1995116567

N95-22984

q-BOSONS
and the
q-ANALOGUE QUANTIZED FIELD

Charles A. Nelson[1]
Department of Physics
State University of New York at Binghamton
Binghamton, N.Y. 13902-6000

Abstract

The q-analogue coherent states $|z>_q$ are used to identify physical signatures for the presence of a q-analogue quantized radiation field in the $|z>_q$ classical limit where $|z|$ is large. In this quantum-optics-like limit, the fractional uncertainties of most physical quantities (momentum, position, amplitude, phase) which characterize the quantum field are $O(1)$. They only vanish as $O(1/|z|)$ when $q = 1$. However,for the number operator, N, and the N-Hamiltonian for a free q-boson gas, $H_N = \hbar\omega(N + 1/2)$, the fractional uncertainties do still approach zero . A signature for q-boson counting statistics is that $(\Delta N)^2/ < N > \to 0$ as $|z| \to \infty$. Except for its $O(1)$ fractional uncertainty, the q-generalization of the Hermitian phase operator of Pegg and Barnett, $\hat{\phi}_q$, still exhibits normal classical behavior. The standard number-phase uncertainty-relation, $\Delta N \, \Delta\hat{\phi}_q = 1/2$, and the approximate commutation relation, $[N, \hat{\phi}_q] = i$, still hold for the single-mode q-analogue quantized field. So, N and $\hat{\phi}_q$ are almost canonically conjugate operators in the $|z>_q$ classical limit. The $|z>_q$ CS's minimize this uncertainty relation for moderate $|z|^2$.

1 Motivation and Introduction

In considering the potential importance of quantum algebras to quantum field theory and to physics[1], I am reminded of the twenty year development of Yang-Mills theory and the strong interactions (now called QCD or quantum chromodynamics):

- 1954: YM theory was proposed to generalize U(1) QED to an $SU(2)_{Isospin}$ theory for the strong interactions with the ρ meson as the analogue of the photon.
- 1966: Nambu suggested that YM theory may be relevant to the color degree of freedom of constituent quarks.

[1]CNELSON@BINGVMB.BITNET cnelson@bingvmb.cc.binghamton.edu

- 1968: Experiments at SLAC discovered scaling of the strong interactions at short-distances.
- 1972-3: Asymptotic freedom was discovered for $SU(3)_{Color}$ YM theory (i.e. the weak coupling of the strong interactions at short distances).

In 1954, both the ultra-violet and infra-red(if the ρ were taken massless) properties of YM theory were regarded as *complicated*. But inspite of the theory's mathematical beauty, it took 20 years for theorists to discover its important physical property of asymptotic freedom; and, in fact, this occurred only after the hint provided by a Nobel prize winning experiment!

For comparison, the recent history of quantum algebras is

- 1979-87: q-algebra symmetries investigated in quantum and statistical mechanical models [1].
- 1989: q-oscillators introduced to realize the new symmetries of q-algebras [2].
- ????: ???

If this historical parallel is of significance, we need to know the physical implications of these novel symmetry structures. If there are q-oscillators in nature which realize these new algebras, it seems reasonable to expect that there will also exist a q-analogue quantum field which has such q-oscillators as its normal modes[4]. We need to know its canonical physical properties—what are its number and phase signatures? Since the usual quasi-classical coherent states (CS) approximately characterize many types of cooperative behavior in the q=1 case, it is natural to use the q-CS's to investigate and identify empirical signatures[4,6] of a generic q-field for cooperative phenomena, whether in quantum optics, many body physics, particle physics ....

The q-analogue coherent states $|z>_q$ satisfy $a|z>_q = z|z>_q$ where the q-oscillator algebra is ($q \to 1$, usual bosons)

$$aa^\dagger - q^{\pm 1/2}a^\dagger a = q^{\mp N/2} \tag{1}$$

with $[N, a^\dagger] = a^\dagger$, $[N, a] = -a$, and the physically important bosonic $[a, a] = 0$. We take q real, and $0 < q < 1$.

