

# WIGNER DISTRIBUTION FUNCTION AND ENTROPY OF THE DAMPED HARMONIC OSCILLATOR WITHIN THE THEORY OF OPEN QUANTUM SYSTEMS

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## Abstract

The harmonic oscillator with dissipation is studied within the framework of the Lindblad theory for open quantum systems. By using the Wang-Uhlenbeck method, the Fokker-Planck equation, obtained from the master equation for the density operator, is solved for the Wigner distribution function, subject to either the Gaussian type or the  $\delta$ -function type of initial conditions. The obtained Wigner functions are two-dimensional Gaussians with different widths. Then a closed expression for the density operator is extracted. The entropy of the system is subsequently calculated and its temporal behaviour shows that this quantity relaxes to its equilibrium value.

## 1 Introduction

In the last two decades, the problem of dissipation in quantum mechanics, i.e. the consistent description of open quantum systems, was investigated by various authors [1, 2, 3, 4, 5]. Because dissipative processes imply irreversibility and, therefore, a preferred direction in time, it is generally thought that quantum dynamical semigroups are the basic tools to introduce dissipation in quantum mechanics. In the Markov approximation the most general form of the generators of such semigroups was given by Lindblad [6]. This formalism has been studied for the case of damped harmonic oscillators [7, 8, 9] and applied to various physical phenomena, for instance, the damping of collective modes in deep inelastic collisions in nuclear physics [10] and the interaction of a two-level atom with the electromagnetic field [11].

In the present work, also dealing with the damping of the harmonic oscillator within the Lindblad theory for open quantum systems, we will explore the physical aspects of the Fokker-Planck equation which is the  $c$ -number equivalent equation to the master equation for the density operator. Generally the master equation gains considerably in clarity if it is represented in terms of the Wigner distribution function which satisfies the Fokker-Planck equation. It is worth mentioning that these master and Fokker-Planck equations agree in form with the corresponding equations formulated in quantum optics [12, 13, 14, 15, 16].

The content of the paper is arranged as follows. In Sec. 2 we review the derivation of the master equation of the harmonic oscillator. In Sec. 3 we transform the master equation into the Fokker-Planck equation by means of the well-known methods [17, 18, 19]. Then the Fokker-Planck equation for the Wigner distribution, subject to either the Gaussian type or the  $\delta$ -function type of initial conditions, is solved by the Wang-Uhlenbeck method. Sec. 4 derives an explicit form of the density operator involved in the Lindblad master equation, formulates the entropy using the explicit form of the density operator and discusses its temporal behaviour. Finally, concluding remarks are given in Sec. 5.

## 2 Master equation for the damped harmonic oscillator

The rigorous formulation for introducing the dissipation into a quantum mechanical system is that of quantum dynamical semigroups [2, 3, 6]. According to the axiomatic theory of Lindblad [6], the usual von Neumann-Liouville equation ruling the time evolution of closed quantum systems is replaced in the case of open systems by the following equation for the density operator  $\rho$ :

$$\frac{d\Phi_t(\rho)}{dt} = L(\Phi_t(\rho)). \quad (1)$$

Here,  $\Phi_t$  denotes the dynamical semigroup describing the irreversible time evolution of the open system in the Schrödinger representation and  $L$  the infinitesimal generator of the dynamical semigroup  $\Phi_t$ . Using the structural theorem of Lindblad [6] which gives the most general form of the bounded, completely dissipative Liouville operator  $L$ , we obtain the explicit form of the most general time-homogeneous quantum mechanical Markovian master equation:

$$\frac{d\rho(t)}{dt} = L(\rho(t)) = -\frac{i}{\hbar}[H, \rho(t)] + \frac{1}{2\hbar} \sum_j ([V_j \rho(t), V_j^+] + [V_j, \rho(t) V_j^+]). \quad (2)$$

Here  $H$  is the Hamiltonian of the system and the operators  $V_j$  and  $V_j^+$  are bounded operators on the Hilbert space of the Hamiltonian.

We should like to mention that the Markovian master equations found in the literature are of this form after some rearrangement of terms, even for unbounded Liouville operators. In this connection we assume that the general form of the master equation given by (2) is also valid for unbounded Liouville operators.

