LOCALIZATION IN DEFORMED QUANTUM MECHANICS

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Abstract

In this talk it is shown that in one version of q-algebras there exists states -a subset of the coherent states- that have negligible dispersion in energy, position and momentum.

1 General remarks

This talk is devoted to the construction of localized states in deformed quantum mechanics. These states will be exhibited explicitly. A localized state is defined as one whose dispersion vanishes or that is at least near zero. To start with I will say that the version of q-algebras [1, 2, 3, 4, 5] that will be used in the sequel is $AA^\dagger - qA^\dagger A = I$ where $q : 0 \to 1$. The operators A and $A^\dagger$ are realized as operators on a space of analytic functions of a complex variable z as follows

$$Af(z) = \frac{f(z) - f(qz)}{(1-q)z} \equiv Df(z) \tag{1}$$

$$A^\dagger f(z) = zf(z) \tag{2}$$

The space of functions -denoted H_q- has an inner product (f,g) defined by [10, 1, 4] see also [6, 9, 7, 8]

$$(f,g) = \int D^2 z f^*(z) m(|\, z\,|^2) g(z) = \pi^{-1} . \int_0^{[\infty]} D(|\, z\,|^2) \int_0^{2\pi} d\phi f^*(z) m(|\, z\,|^2) g(z) \tag{3}$$

where the kernel $m(|\, z\,|^2)$ is fixed by the requirement that A has to be the hermitian conjugate of $A^\dagger$. The explicit form of the kernel $m(|\, z\,|^2)$ is

$$m(|\, z\,|^2) = \frac{1}{exp_q(q\,|\, z\,|^2)} \tag{4}$$

where the deformed exponential $exp_q(z)$ is defined as the solution to the equation $Df(z) = f(z)$ and has the explicit expression

$$exp_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]!} \tag{5}$$

The box symbol $[n]$ is defined as $[n] = \frac{1-q^n}{1-q}$; the special value with $n = \infty$ is given by $[\infty] = \frac{1}{1-q}$ and the deformed factorial is defined as $[n]! = [1] \cdots [n]$ and $[1]! = 1$. At this point it is convenient to remark that the form of the box symbol is intimately related to the particular version of the q-algebra that is being used while the realization 1 and 2 is not. See [11].

Under the inner product 3 the set of functions $u_n(z) = \frac{z^n}{[n]!^{\frac{1}{2}}}, n = 0, \ldots, \infty$ are orthonormal. It is clear that $u_n(z)$ is an eigenfunction of the operator $z\frac{d}{dz}$ with eigenvalue n; in fact $z\frac{d}{dz}$ is the number operator. To each function $u_n(z)$ there corresponds a ket $\mid n >$ which is an eigenket of the number operator. As a further remark, it is clear from the definition of the deformed exponential function that its series expansion converges in a finite region of the complex z-plane; this region is defined by $\mid z \mid < [\infty]$.

2 Coherent states

Coherent states are defined as eigenfunctions of the anihilation operator. Each function will be labelled with a complex number β so that the function $f_\beta(z)$ represents the coherent state $\mid \beta >$ in the ket notation. The explicit expression for the coherent state $f_\beta(z)$ is

$$f_\beta(z) = C(\beta) exp_q(\beta z) = C(\beta) \sum_{n=0}^{\infty} \frac{\beta^n u_n(z)}{[n]!^{\frac{1}{2}}} \tag{6}$$

where the normalization constant $C(\beta)$ is given by

$$C(\beta)^2 = \frac{1}{exp_q(\mid \beta \mid^2)} \tag{7}$$

The set of all coherent states is overcomplete as seen by the fact that the functions $f_\beta(z)$ are not orthogonal to one another and that a resolution of the identity can be constructed with them. This construction requires that the identity be resolved both in terms of the orthonormal set of functions $u_n(z)$ defined above and in terms of coherent states. This leads to the equation

$$I = \sum_0^\infty u_n^*(z) u_n(z) = \int D^2\beta M(\mid \beta \mid^2) f^*(z) f(z) \tag{8}$$

where the kernel $M(\mid \beta \mid^2)$ is obtained by requiring that the above equation be satisfied. Its explicit expression is [5]

$$M(\mid \beta \mid^2) = \frac{exp_q(\mid \beta \mid^2)}{exp_q(q \mid \beta \mid^2)} \tag{9}$$

At this point it is convenient to reconsider the convergence question. The coherent states are constructed so as to be normalized. This implies that the region in the β -complex plane allowed for the label of the coherent states is $\mid \beta \mid^2 < [\infty]$. The same upper bound is found for $\mid z \mid^2$ to have a convergent power series.As a result it is found that the functions that belong to the Hilbert space H_q are analytic in a finite region of the complex z-plane. This region extends to the whole complex z-plane when q goes to 1 and reduces to the unit circle when q goes to zero.

3 Localization

To study localization two hermitian operators Q and P will be introduced in such a way that their relation to the creation and anihilation operators A and $A^\dagger$ resembles closely the relation valid for q=1. Then Q and P are written in the form [12]

$$Q = sA + s^* A^\dagger, P = rA + r^* A^\dagger \tag{10}$$

From the commutation relations for A and $A^\dagger$ it is found that Q and P satisfy

$$[Q, P] = (rs^* - r^*s)[1 + (q-1)A^\dagger A] \tag{11}$$

which reduces to the usual commutation relation for the position and momentum operators when q=1 after a particular selection of the constants r and s that appear in equation (10). This justifies calling Q the deformed position operator and P the deformed momentum operator.

