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AN ALGORITHM FOR THE BASIS
OF THE FINITE FOURIER TRANSFORM

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Abstract

The Finite Fourier Transformation matrix (F.F.T.) plays a central role in the formulation of quantum mechanics in a finite dimensional space studied by the author over the past couple of decades. An outstanding problem which still remains open is to find a complete basis for F.F.T. In this paper we suggest a simple algorithm to find the eigenvectors of F.F.T.

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I. INTRODUCTION

The finite Fourier transform matrix (F.F.T.) plays a fundamental role in many contexts and has been studied extensively [1-3]. It is central in the discussions on finite dimensional quantum mechanics based on Weyl's commutation relations [4] studied by the author in a series of publications [5]. The eigenvalues of this matrix were determined by Schur [1] and a simple argument to recover this result has been given earlier [6]. The calculation of the eigenvectors is not straightforward and many methods have been given in particular, by Mehta [7]. In Section IV, we present a new algorithm to find the eigenvectors.

II. EIGENVALUES OF S

The F.F.T. matrix S, which is unitary, is defined by

$$S_{\alpha\beta} = \frac{1}{\sqrt{n}} \exp\left[\frac{2\pi i}{n} \alpha \beta\right],$$

$$\alpha, \beta = 0,1,2,\ldots n-1 \tag{2.1}$$

$$i = \sqrt{-1}$$

and has many interesting properties

1) $$(S^2)_{\alpha\beta} \equiv I'_{\alpha\beta} = \delta_{\alpha+\beta,\, o \pmod n} \tag{2.2}$$

Since $S^2 f_\alpha = f_{-\alpha \bmod n}$, for a vector f_α with n components, S^2 is called the parity operator

2) $$(S^4)_{\alpha\beta} = \delta_{\alpha\beta} \tag{2.3}$$

like the usual Fourier transform.

3) The matrix S, which is by definition a symmetric matrix will diagonalize any circulant matrix.

From Equation (?-3), it is clear that the eigenvalues of S are simply ± 1 and $\pm i$. There is then a degeneracy of the eigenvalues. The first problem will be to determine this. Luckily, Equations (2.1)-(2.3) can be repeatedly used to fix this [6]. If k_1, k_2, k_3 and k_4 denote the multiplicity of the eigenvalues taken in the order (1, -1, i, -i), Equation (2.1) implies that

$$\mathrm{Tr}\ S = \frac{1}{\sqrt{n}} \sum_{\ell=0}^{n-1} \left[\exp \frac{2\pi i}{n} \right]^{\ell^2}$$

$$= \frac{1}{2} (1 + i) \left[1 + \exp \left(\frac{-i\pi n}{2} \right) \right], \qquad (2.4)$$

and hence

$$\begin{aligned} \mathrm{Tr}\ S &= (k_1 - k_2) + i(k_3 - k_4) \\ &= 1 \text{ for } n = 4k + 1, \\ &= 0 \text{ for } n = 4k + 2, \\ &= i \text{ for } n = 4k + 3, \\ &= (1 + i) \text{ for } n = 4k, \\ k &= 0,1,2,... \end{aligned} \qquad (2.5)$$

From Equation (2) we infer that

$$\begin{aligned} \mathrm{Tr}\ S^2 &= (k_1 + k_2) - (k_3 + k_4) \\ &= 1 \text{ for } n \text{ odd}, \\ &= 2 \text{ for } n \text{ even}. \end{aligned}$$

We also have

$$\mathrm{Tr}\ S^4 = n = k_1 + k_2 + k_3 + k_4.$$

Equations (2.5), (2.6) and (2.7) can be used to solve for k_1, k_2, k_3 and k_4 and one finds that

	$n = 4k + 1$	$n = 4k + 2$	$n = 4k + 3$	$n = 4k$
k_1	$k + 1$	$k + 1$	$k + 1$	$k + 1$
k_2	k	$k + 1$	$k + 1$	k
k_3	k	k	$k + 1$	k
k_4	k	k	k	$k - 1$

III. EIGENVECTORS OF S

Let us decompose S into its primitive idempotents as

$$S = \sum_{j=1}^{4} i^j B(j),$$

where

$$B(1) = \frac{1}{2}s + \frac{1}{4}(I - I')$$

$$B(2) = -\frac{1}{2}c + \frac{1}{4}(I + I'),$$

$$B(3) = -\frac{1}{2}s + \frac{1}{4}(I - I'),$$

$$B(4) = \frac{1}{2}c + \frac{1}{4}(I + I'), \tag{3.2}$$

$$c_{\alpha\beta} = \frac{1}{\sqrt{n}} \cos\left(\frac{2\pi}{n}\alpha\beta\right)$$

$$s_{\alpha\beta} = \frac{1}{\sqrt{n}} \sin\left(\frac{2\pi}{n}\alpha\beta\right),$$

$$\alpha, \beta = 0,1,2,\ldots n-1 \tag{3.3}$$

It is easily verified that

$$S\ B(j) = i^j\ B(j), \tag{3.4}$$

thus the nonzero columnus of $B(j)$ yield the eigenvectors of S with eigenvalue i^j. Also, in analogy with the standard case, Mehta [7] has been able to express these eigenvectors in terms of Hermite functions with discrete arguments.