In this paper we impose a simple condition to the operators  $H, V_j, V_j^+$  that they are functions of the basic observables  $\hat{q}$  and  $\hat{p}$  of the one-dimensional quantum mechanical system (with  $[\hat{q}, \hat{p}] = i\hbar$ ) of such kind that the obtained model is exactly solvable. A precise version for this last condition is that linear spaces spanned by first degree (respectively second degree) noncommutative polynomials in  $\hat{q}$  and  $\hat{p}$  are invariant to the action of the completely dissipative mapping  $L$ . This condition implies [7] that  $V_j$  are at most first degree polynomials in  $\hat{q}$  and  $\hat{p}$  and  $H$  is at most a second degree polynomial in  $\hat{q}$  and  $\hat{p}$ . Then the harmonic oscillator Hamiltonian  $H$  is chosen of the form

$$H = H_0 + \frac{\mu}{2}(\hat{q}\hat{p} + \hat{p}\hat{q}), \quad H_0 = \frac{1}{2m}\hat{p}^2 + \frac{m\omega^2}{2}\hat{q}^2. \quad (3)$$

With these choices the Markovian master equation can be written [8]:

$$\begin{aligned} \frac{d\rho}{dt} = & -\frac{i}{\hbar}[H_0, \rho] - \frac{i}{2\hbar}(\lambda + \mu)[\hat{q}, \rho\hat{p} + \hat{p}\rho] + \frac{i}{2\hbar}(\lambda - \mu)[\hat{p}, \rho\hat{q} + \hat{q}\rho] \\ & - \frac{D_{pp}}{\hbar^2}[\hat{q}, [\hat{q}, \rho]] - \frac{D_{qq}}{\hbar^2}[\hat{p}, [\hat{p}, \rho]] + \frac{D_{pq}}{\hbar^2}([\hat{q}, [\hat{p}, \rho]] + [\hat{p}, [\hat{q}, \rho]]), \end{aligned} \quad (4)$$

where  $D_{pp}$ ,  $D_{qq}$  and  $D_{pq}$  are the diffusion coefficients and  $\lambda$  the friction constant. They satisfy the following fundamental constraints [8]:

$$\text{i) } D_{pp} > 0, \quad \text{ii) } D_{qq} > 0, \quad \text{iii) } D_{pp}D_{qq} - D_{pq}^2 \geq \frac{\lambda^2 \hbar^2}{4}. \quad (5)$$

In the particular case when the asymptotic state is a Gibbs state

$$\rho_G(\infty) = e^{-\frac{H_0}{kT}} / \text{Tre}^{-\frac{H_0}{kT}}, \quad (6)$$

these coefficients reduce to

$$D_{pp} = \frac{\lambda + \mu}{2} \hbar m \omega \coth \frac{\hbar \omega}{2kT}, \quad D_{qq} = \frac{\lambda - \mu}{2} \frac{\hbar}{m \omega} \coth \frac{\hbar \omega}{2kT}, \quad D_{pq} = 0, \quad (7)$$

where  $T$  is the temperature of the thermal bath.

### 3 Wigner distribution function

One useful way to study the consequences of the master equation (4) for the density operator of the one-dimensional damped harmonic oscillator is to transform it into more familiar forms, such as the equations for the  $c$ -number quasiprobability distributions Glauber  $P$ , antinormal ordering  $Q$  and Wigner  $W$  associated with the density operator [20]. In this case the resulting differential equations of the Fokker-Planck type for the distribution functions can be solved by standard methods [17, 19, 21] employed in quantum optics and observables directly calculated as correlations of these distribution functions.

