GEOMETRIC PHASES, EVOLUTION LOOPS AND GENERALIZED OSCILLATOR POTENTIALS

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Abstract

The geometric phases for dynamical processes where the evolution operator becomes the identity (evolution loops) are studied. The case of time-independent Hamiltonians with equally spaced energy levels is considered; special emphasis is made on the potentials having the same spectrum as the harmonic oscillator potential (the generalized oscillator potentials) and their recently found coherent states.

1 Introduction

Departing from Berry's work [1], a *geometric phase* β has been associated to the cyclic evolution of a vector state $|\psi(t)\rangle$, i.e., $|\psi(\tau)\rangle = e^{i\phi}|\psi(0)\rangle$, where τ is the period, $\langle\psi(t)|\psi(t)\rangle = 1$, and $\phi \in \mathbf{R}$. For a non-relativistic system with Hamiltonian $H(t)$, β takes the form [2]:

$$\beta = \phi + i\int_0^\tau \langle\psi(t)|\frac{d}{dt}|\psi(t)\rangle dt = \phi + \hbar^{-1}\int_0^\tau \langle\psi(t)|H(t)|\psi(t)\rangle dt. \tag{1}$$

The geometric phase describes some curvature effects arising on the projective space $\mathcal{P}$ associated to the system's Hilbert space $\mathcal{H}$: β turns out to be the holonomy of the horizontal lifting of the closed trajectory $|\psi(t)\rangle\langle\psi(t)| \in \mathcal{P}$ to $\mathcal{H}$.

Eq.(1) is valid for *any* cyclic evolution, regardless of whether or not it is induced by a time-dependent Hamiltonian. There is a widespread believing, however, that β becomes non-null just when the Hamiltonian inducing the cyclic evolution is time-dependent. This could be understood if one realizes the great influence of Berry's article; so one could think of Eq.(1) as applied to the cyclic evolutions of the eigenstates of a cyclic $H(t)$ changing adiabatically in time [1]. Making use of this idea, $\beta = 0$ for the eigenstates of a time-independent Hamiltonian H. In this paper we are going to show that for any H having at least two bounded states there are a lot of cyclic evolutions for which $\beta \neq 0$.

On the other hand, some developments in the analysis of the dynamics of a quantum system led to the concept of *evolution loop* (EL) [3, 4]. An evolution loop is a specific dynamical process, induced by time-dependent [3, 4] or time-independent Hamiltonians [5], whose evolution operator becomes the identity $\mathbf{1}$ (modulo phase) for a certain time $\tau > 0$ (the loop period):

$$U(\tau) = e^{i\phi}\mathbf{1}, \tag{2}$$

where $U(0) = 1$. The EL is interesting because, if perturbed by some additional external fields, it can induce any unitary transformation of $\mathcal{H}$ as the result of the small precessions of the distorted loop [6]. There is, moreover, an obvious interelation between the evolution loops and the geometric phases.

2 Geometric phases and evolution loops

In this work, we restrict the discussion to systems with a time–independent Hamiltonian whose evolution operator performs an evolution loop. The main property of these systems is that *any* state evolves cyclically from $t = 0$ until $t = \tau$:

$$|\psi(\tau)\rangle = e^{i\phi}|\psi(0)\rangle. \tag{3}$$

According to (1), $|\psi(t)\rangle$ will have associated, in general, a non-null geometric phase. Indeed, because $U(t) = e^{-iHt/\hbar}$ commutes with H we have:

$$\beta = \phi + \hbar^{-1}\int_0^\tau \langle\psi(0)|U^\dagger(t)HU(t)|\psi(0)\rangle dt = \phi + \hbar^{-1}\tau\langle H\rangle, \tag{4}$$

where $\langle H\rangle = \langle\psi(0)|H|\psi(0)\rangle$. In terms of the basis $\{|E_m\rangle\}$ of eigenstates of H, $|\psi(0)\rangle = \sum_m c_m|E_m\rangle$ with $c_m = \langle E_m|\psi(0)\rangle$, and Eq.(4) becomes:

$$\beta = \phi + \hbar^{-1}\tau\sum_m |c_m|^2 E_m. \tag{5}$$

There are some interesting systems whose time–independent Hamiltonian induces evolution loops (see, e.g., [7, 5, 8, 9]). We will illustrate this assertion with the simplest generic case. Suppose that H has an equally spaced spectrum of the form:

$$E_n = E_0 + n\Delta E, \tag{6}$$

where ΔE is the level's spacing, E_0 is the ground state energy and $n = 0, 1, \cdots, N$, being N either finite or infinite. The evolution operator reads:

$$U(t) = \sum_{n=0}^{N} e^{-iE_n t/\hbar}|E_n\rangle\langle E_n|. \tag{7}$$

As can be seen, an evolution loop is present at $\tau = 2\pi\hbar/\Delta E$:

$$U(\tau) = \sum_{n=0}^{N} e^{-i2\pi(E_0+n\Delta E)/\Delta E}|E_n\rangle\langle E_n| = e^{-i2\pi E_0/\Delta E}\mathbf{1}. \tag{8}$$

