1995116569

N95- 22986

A NOTE ON THE ANGULAR FOURIER TRANSFORMATION

Balu Santhanam
The DSP Laboratory
School of Electrical Engineering
Georgia Institute of Technology
Atlanta, Georgia, 30332, U.S.A.

and

T. S. Santhanam
Parks College of Saint Louis University
Cahokia, Illinois, 62206, U.S.A.

ABSTRACT

It is demonstrated that an angular Fourier transformation is obtained by making a rotation around the non-compact axis of So(2,1), the Lorentz group in three dimensions.

Talk presented by the second author in the Interworkshop on 'Harmonic oscillators' held in Cocoyoc, Mexica during 23-25 March, 1994. To appear in the Proceedings.

I. INTRODUCTION

The conventional fourier transformation has been at the root of quantum mechanics. If $\hat{q}$, $\hat{p}$ represent the position and momentum self adjoint operators of quantum mechancis, they satisfy the commutation relation *[1]

$$[\hat{q}, \hat{p}] = \hat{q}\hat{p} - \hat{p}\hat{q} = i \quad . \tag{1}$$

It is also well known that this relation implies that

$$(\Delta p)^2 \ (\Delta q)^2 \geq \frac{1}{4} , \tag{2}$$

where

$$(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2 \ . \tag{3}$$

Equation (1) is known to imply that

$$|p\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(ipq)|q\rangle dq \ .$$

i.e. the basis $|q\rangle$ in which the operator $\hat{q}$ is diagonal is related to the basis in which the operator $\hat{p}$ is diagonal through the Fourier Transform operator $\hat{S}$

*We use the unit where units $\frac{h}{2\pi} = 1$, where he is the plank constant

It is also known [2] that the classical fourier transform operator S can be represented as,

$$\hat{S} = \exp\{i\Pi\{(\hat{p}^2 + \hat{q}^2)/4 - \frac{1}{4}\}\} , \tag{5}$$

where $\hat{S}$ is defined as

$$\langle q|\hat{S}\Psi\rangle = \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} \exp(iqq')\langle q'|\Psi\rangle dq' , \tag{6}$$

where Ψ is the wave function which satisfies the Schröedinger wave euqation [1].

In this paper we show that the conventional fourier transform operator $\hat{S}$ when rotated by an angle θ through the non compact generator Γ_4 of the Lorentz group SO(2,1) yields the Angular Fourier Transformation (AFT). We also analyze the properties of the AFT from this perspective and relate it to the recent work of L. B. Almeida [23] who has derived the AFT from a different point of view.

In Section 2 we summarize the properties of the group SO(2,1). In section 3, we study some properties of the AFT from this perspective and relate this to the work of Almeida. In the last section we offer some conclusions on the discretization of the transform.

II. THE LORENTZ GROUP SO(2,1)

We define the three operators by Γ_4, Γ_0, T as

$$\begin{aligned} \Gamma_0 &= (1/4) \times \{\hat{p}^2 + \hat{q}^2\} , \\ \Gamma_4 &= (-1/4) \times \{\hat{p}^2 - \hat{q}^2\} , \\ T &= (-1/4) \times \{\hat{q}\,\hat{p} + \hat{p}\,\hat{q}\} = (-1/2) \times \{\hat{p}\,\hat{q} + \frac{i}{2}\} . \end{aligned} \tag{7}$$

It is easily verified that Γ_4, Γ_0, T satisfy the following commutation relations.