The Fokker-Planck equation, obtained from the master equation and satisfied by the Wigner distribution function  $W(x_1, x_2, t)$  of real variables  $x_1, x_2$  corresponding to the operators  $\hat{q}, \hat{p}$

$$x_1 = \sqrt{\frac{m\omega}{2\hbar}} q, \quad x_2 = \frac{1}{\sqrt{2\hbar m\omega}} p, \quad (8)$$

has the form [20]:

$$\frac{\partial W}{\partial t} = \sum_{i,j=1,2} A_{ij} \frac{\partial}{\partial x_i} (x_j W) + \frac{1}{2} \sum_{i,j=1,2} Q_{ij}^W \frac{\partial^2}{\partial x_i \partial x_j} W, \quad (9)$$

where

$$A = \begin{pmatrix} \lambda - \mu & -\omega \\ \omega & \lambda + \mu \end{pmatrix}, \quad Q^W = \frac{1}{\hbar} \begin{pmatrix} m\omega D_{qq} & D_{pq} \\ D_{pq} & D_{pp}/m\omega \end{pmatrix}. \quad (10)$$

Since the drift coefficients are linear in the variables  $x_1$  and  $x_2$  and the diffusion coefficients are constant with respect to  $x_1$  and  $x_2$ , Eq. (9) describes an Ornstein-Uhlenbeck process [22, 23]. Following the method developed by Wang and Uhlenbeck [23], we shall solve this Fokker-Planck equation, subject to either the wave-packet type or the  $\delta$ -function type of initial conditions.

1) When the Fokker-Planck equation is subject to a Gaussian (wave-packet) type of the initial condition ( $x_{10}$  and  $x_{20}$  are the initial values of  $x_1$  and  $x_2$  at  $t = 0$ , respectively)

$$W_w(x_1, x_2, 0) = \frac{1}{\pi \hbar} \exp\{-2[(x_1 - x_{10})^2 + (x_2 - x_{20})^2]\}, \quad (11)$$

the solution is found to be

$$W_w(x_1, x_2, t) = \frac{\Omega}{\pi \hbar \omega \sqrt{|B_w|}} \exp\left\{-\frac{1}{B_w}[\phi_w(x_1 - \bar{x}_1)^2 + \psi_w(x_2 - \bar{x}_2)^2 + \chi_w(x_1 - \bar{x}_1)(x_2 - \bar{x}_2)]\right\}, \quad (12)$$

where

$$B_w = g_1 g_2 - \frac{1}{4} g_3^2, \quad g_1 = g_2^* = \frac{\mu a}{\omega} e^{2\Lambda t} + \frac{d_1}{\Lambda} (e^{2\Lambda t} - 1), \quad g_3 = 2[e^{-2\lambda t} + \frac{d_2}{\lambda} (1 - e^{-2\lambda t})], \quad (13)$$

$$\phi_w = g_1 a^{*2} + g_2 a^2 - g_3, \quad \psi_w = g_1 + g_2 - g_3, \quad \chi_w = 2(g_1 a^* + g_2 a) - g_3(a + a^*). \quad (14)$$

We have put  $a = (\mu - i\Omega)/\omega$ ,  $\Lambda = -\lambda - i\Omega$  and  $d_1 = (a^2 m \omega D_{qq} + 2a D_{pq} + D_{pp}/m\omega)/\hbar$ ,  $d_2 = (m\omega D_{qq} + 2\mu D_{pq}/\omega + D_{pp}/m\omega)/\hbar$  and  $\Omega^2 = \omega^2 - \mu^2$ . The functions  $\bar{x}_1$  and  $\bar{x}_2$ , which are also oscillating functions, are given by

$$\bar{x}_1 = e^{-\lambda t} [x_{10}(\cos \Omega t + \frac{\mu}{\Omega} \sin \Omega t) + x_{20} \frac{\omega}{\Omega} \sin \Omega t], \quad (15)$$

$$\bar{x}_2 = e^{-\lambda t} [x_{20}(\cos \Omega t - \frac{\mu}{\Omega} \sin \Omega t) - x_{10} \frac{\omega}{\Omega} \sin \Omega t]. \quad (16)$$

2) If the Fokker-Planck equation (9) is subject to the  $\delta$ -function type of initial condition, the Wigner distribution function is given by

$$W(x_1, x_2, t) = \frac{\Omega}{\pi \hbar \omega \sqrt{|B|}} \exp\left\{-\frac{1}{B}[\phi_d(x_1 - \bar{x}_1)^2 + \psi_d(x_2 - \bar{x}_2)^2 + \chi_d(x_1 - \bar{x}_1)(x_2 - \bar{x}_2)]\right\}, \quad (17)$$

where

$$B = f_1 f_2 - f_3^2, \quad f_1 = f_2^* = \frac{d_1}{\Lambda} (e^{2\Lambda t} - 1), \quad f_3 = \frac{d_2}{\lambda} (1 - e^{-2\lambda t}), \quad (18)$$

$$\phi_d = f_1 a^{*2} + f_2 a^2 - 2f_3, \quad \psi_d = f_1 + f_2 - 2f_3, \quad \chi_d = 2[f_1 a^* + f_2 a - f_3(a + a^*)]. \quad (19)$$