By comparing with (2), $\phi = -2\pi E_0/\Delta E$, and according with (4–5) the geometric phase for the cyclic state $|\psi(t)\rangle$ is:

$$\beta = 2\pi\frac{(\langle H\rangle - E_0)}{\Delta E} = 2\pi\sum_{n=1}^{N} n|c_n|^2 \geq 0. \tag{9}$$

By restricting β (modulo 2π) to the interval $[0, 2\pi)$ one can interpret (9) in the following way: β measures the energy excess in dimensionless units of $\langle H \rangle$ with respect to its nearest energy level E_k (see Fig.1).

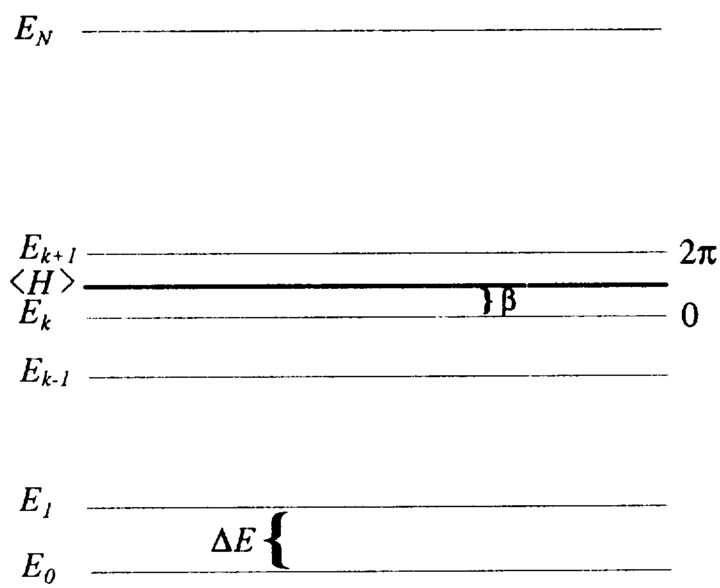

FIG. 1. Schematic representation of the $N + 1$ energy levels and the geometric phase for a system with equally spaced spectrum.

Suppose now that, due to some physical reasons, we are faced with a situation involving just two energy levels of H. Restricting considerations to the subspace $\mathcal{E}_2$ generated by the two eigenstates $|E_0\rangle$ and $|E_1\rangle$ it can be shown that the evolution operator performs an evolution loop. Formulae (6–9) are valid in this situation with $N = 1$ and $\tau = 2\pi\hbar/\Delta E$. In particular, (9) becomes $\beta = 2\pi|c_1|^2$, where c_1 is the component along $|E_1\rangle$. As there are an infinite number of linear combinations $c_0|E_0\rangle + c_1|E_1\rangle$ such that $|c_0|^2 + |c_1|^2 = 1$, $c_0 \neq 0$ and $c_1 \neq 1$, we have shown the following: for any H having at least two bounded states there are an infinity of cyclic evolutions for which $\beta \neq 0$ (see also [10]).

Other examples for which formulae (6–9) can be applied are the following: a spin-j system interacting with a constant homogeneous magnetic field $\mathbf{B}$; the harmonic oscillator potential and all the Hamiltonians having the same spectrum as the harmonic oscillator (generalized oscillators). Next, we will derive the geometric phases for a family of generalized oscillator Hamiltonians.

3 The generalized oscillator potentials

The simplest method to derive a family of generalized oscillator potentials was introduced by Mielnik by means of a modification of the well known factorization method [11]. Consider the *classical* factorization of the harmonic oscillator Hamiltonian in dimensionless coordinates $m = \omega = \hbar = 1$:

$$H = \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2\right), \quad aa^\dagger = H + \frac{1}{2}, \quad a^\dagger a = H - \frac{1}{2}, \tag{10}$$

where $a = (d/dx + x)/\sqrt{2}$, and $a^\dagger = (-d/dx + x)/\sqrt{2}$ are the ordinary ladder operators with $[a, a^\dagger] = \mathbf{1}$. The eigenfunctions and eigenvalues of the harmonic oscillator can be constructed using the relations