In the $|n>_q$ basis, $<m|n> = \delta_{mn}$ and[2]

$$a^\dagger|n> = \sqrt{[n+1]}|n+1> \qquad a|n> = \sqrt{[n]}|n-1> \qquad a|0> = 0 \tag{2}$$

where $[x]_q = [x] \equiv (q^{x/2} - q^{-x/2})/(q^{1/2} - q^{-1/2})$ is the "q-deformation" of x. More simply $[x] = \sinh(sx/2)/\sinh(s/2)$ where $q = \exp s$. Note that

$$a^\dagger a|n> = [N]|n> = [n]|n> \qquad N|n> = n|n> \qquad a|0> = 0 \tag{3}$$

It follows that with $< z|z> = 1$ the q-CS's are

$$|z>_q = N(z)\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{[n]!}}|n>, \qquad N(z) = e_q(|z|^2)^{-1/2} \tag{4}$$

[2]From now on the sub-q's are usually implicit!

in terms of the "q-exponential function"

$$e_q(z) \equiv \sum_{n=0}^{\infty} \frac{z^n}{[n]!}, \qquad [n]! \equiv [n][n-1]\cdots[1], \qquad [0]! = 1 \tag{5}$$

which is an order zero entire function [5], and $|e_q(z)| \leq e_q(|z|) \leq exp(|z|)$. For $x > 0$, it's positive, but for $x < 0$ and $q < (q_1{}^* \sim 0.14)$ there are an infinite number of increasing amplitude oscillations of decreasing frequency as $x \to (-\infty)$. The infinite number of real zeros are approximately at $\tilde{\mu}_n = -q^{(1-n)/2}/(1-q)$; $n = 1, 2, \ldots$. As q increases, these zeros collide in pairs and move off the real axis as a complex conjugate pair. In this manner, $e_q(z) \to exp(z)$ as $q \to 1$.

In analyzing the q-boson field in the $|z>_q$ classical limit, we use the Heisenberg representation, consider a specific mode, and suppress the $\overrightarrow{k}$ mode and $\hat{\epsilon}$ polarization indices for the generic electric and magnetic fields, etc. . Notice that the q-analogue coherent states $|z>_q$ are good candidates for studying the classical limit of the q-analogue quantized radiation field because they are minimum uncertainty states. They minimize the fundamental commutation relation

$$U_{Q,P} \equiv \frac{2\Delta Q \Delta P - |<[Q,P]>|}{|<[Q,P]>|} \geq 0 \tag{6}$$

with $U|_{|z>} = 0$, but $U|_{|n> \neq |0>} = \frac{(3[n]+[n+1])}{([n+1]-[n])}$. Also, the n^{th} order correlation function factorizes, i.e.

$$Tr(\rho E^-(x)E^+(y)) = \mathcal{E}^-(x)\mathcal{E}^+(y), \ldots \tag{7}$$

In addition, there exists a resolution of unity[3-5] for the q-CS's

$$\int |z><z|\, d\mu(z) + \int |\tilde{z}><\tilde{z}|\, d\tilde{\mu} = I \tag{8}$$

with, respectively, a continuous (q-integration) measure

$$d\mu(z) = \frac{1}{2\pi} e_q(|z|^2) e_q(-|z|^2) d_q|z|^2\, d\theta \tag{9}$$

and a discrete measure

$$d\tilde{\mu}_k = \frac{1}{2\pi} e_q(q^{1/2}|\tilde{z}_k|^2) e_q(-|\tilde{z}_k|^2) d\theta. \tag{10}$$

Note that $|\tilde{z}_k|^2 = q^{k/2}\zeta_i$ with $k = 0, 1, \ldots$ and ζ_i = minus the i^{th} zero of $e_q(z)$. The q-discrete auxiliary states,[4], $|\tilde{z}_k>_q$ satisfy

$$\tilde{a}_k|\tilde{z}_k>_q = (q^{1/4}\tilde{z}_k)|\tilde{z}_k>_q \tag{11}$$

The $\tilde{a}_k$ obey the q-commutation relations, (1).

Consequently the q-CS's are non-orthogogonal and overcomplete. There are q-analogue generalizations[4,6] of the P-, Q-, and W-phase space representations of quantum optics. However, as we next discuss, there also are important differences in the $|z>_q$ basis for other coherence and uncertainty properties of the q-analogue quantized field[3] .

[3]For more details see [6].