From the commutation relation equation(11) for Q and P it follows the uncertainty relation

$$(\Delta Q)_f^2 (\Delta P)_f^2 \geq \frac{|< [Q, P] >_f|^2}{4} \tag{12}$$

In equation (12) $(\Delta Q)_f^2 =< Q^2 >_f - < Q >_f^2$ and $< Q >_f = (f, Qf)$; f is any function in H_q. Now the expectation values and dispersions will be computed using the coherent state basis (that means that $f(z)$ is taken as $f_\beta(z)$). The results are

$$(\Delta Q)_\beta^2 =| s |^2 [1 + (q-1) | \beta |^2] \tag{13}$$

$$(\Delta P)_\beta^2 =| r |^2 [1 + (q-1) | \beta |^2] \tag{14}$$

$$< PQ >_\beta - < QP >_\beta = (rs^* - r^*s)[1 + (q-1) | \beta |^2] \tag{15}$$

From these results it follows that, unlike the non-deformed (q=1) case, the uncertainties depend on the label β of the particular coherent state used to compute them. Notice that if q=1 then all uncertainties are constant. The fact that the uncertainties depend on β is the crucial result to exhibit localization; in fact, if $| \beta |^2$ has a value near $\frac{1}{1-q}$ which is an annulus near the boundary of the convergence region then all uncertainties in equations(13), ('14) and (15) are negligible. So those coherent states whose labels are near the boundary show localization according to the definition stated above. Moreover, near the boundary the operators Q and P are commuting, at least in the weak sense that $< PQ >_\beta - < QP >_\beta$ tends to 0. Those coherent states that are localized behave as classical states in a much closer way than the usual (q=1) coherent states which exhibit minimum non-vanishing uncertainty. The deformed coherent states are in this sense a better answer to the original Schroedinger question of finding those states that resemble classical states than the ordinary coherent states.

Now I will show that the deformed coherent states are minimum uncertainty states and that they can be generated by the action of a shifting operator acting on the vacuum. To start with, the right-hand side of 12 (T denotes the right-hand side of 11) is found to be

$$|< T >_\beta|^2 =| r |^2 | s |^2 [2 - exp2i(\phi_r - \phi_s) - exp2i(\phi_s - \phi_r)][1 + (q-1) | \beta |^2] \tag{16}$$

which tends to zero when $|\beta|^2$ tends to $\frac{1}{1-q}$; in the above equation ϕ_r and ϕ_s denote the phases of the complex numbers r and s, respectively. If $exp2i(\phi_r - \phi_s) = -1$ then the equality sign is valid in equation (12) so that for any fixed value of β the corresponding coherent state has minimum uncertainty; on the other hand, if the boundary of the convergence region is approached both sides of 12 tend to zero.

4 Shifting operator

Next, turn to the shifting operator. Notice that the function $f_\beta(z)$ representing the coherent state labelled by β can be written

$$f_\beta(z) = C(\beta) exp_q(\beta A^\dagger) f_0(z) \tag{17}$$

where $f_0(z) = 1$ represents the vacuum state. Then

$$f_\beta = C(\beta) D_q(A, A^\dagger; \beta) f_0 \tag{18}$$

where

$$D_q(A, A^\dagger; \beta) = exp_q(\beta A^\dagger) \frac{1}{exp_q(\beta^* A)} \tag{19}$$

$D_q(A, A^\dagger; \beta)$ is the shifting operator. It follows that $D_q^{-1} \neq D_q^\dagger$ so that D_q is non-unitary. The action of $D_q(A, A^\dagger; \beta)$ on $f_0(z)$ generates unnormalized coherent states; that is why the factor $C(\beta)$ appears in an explicit way. Finally, it is easy to verify that $D_q(A, A^\dagger; \beta)$ satisfies

$$D_q(A, A^\dagger; \beta) A D_q^{-1}(A, A^\dagger; \beta) = \frac{exp_q(\beta A^\dagger)}{exp_q(q\beta A^\dagger)} [A - \beta I] \tag{20}$$

which gives the usual result when $q = 1$. For labels α , β and γ

$$D_q(A, A^\dagger; \alpha) D_q(A, A^\dagger; \beta) \neq D_q(A, A^\dagger; \gamma) \tag{21}$$

5 Hamiltonian

The last point concerns the hamiltonian of the system. This is constructed from the commutation relation for Q and P and has the form

$$h_q = AA^\dagger + A^\dagger A = \frac{Q^2}{2|s|^2} + \frac{P^2}{s|r|^2} \tag{22}$$

whose uncertainty is

$$(\Delta h_q)^2_\beta = |\beta|^2 [1 + (q-1)|\beta|^2] \tag{23}$$

which tends to zero when $|\beta|^2$ approaches $\frac{1}{1-q}$. This is another indication that the system described by the coherent states near the boundary of the convergence region resembles a classical system.

To summarize: when $|\beta|^2$ is near $\frac{1}{1-q}$ then $(\Delta Q)^2_\beta$, $(\Delta P)^2_\beta)$, $(\Delta h_q)^2_\beta$ and $< PQ >_\beta - < QP >_\beta$ all tend to zero. This corresponds closely to the behavior of a classical system.

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