$$Ha^\dagger = a^\dagger(H+1), \quad Ha = a(H-1). \tag{11}$$

The ground state $\psi_0(x)$ has eigenvalue $E_0 = 1/2$ and satisfies $a\psi_0(x) = 0 \Rightarrow \psi_0(x) \propto e^{-x^2/2}$, while the $\psi_n(x)$'s associated to $E_n = n + 1/2$ are:

$$\psi_n(x) = \frac{(a^\dagger)^n}{\sqrt{n!}}\psi_0(x). \tag{12}$$

The *generalized* factorization method [11] consists in looking for more general operators

$$b = \frac{1}{\sqrt{2}}\left(\frac{d}{dx} + \beta(x)\right), \quad b^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{d}{dx} + \beta(x)\right), \tag{13}$$

satisfying just one of relations (10):

$$bb^\dagger = H + \frac{1}{2}. \tag{14}$$

Hence, the unknown function $\beta(x)$ obeys the Riccati equation

$$\beta' + \beta^2 = 1 + x^2, \tag{15}$$

whose general solution is

$$\beta(x) = x + \frac{e^{-x^2}}{\lambda + \int_0^x e^{-y^2}dy}, \quad \lambda \in \mathbf{R}. \tag{16}$$

Now, the point is that $b^\dagger b$ is not related with the harmonic oscillator Hamiltonian, but it leads to a new operator H_λ:

$$b^\dagger b = H_\lambda - \frac{1}{2}, \tag{17}$$

where

$$H_\lambda = -\frac{1}{2}\frac{d^2}{dx^2} + V_\lambda(x), \tag{18}$$

with

$$V_\lambda(x) = \frac{x^2}{2} - \frac{d}{dx}\left(\frac{e^{-x^2}}{\lambda + \int_0^x e^{-y^2}dy}\right) = \left(x + \frac{e^{-x^2}}{\lambda + \int_0^x e^{-y^2}dy}\right)^2 - \frac{x^2}{2}, \quad |\lambda| > \sqrt{\pi}/2. \tag{19}$$

The relationships analogous to (11) provide now the way to obtaining the eigenfunctions and eigenvalues of H_λ:

$$H_\lambda b^\dagger = b^\dagger(H+1), \quad Hb = b(H_\lambda - 1). \tag{20}$$

Hence, the states $\theta_n(x) = b^\dagger\psi_{n-1}(x)/\sqrt{n}$, $n = 1, 2, \cdots$ form an orthonormal set of eigenfunctions of H_λ with eigenvalues $E_n = n + 1/2$. However, $\{\theta_n(x),\ n = 1, 2, \cdots\}$ is not a basis of $L^2(\mathbf{R})$. There is a missing vector $\theta_0(x)$, orthogonal to $\theta_n(x), n = 1, 2, \cdots$. It turns out to be an eigenfunction of H_λ with eigenvalue $E_0 = 1/2$ satisfying $b\theta_0(x) = 0$, and taking the form:

$$\theta_0(x) \propto \exp\left(-\int_0^x \beta(y)dy\right). \tag{21}$$

The set $\{\theta_n(x),\ n = 0, 1, 2, \cdots\}$ forms an orthonormal basis in $L^2(\mathbf{R})$; then $\{H_\lambda : |\lambda| > \sqrt{\pi}/2\}$ is a family of Hamiltonians distinct of the harmonic oscillator one but having exactly the same spectrum as the oscillator. In the limit $|\lambda| \to \infty$, the harmonic oscillator potential is recovered, $V_\lambda(x) \to x^2/2$.

We return now to our original subject. Due to the kind of spectrum of H_λ, relations (6–9) involving the evolution loops and the geometric phases can be applied here with $E_0 = 1/2$, $\Delta E = 1$, $\tau = 2\pi$, $\phi = -\pi$ and $N = \infty$. In particular, $\beta = 2\pi(\langle H_\lambda\rangle - 1/2)$, and when applied to the cyclic states $\{\theta_n(x), n = 0, 1, 2\cdots\}$ we recover again $\beta = 2n\pi$. Is there any other set of generic states for which we can evaluate explicitly the geometric phase?