2 Fractional Uncertainties in the $|z>_q$ Classical Limit

With the usual definitions $\hat{P} = -i(\hbar\omega/2)^{1/2}(a - a^\dagger)$, $\hat{Q} = (\hbar/2\omega)^{1/2}(a + a^\dagger)$, the fractional uncertainties $\frac{\Delta\hat{Q}}{|<\hat{Q}>|}$ and $\frac{\Delta\hat{P}}{|<\hat{P}>|}$ are of $O(1)$ for $|z| \to \infty$ and

$$< z|[Q,P]|z >= i\hbar < z|[a,a^\dagger]|z >= i\hbar < z|\hat{\Lambda}|z >= i\hbar\lambda(z) \geq i\hbar \tag{12}$$

This defines the resolution operator $\hat{\Lambda} \equiv [a,a^\dagger]$. The q-boson "resolution function" ($q = \exp s$, and $N(z)$ is the CS norm.)

$$\lambda(z) \equiv N(z)^2 \sum_{n=0}^{\infty} \frac{|z|^{2n}\cosh(s(2n+1)/4)}{[n]!\cosh(s/4)} \tag{13}$$

goes as $(q^{-1/2} - 1)|z|^2 + 1$ as $|z| \to \infty$. This follows because

$$\lambda(z) \equiv < z|[N+1]|z > - < z|[N]|z >= ((q^{-1/2} - 1)|z|^2 + (e_q(q^{1/2}|z|^2)/e_q(|z|^2)) \tag{14}$$

Note that $\lambda(z)$ is bounded from above and below.

For the generic q-electromagnetic field, the fractional uncertainties in amp $\hat{E}$, in amp $\hat{B}$, and in the "Hermitian" Pegg-Barnett phase operator, $\hat{\phi}_q$, are also of $O(1)$ [7,4,6].

Note[4] that the quadratic $\hat{P}$,$\hat{Q}$ single-mode hamiltonian, which has an $O(1)$ fractional uncertainty,

$$H_{\hat{P},\hat{Q}} = (1/2)\hbar\omega(a^\dagger a + aa^\dagger) = (1/2)(\hat{P}^2 + \omega^2\hat{Q}^2). \tag{15}$$

is proportional to the anti-commutator. Hence for $q \neq 1$, $H_{\hat{P},\hat{Q}}$ is not mathematically independent of the basic commutator $\hat{\Lambda} \equiv [a,a^\dagger]$ because of the fundamental operator identity

$$(-(i/\hbar)[Q,P]\cosh(s/4))^2 - ((2/\hbar\omega)H_{\hat{P},\hat{Q}}\sinh(s/4))^2 = 1. \tag{16}$$

In striking contrast to these $O(1)$ fractional uncertainties, both the usual N operator and the elementary N-Hamiltonian operator

$$H_N = \hbar\omega(N + 1/2) \tag{17}$$

possess zero fractional uncertainties as $|z| \to \infty$. Also, H_N does indeed possess the conventional field-theoretic properties of the classic $q = 1$ Hamiltonian operator.

[4]For $H_{\hat{P},\hat{Q}}$, the energy is not additive for two widely separated systems, violating the usual cluster decomposition "axiom" in quantum field theory. For q-quanta this is not so surprising since the fractional uncertainty in the energy based on $H_{\hat{P},\hat{Q}}$ is $O(1)$ in the $|z>$ basis and the quanta by (1) are compelled to be always interacting,i.e. by exclusion-principle-like q-forces! So it is doubtful that $H_{\hat{P},\hat{Q}}$ permits the usual physical interpretation based on a smooth limit to a conventional, free quantized field.

3 q-Boson Counting Statistics

The physically important $[a, a] = 0$ implies that the usual Bose-Einstein energy distribution still follows for a free q-boson gas . Note that (9) above does imply a non-degenerate equally-spaced spectrum. On the other hand, the q-CS's do not give a Poisson number distribution for $q \neq 1$ since [2,8]

$$P_n{}^q(z) = |<n|z>|^2 = \frac{|z|^{2n}}{[n]!e_q(|z|^2)}. \tag{18}$$

Note that for $q \neq 1$, $|z|^2$ is the eigenvalue of the deformed number operator, [N], in the $|z>_q$ basis. The mean value of usual number operator N goes as $<N>= 2\alpha_q log|z|+\beta_q$ for $1 < |z|^2 < few100$, where $\alpha_q$andβ_q are q-dependent constants. For fixed $|z|^2$, as q decreases the peak of $P_n{}^q(z)$ narrows and shifts to smaller n. Therefore, the behavior of the fractional uncertainty $(\Delta N)/<N>$ is not very q-dependent.