IV. EIGENVECTORS OF S; AN ALTERNATE METHOD

Since the F.F.T. matrix S satisfies Equation (2.1) we construct the matrix [10]

$$T = S^3 + S^2 S_d + S S_d^2 + S_d^3$$

$$= I' (S + S_d) + (S + S_d) S_d^2 , \qquad (4.1)$$

where

$$S_d = \text{diagonal } S . \qquad (4.2)$$

We find that

$$S T = S (S^3 + S^2 S_d + S S_d^2 + S_d^3)$$

$$= (I + S^3 S_d + S^2 S_d^2 + S S_d^3)$$

$$= (S_d^3 + S^3 + S^2 S_d + S S_d^2) S_d$$

$$= T S_d . \qquad (4.3)$$

If T is nonsingular,

$$T^+ S T = S_d \qquad (4.4)$$

Therefore, the columns of T automatically provide the eigenvectors of S. The degenerate eigenvectors of S corresponding to the repeated eigenvalues can be made orthonormal by using Gram-Schmidt process. This will render T unitary. While the process is quite general, we shall illustrate this for some special cases

<u>case of n = 2</u>

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} , \qquad (4.5)$$

and

$$S_d = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \tag{4.6}$$

$$\text{Since } S^2 = S_d^2 = I, \tag{4.7}$$

We get from Equation (4.1)

$$T = 2 \quad (S + S_d),$$

$$= 2 \begin{pmatrix} 1 + \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -1 - \frac{1}{\sqrt{2}} \end{pmatrix} \tag{4.8}$$

We unitarized matrix of the eigenvectors of S is therefore

$$U_2 = \frac{1}{\sqrt{2\sqrt{2}\ (\sqrt{2} + 1)}} \begin{pmatrix} \sqrt{2} + 1 & 1 \\ 1 & -(\sqrt{2} + 1) \end{pmatrix} \tag{4.9}$$

<u>case of n = 3</u>

$$S = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \varepsilon & \varepsilon^2 \\ 1 & \varepsilon^2 & \varepsilon \end{pmatrix},$$

$$\varepsilon = \exp \frac{2\pi i}{3} . \tag{4.10}$$

From Equation (2.8) we see that

$$S_d = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & i \end{pmatrix} \tag{4.11}$$

one finds from Equation (4.1) that the unitarized matrix of the eigen-vectors of S is

$$U_3 = \frac{1}{\sqrt{\sqrt{2}\ \sqrt{3}\ (\sqrt{3} + 1)}} \begin{pmatrix} \sqrt{3} + 1 & \sqrt{2} & 0 \\ 1 & \frac{1 + \sqrt{3}}{\sqrt{2}} & i\sqrt{3 + \sqrt{3}} \\ 1 & \frac{1 + \sqrt{3}}{\sqrt{2}} & -i\sqrt{3 + \sqrt{3}} \end{pmatrix}. \tag{4.12}$$

case of $n = 4$

In this case we have

$$S = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -i \\ 1 & -i & -1 & i \end{pmatrix} \tag{4.13}$$

and

$$S_d = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \tag{4.14}$$

It is easily calculated that

$$T = \begin{pmatrix} 3 & 1 & 1 & 0 \\ 1 & 1 & -1 & 2i \\ 1 & -1 & -1 & 0 \\ 1 & 1 & -1 & -2i \end{pmatrix} \tag{4.15}$$

The first two column vectors correspond to the eigenvalue = +1, the third one to -1 and the last to -i.

By a simple use of Gram-Schmidt orthogonalization procedure one can find the unitarized matrix corresponding to the eigenvectors of S as

$$U_4 = \frac{1}{\sqrt{2\sqrt{4}\ (\sqrt{4} + 1)}} \begin{pmatrix} 3 & 0 & \sqrt{3} & 0 \\ 1 & \sqrt{2} & -\sqrt{3} & i\sqrt{6} \\ 1 & -2\sqrt{2} & -\sqrt{3} & 0 \\ 1 & \sqrt{2} & -\sqrt{3} & -i\sqrt{6} \end{pmatrix} \tag{4.16}$$

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[9] See, for instance, F. R. Gantmacher, Matrix Theory (Chelsea, New York, 1959), Vol. I, p. 239.

[10] This can be easily generalized to $A^{n-1} + A^{n-2} A_d + A^{n-3} A_d^2 + \ldots + A A_d^{n-2} + A_d^{n-1}$ for the case of a general involution matrix satisfying the relation $A^n = I$.