The answer turns out positive if we consider the recently found coherent states of H_λ (the generalized coherent states GCS) [12]. Let's denote them as $|z\rangle$ with $z \in \mathbf{C}$. The annihilation and creation operators of the system can be identified as:

$$A = b^\dagger a b, \quad A^\dagger = b^\dagger a^\dagger b. \tag{22}$$

Define now $|z\rangle$ by $A|z\rangle = z|z\rangle$. A direct calculation leads to:

$$|z\rangle = \frac{1}{\sqrt{{}_0F_2(1, 2; |z|^2)}} \sum_{n=0}^{\infty} \frac{z^n}{n!\sqrt{(n+1)!}} |\theta_{n+1}\rangle, \tag{23}$$

where $|\theta_n\rangle$ represents to $\theta_n(x)$ and ${}_0F_2(1, 2; y)$ is a generalized hypergeometric function [13]. Each $z \neq 0$ is a non–degenerate eigenvalue. However, $z = 0$ is a double degenerate eigenvalue of A with eigenvectors $|\theta_0\rangle$ and $|z = 0\rangle = |\theta_1\rangle$. It is possible to find a measure in the complex plane such that $\{|\theta_0\rangle, |z\rangle\}$ is complete in $\mathcal{H}$.

To evaluate the geometric phase β_{GCS}, $\langle z|H_\lambda|z\rangle$ is needed. A direct calculation leads to:

$$\langle H_\lambda\rangle = \langle z|H_\lambda|z\rangle = 1/2 + \frac{{}_0F_2(1, 1; |z|^2)}{{}_0F_2(1, 2; |z|^2)}. \tag{24}$$

Finally:

$$\beta_{GCS} = 2\pi \frac{{}_0F_2(1, 1; |z|^2)}{{}_0F_2(1, 2; |z|^2)}. \tag{25}$$

The behaviour of β_{GCS}, is shown in Fig.2. Notice that β_{GCS} is independent of λ. Moreover, its behaviour is quite different compared with the standard coherent state (SCS) of the harmonic oscillator for which $\beta_{SCS} = 2\pi|z|^2$ (see Fig.2). The difference rests on the fact that the GCS do not tend to the SCS when $\lambda \to \infty$ and $A_\infty \equiv \lim_{\gamma\to\infty} A = a^\dagger a^2 \neq a$ even though $V_\lambda(x) \to x^2/2$ in this limit.

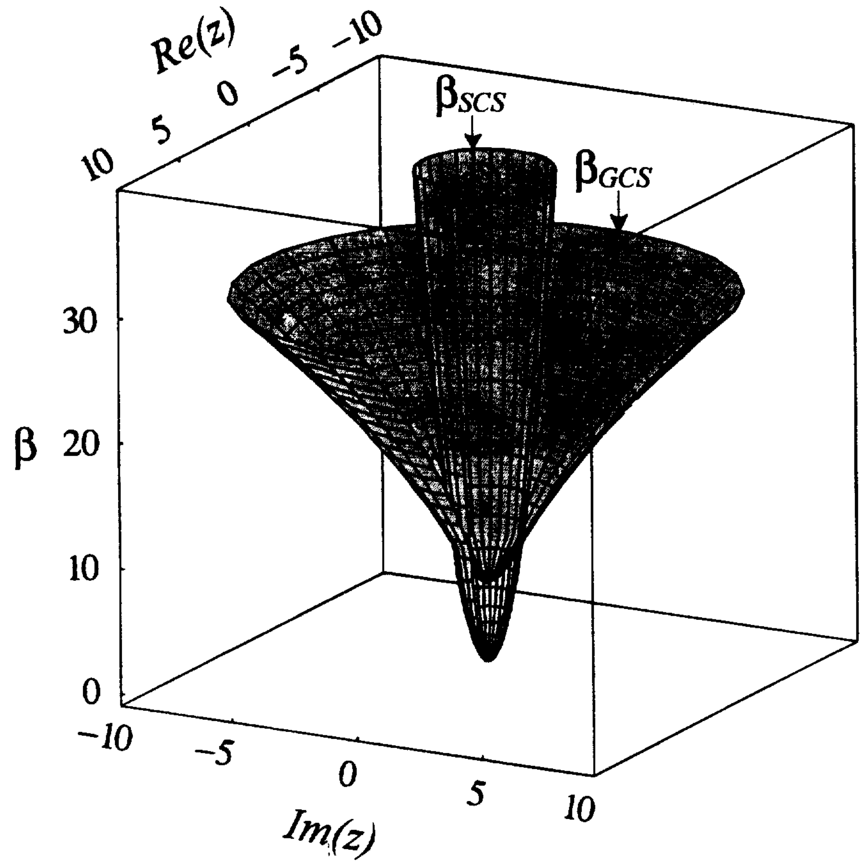

FIG. 2. The geometric phases versus z for the standard coherent states of the harmonic oscillator (β_{SCS}) and the coherent states of the generalized oscillator (β_{GCS}).

Acknowledgments

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