$$[\Gamma_0, \Gamma_4] = iT \quad , \tag{8}$$

$$[T , \Gamma_0] = i\Gamma_4 \quad , \tag{9}$$

$$[T , \Gamma_4] = i\Gamma_0 \quad , \tag{10}$$

and the Lie algebra so obtained is that of the Lorentz group SO(2,1) in (2+1) dimensions [4]. It is recognized that the classical Fourier Transform operator in Equation (5) can be rewritten as

$$\hat{S} = \exp\{i\Pi(\Gamma_0 - \frac{1}{4})\} \quad . \tag{11}$$

The generator Γ_4 is called the non compact generator of the Lorentz group SO(2,1), reflecting the fact that it is not bounded in support. From the commutation relations we can verify using Equations (8, 9, 10) that

$$\hat{K}_\theta(\hat{p}, \hat{q}) = \exp(i\theta\Gamma_4).(\hat{S}).\exp(i\theta\Gamma_4) \quad , \tag{12}$$

$$\hat{K}_\theta(\hat{p}, \hat{q}) = \exp(-i\frac{\pi}{4}).\exp(\frac{\pi}{4}\sinh\theta).\exp(i\pi\{\{\frac{\hat{p}^2 + \hat{q}^2}{4}\cosh\theta - \{\hat{p}\ \hat{q}\}\sinh\theta\}). \tag{13}$$

It may also be verified that

$$\langle q|\hat{K}_\theta\Psi\rangle = N_\theta \int_{-\infty}^{\infty} \exp(i\{\{\frac{q^2 + q'^2}{2}\}\sinh\theta - \{qq'\}\cosh\theta\})\langle q'|\Psi\rangle dq', \tag{14}$$

Where N_θ is a normalization constant that is dependent on θ.

If we now set $\sinh\theta = \cot\alpha$ and $\cosh\theta = \mathrm{cosec}\alpha$ then we obtain the kernel of L. B. Almeida where the variables are (t,u) instead of (q,q'). Thus the kernel of the AFT gets a meaning as a rotation in the (t,w) plane. The variables (q,q') are the canonical variables and can be substituted with any pair of variables that satisfy equation (1).

III. PROPERTIES OF THE ANGULAR FOURIER TRANSFORM

It becomes clear after substituting $\sinh\theta = \cot\alpha$ and $\cosh\theta = \mathrm{cosec}\alpha$ that

$$\theta = \ell n(\cot(\tfrac{\alpha}{2})) \quad , \tag{15}$$

which implies that as $\theta \to \infty$, $\alpha \to 2n\Pi$ and $\theta \to -\infty$, $\alpha \to (2n+1)\Pi$. With the above identifications our kernel in Equation (13) is identical to that of Almeida who has shown that the kernel exhibits the following properties

$$(1) \quad K\alpha(t,u) = K\alpha(u,t) \quad , \tag{16}$$

$$(2) \quad \int_{-\infty}^{\infty} K\alpha(t,u)K^{*}\alpha(t,u')dt = \delta(u-u') \quad , \tag{17}$$

$$(3) \quad Ko(t,\omega) = \frac{1}{\sqrt{2\pi}} \exp\{-it\omega\} \quad . \tag{18}$$

For further properties use reference [3]. As envisaged in reference [3] the AFT can be applied to the study of frequency swept filters.

IV. CONCLUSIONS

In this paper we have used the properties of the group SO(2,1) to define the AFT as a rotation of the fourier transform operator $\hat{S}$ by an angle θ through the non compact generator Γ_4 of the group, which will reduce to the conventional Fourier Transform as $\theta \to \infty$.

The study of a discrete version of this transform and fast algorithms for it's computation is of great interest and has been carried out [5].

REFERENCES

[1] P.A.M. Dirac, The Principles of Quantum Mechanics, Oxford University Press, London (1930).

[2] See for example,

J. M. Jauch, Foundations of Quantum Mechanics, Addison-Wesley, Reading, MA. See also,

K. B. Wolf, Integral Transforms in Science and Engineering, Plenum Press, NY, (1972).

[3] L. B. Almeida, An Introduction to the Angular Fourier Transform, IEEE III, (1993), p. 253-260.

[4] See for example,

A. O. Barut and R. Raczka, Theory of Group Representations and Applications, Plan-Polish Scientific Publishers, Warszawa, 1977.

[5] Balu. Santhanam and J. H. McClellan, preprint, Georgia Institute of Technology, 1994.

C-4.