So, the Wigner functions are 2-dimensional Gaussian distributions with the average values  $\bar{x}_1$  and  $\bar{x}_2$  and different widths.

When time  $t \rightarrow \infty$ ,  $\bar{x}_1$  and  $\bar{x}_2$  vanish and we obtain the steady state solution:

$$W(x_1, x_2) = \frac{1}{2\pi \sqrt{\det \sigma^W(\infty)}} \exp\left[-\frac{1}{2} \sum_{i,j=1,2} (\sigma^W)_{ij}^{-1}(\infty) x_i x_j\right]. \quad (20)$$

The stationary covariance matrix  $\sigma^W(\infty)$  can be determined from the algebraic equation

$$A \sigma^W(\infty) + \sigma^W(\infty) A^T = Q^W. \quad (21)$$

## 4 Entropy and effective temperature

Entropy is a quantity which may be visualized physically as a measure of the lack of knowledge of the system. When we denote by  $\rho(t)$  the density operator in the Schrödinger picture for the harmonic oscillator, the entropy  $S(t)$  is given by

$$S(t) = -k\text{Tr}(\rho \ln \rho). \quad (22)$$

For calculating the entropy we shall compute straightway the expectation value of the logarithmic operator  $\langle \ln \rho \rangle = \text{Tr}(\rho \ln \rho)$ . Accordingly, the problem amounts to derive the explicit form of the density operator for the damped harmonic oscillator.

To get the explicit expression for the density operator, we use the relation  $\rho = 2\pi\hbar\mathbf{N}\{W_s(q, p)\}$ , where  $W_s$  is the Wigner distribution function in the form of standard rule of association and  $\mathbf{N}$  is the normal ordering operator [17, 24] which acting on the function  $W_s(q, p)$  moves all  $p$  to the right of the  $q$ . By the standard rule of association is meant the correspondence  $p^m q^n \rightarrow \hat{q}^n \hat{p}^m$  between functions of two classical variables  $(q, p)$  and functions of two quantum mechanical canonical operators  $(\hat{q}, \hat{p})$ . The calculation of the density operator is then reduced to a problem of transformation of the Wigner distribution function by the  $\mathbf{N}$  operator, provided that  $W_s$  is known. A special care is necessary for the  $\mathbf{N}$  operation when the Wigner function is in the exponential form of a second order polynomial of  $q$  and  $p$ . The Wigner distribution function previously obtained corresponds however to the form of the Weyl rule of association [25]. The solution (12) of the Fokker-Planck equation (9), subject to the wave-packet type of initial condition (11) can be written in terms of the coordinate and momentum as:

$$W(q, p, t) = \frac{1}{2\pi\sqrt{\delta}} \exp\left\{-\frac{1}{2\delta}[\phi(q - \langle \hat{q} \rangle)^2 + \psi(p - \langle \hat{p} \rangle)^2 - 2\chi(q - \langle \hat{q} \rangle)(p - \langle \hat{p} \rangle)]\right\}, \quad (23)$$

where

$$\langle \hat{q} \rangle = \sqrt{\frac{2\hbar}{m\omega}} \bar{x}_1, \quad \langle \hat{p} \rangle = \sqrt{2\hbar m\omega} \bar{x}_2, \quad (24)$$

$$\phi \equiv \sigma_{pp} = \langle \hat{q}^2 \rangle - \langle \hat{q} \rangle^2 = -\frac{\hbar\omega^2 + 1}{4\Omega^2 m\omega} \psi_w, \quad (25)$$

$$\psi \equiv \sigma_{qq} = \langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2 = -\frac{\hbar\omega^2}{4\Omega^2} m\omega \phi_w, \quad (26)$$