However, since $\Delta N \rightarrow \eta_q$ as $|z| \rightarrow \infty$, where η_q is a q-dependent constant for $q \neq 1$, there is the very important signature for q-boson counting statistics that

$$(\Delta N)^2/<N> \rightarrow 0 \tag{19}$$

as $|z| \rightarrow \infty$. This is in contrast to a thermal source where the "rhs" of (19) equals $<N+1>$ for all $|z|$, and for laser light (and q=1 CS's) where the "rhs" equals "one" as $|z| \rightarrow \infty$. So in principle it is possible by q-boson counting experiments to very simply identify a q-boson gas in this limit in spite of the ordinary Bose-Einstein frequency distribution.

4 The q-Analogue of the Pegg-Barnett Phase Operator,$\hat{\phi}_q$

Recall $z = |z| \exp(i\theta)$. While mathematically a hermitian phase operator conjugate to N, or to $[N] \equiv a^\dagger a$ does not exist [9], q-generalizations of the phase operators of Susskind-Glogower [9,10] and of Pegg-Barnett [7] have been constructed [4,6]. The q-generalization of the Pegg and Barnett operator[5] is obtained by introducing a complete, orthonormal basis of $(s+1)$ phase states $|\theta_m>_q= (s+1)^{-1/2}\sum_{m=0}^{s} \exp(in\theta_m)|n>_q$, $\theta_m = \theta_0 + 2m\pi/(s+1)$, with $m = 0, 1, \ldots, s,$. These are eigenstates of the respectively hermitian and unitary

$$\hat{\phi}_q \equiv \sum_{m=0}^{s} \theta_m |\theta_m><\theta_m| \tag{20}$$

$$\exp(i\hat{\phi}_q) \equiv |0><1| + \cdots + |s-1><s| + \exp(i(s+1)\theta_0)|s><0| \tag{21}$$

which is manifestly q-independent and unitary. In the analysis of $SU(2)_q$ Chaichian and Ellinas[11] introduce a polar decomposition operator that is the same as $\exp(i\hat{\phi})_q$ when the reference phase is chosen to be $\phi_R = (s+1)\theta_0$.

For arbitrary q, it still follows that

$$[cos\hat{\phi}_q, sin\hat{\phi}_q] = 0 \qquad cos^2\,\hat{\phi}_q + sin^2\,\hat{\phi}_q = 1 \tag{22}$$

[5]The number-phase properties of the q-generalized SG operators are treated in [6]. For research prior to PB on phase operators in spaces of finite dimension see T.S. Santhanam (*this conference*) and see the two recent general reviews of phase operators [12].

and that $< n|cos^2\hat{\phi}_q|n >=< n|sin^2\hat{\phi}_q|n >= 1/2$ for $n = 1, 2, \ldots$ In particular, the q-boson vacuum state $|0>_q$ has a random phase.

The mean-value of $\hat{\phi}_q$ in the $|z>_q$ basis is

$$< \hat{\phi}_q >= \frac{1}{2\pi}\int_0^{2\pi} \theta_m \bar{P}_q(\theta_m)\, d\theta_m \; = \; \theta \; = \; Arg(z)(23)$$

in terms of the q-boson phase distribution(the conjugate distribution to $P_n{}^q(z)$)

$$\bar{P}_q(\theta_m) \; = \; \lim_{s\to\infty}(s+1)| < \theta_m|z>_q|^2 \tag{24}$$

with the normalization $\frac{1}{2\pi}\int_0^{2\pi}\bar{P}_q(\theta_m)\, d\theta_m \; = \; 1$. The variance of the phase operator

$$(\Delta\hat{\phi}_q)^2 \to 1/(2\eta_q)^2 \tag{25}$$

as $|z| \to \infty$, where η_q is the same q-dependent constant found for ΔN as $|z| \to \infty$.

