+m}} &= \sqrt{[n]_q[m]_q},
\end{aligned}$$

where, for simplicity's sake, the row labels of the coupled $u_q(n) + u_q(m)$ irreps have been dropped. In (41), the coupled q-commutator of two double irreducible tensors $T^{[\lambda_1]_n[\lambda_2]_m}$ and $U^{[\lambda'_1]_n[\lambda'_2]_m}$ is defined by [14]

$$\begin{aligned}
&\left[T^{[\lambda_1]_n[\lambda_2]_m}, U^{[\lambda'_1]_n[\lambda'_2]_m}\right]^{[\Lambda_1]_n[\Lambda_2]_m}_{(M_1)_n(M_2)_m q^\alpha} \\
&= \left[T^{[\lambda_1]_n[\lambda_2]_m} \times U^{[\lambda'_1]_n[\lambda'_2]_m}\right]^{[\Lambda_1]_n[\Lambda_2]_m}_{(M_1)_n(M_2)_m} - (-1)^\epsilon q^\alpha \left[U^{[\lambda'_1]_n[\lambda'_2]_m} \times T^{[\lambda_1]_n[\lambda_2]_m}\right]^{[\Lambda_1]_n[\Lambda_2]_m}_{(M_1)_n(M_2)_m}. \qquad (42)
\end{aligned}$$

Here

$$\begin{aligned}
\epsilon &= \phi([\lambda_1]_n) + \phi([\lambda'_1]_n) - \phi([\Lambda_1]_n) + \phi([\lambda_2]_m) + \phi([\lambda'_2]_m) - \phi([\Lambda_2]_m), \\
\phi([\lambda_1]_n) &= \tfrac{1}{2}\sum_{i=1}^{n}(n+1-2i)\lambda_{1i}, \qquad \phi([\lambda_2]_m) = \tfrac{1}{2}\sum_{s=1}^{m}(m+1-2s)\lambda_{2s}, \qquad (43)
\end{aligned}$$

and

$$\left[T^{[\lambda_1]_n[\lambda_2]_m} \times U^{[\lambda'_1]_n[\lambda'_2]_m}\right]^{[\Lambda_1]_n[\Lambda_2]_m}_{(M_1)_n(M_2)_m}$$
$$= \sum_{(\mu_1)_n(\mu_2)_m(\mu'_1)_n\mu'_2)_m} \langle[\lambda_1]_n(\mu_1)_n, [\lambda'_1]_n(\mu'_1)_n|[\Lambda_1]_n(M_1)_n\rangle_q \langle[\lambda_2]_m(\mu_2)_m, [\lambda'_2]_m(\mu'_2)_m|[\Lambda_2]_m(M_2)_m\rangle_q$$
$$\times T^{[\lambda_1]_n[\lambda_2]_m}_{(\mu_1)_n(\mu_2)_m} U^{[\lambda'_1]_n[\lambda'_2]_m}_{(\mu'_1)_n(\mu'_2)_m}, \tag{44}$$

where $\langle\ ,\ |\ \rangle_q$ denotes a $u_q(n)$ or $u_q(m)$ Wigner coefficient.

By using the values of the latter, eq. (41) can be written in an explicit form [14]. The resulting relations coincide with eqs. (31) and (32), thereby proving the equivalence of the two constructions of $\mathcal{A}_q(n,m)$ based upon $SU_q(n) \times SU_q(m)$ and $u_q(n) + u_q(m)$, respectively.

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