5 Approximate $[N, \hat{\phi}_q] = i$ in $|z>_q$ Classical Limit

Thus, from the reciprocal-dependencies on η_q of ΔN and $\Delta\hat{\phi}_q$, it follows that there are the ususal (though approximate) number-phase and energy-phase uncertainty relations

$$\Delta N\, \Delta\hat{\phi}_q \geq 1/2 \qquad \Delta H_N\, \Delta\hat{\phi}_q \geq \hbar\omega/2 \tag{26}$$

In the $|z>_q$ basis, the q-boson phase distribution $\bar{P}_q(\theta_m)$ function also appears in Dirac's approximate number-phase commutation relation

$$< z|[N, \hat{\phi}_q]|z >= i \; - \; i\, \bar{P}_q(\theta_0) \tag{27}$$

where θ_0 is the Pegg-Barnett indicial angle used above in (20). So for large $|z|$, for $q \neq 1$,

$$\lim_{s\to\infty} < z|[N, \hat{\phi}_q]|z > \; = \; i \; - \; i\, 2\pi\, \delta_q(\theta - \theta_0) \tag{28}$$

for $\hat{\phi}_q$ eigenvalues from the indicial θ_0 to $(\theta_0 + 2\pi)$. This extra δ_q term is a "bell-shaped" function. This term serves a physical role analogous to that of a smeared "magnetic monopole" string in that it appears in the classical limit to uniquely specify the classical phyase angle. For $q = 1$, the smearing is absent and δ_q is replace by a Dirac-delta-function distribution. This smearing is in agreement with the greater fractional uncertainty of $\hat{\phi}_q$ for $q \neq 1$.

So, neglecting the indicial-referencing term, we conclude that the $|z>_q$ coherent states both *give* and *minimize* Dirac's commutation relation, i.e. in $|z>_q$ basis for $|z|$ large

$$[N, \hat{\phi}_q] = i \tag{29}$$

Hence, for the q-boson quantum field the operators N and $\hat{\phi}_q$ are almost canonically conjugate in the $|z>_q$ classical limit. This is in contrast to the extra $\lambda(z)$ "resolution factor" in the commutation relation for the position and momentum operators. Given the physical importance of Dirac's commutation relation to cooperative phenomena in many different fields of physics, it is very encouraging that for arbitrary q values Eq.(29) still holds for the q-boson quantum field [13].

This is based on work with M. Fields. We thank C. Zachos for discussions; R. Lynch and D.T. Pegg for correspondence; the Argonne, Cornell, and Fermilab theory groups for intellectual stimulation; and U.S. Dept. of Energy Contract No. DE-FG 02-86ER40291 for support.

References

1. See papers of Faddeev, Kulish, Reshetikhin, Sklyanin, Takhtajan, Drinfeld, and Jimbo; and books on Hopf algebras.
2. A. Macfarlane, J. Phys. **A22**, 4581(1989); L. Biedenharn, J. Phys. **A222**, L873(1989); C.-P. Sun and H.-C. Fu, J. Phys. **A22**, L983(1989); M. Chaichian and P. Kulish, Phys. Lett. **B234**, 72(1990); and see M. Arik and D. D. Coon, J. Math. Phys. **17**, 4(1976).
3. R. W. Gray and C. A. Nelson, J. Phys. **A23**, L945(1990); A. J. Bracken, D. S. McAnally, R. B. Zhang and M. D. Gould, J. Phys. **A24**, 1379(1991); B. Jurco, Lett. Math. Phys. **21**, 51(1991).
4. S.-H. Chiu, R. W. Gray, C. A. Nelson, Phys. Lett. **A164**, 237(1992); C. A. Nelson, SUNY BING 9/7/92 (to appear in "Sym. in Sci. VI", ed. B. Gruber).
5. C. A. Nelson and M. G. Gartley, J. Phys. **A27**,(1990).
6. C. A. Nelson and M. H. Fields, SUNY BING 12/15/93(submitted to Phys. Rev. A).
7. D. T. Pegg and S. M. Barnett, Europhys. Lett. **6**, 483(1988) ; J. Mod. Opt. **36**, 7(1989).
8. V. Buzek, J. Mod. Optics **38**, 801(1991).
9. L. Susskind and J. Glogower, Physics **1**, 49,(1964). W. H. Louisell, Phys. Lett. **7**, 60(1963).
10. P. Carruthers and M. M. Nieto, Phys. Rev. Lett. **14**, 387(1965); Rev. Mod. Phys.**40**, 411(1968).
11. M. Chaichian and D. Ellinas, J. Phys. **A23**, L291(1990).
12. R. Lynch, The Quantum Phase Problem: A Critical Review,(submitted to Phys. Reports); and Phys. Scripta **T48** (1993), eds. W. P. Schleich and S. M. Barnett.
13. See related talks by A. Solomon, C. Quesne,V.I. Manko, S. Codriansky, and T.S. Santhanam.