$$\chi \equiv \sigma_{pq}(t) = \frac{1}{2} \langle \hat{q}\hat{p} + \hat{p}\hat{q} \rangle - \langle \hat{q} \rangle \langle \hat{p} \rangle = \frac{\hbar\omega^2}{8\Omega^2} \chi_w, \quad \delta = \phi\psi - \chi^2 \quad (27)$$

and  $\langle \hat{A} \rangle = \text{Tr}(\rho \hat{A})$  denotes the expectation value of an operator  $\hat{A}$ . The Wigner distribution function (23) can be transformed into the form of standard rule of association [26] by

$$W_s(q, p) = \exp\left(\frac{1}{2}i\hbar \frac{\partial^2}{\partial p \partial q}\right) W(q, p). \quad (28)$$

Upon performing the operation on the right-hand side, we get the Wigner distribution function  $W_s$ , which has the same form as the original  $W$  multiplied by  $\hbar$  but with  $\chi - i\hbar/2$  in place of  $\chi$ . The normal ordering operation of the Wigner function  $W_s$  in Gaussian form can be carried out by applying McCoy theorem [24, 27, 28]. The explicit form of the density operator is the following:

$$\rho = \frac{\hbar}{\sqrt{\xi}} \exp\left[\frac{1}{2} \ln \frac{\xi}{\xi - i\hbar\chi'} - \frac{1}{2\hbar\sqrt{\xi - i\hbar\chi' + \frac{1}{4}\hbar^2}} \cosh^{-1}\left(1 + \frac{\hbar^2}{2(\xi - i\hbar\chi')}\right)\right] \\ \times \{\phi(\hat{q} - \langle \hat{q} \rangle)^2 + \psi(\hat{p} - \langle \hat{p} \rangle)^2 - (\chi' + i\frac{\hbar}{2})[2(\hat{q} - \langle \hat{q} \rangle)(\hat{p} - \langle \hat{p} \rangle) - i\hbar]\}, \quad (29)$$

where

$$\xi = \phi\psi - \chi'^2, \quad \chi' = \chi - i\frac{\hbar}{2}. \quad (30)$$

The density operator (29) is in a Gaussian form, as was expected from the initial form of the Wigner distribution function. While the density operator is expressed in terms of operators  $\hat{q}$  and  $\hat{p}$ , the Wigner distribution is a function of real variables  $q$  and  $p$ . When time  $t$  goes to infinity, the density operator approaches to

$$\rho(\infty) = \frac{\hbar}{\sqrt{\sigma - \frac{\hbar^2}{4}}} \exp\left[-\frac{1}{2\hbar\sqrt{\sigma}} \ln \frac{2\sqrt{\sigma} + \hbar}{2\sqrt{\sigma} - \hbar} [\sigma_{pp}(\infty)\hat{q}^2 + \sigma_{qq}(\infty)\hat{p}^2 - \sigma_{pq}(\infty)(\hat{q}\hat{p} + \hat{p}\hat{q})]\right], \quad (31)$$

where  $\sigma = \sigma_{pp}(\infty)\sigma_{qq}(\infty) - \sigma_{pq}^2(\infty)$  and [8]:

$$\sigma_{pp}(\infty) = \frac{1}{2\lambda(\lambda^2 + \omega^2 - \mu^2)} ((m\omega)^2\omega^2 D_{qq} + (2\lambda(\lambda - \mu) + \omega^2)D_{pp} - 2m\omega^2(\lambda - \mu)D_{pq}), \quad (32)$$

$$\sigma_{qq}(\infty) = \frac{1}{2(m\omega)^2\lambda(\lambda^2 + \omega^2 - \mu^2)} ((m\omega)^2(2\lambda(\lambda + \mu) + \omega^2)D_{qq} + \omega^2 D_{pp} + 2m\omega^2(\lambda + \mu)D_{pq}), \quad (33)$$

$$\sigma_{pq}(\infty) = \frac{1}{2m\lambda(\lambda^2 + \omega^2 - \mu^2)} (-(\lambda + \mu)(m\omega)^2 D_{qq} + (\lambda - \mu)D_{pp} + 2m(\lambda^2 - \mu^2)D_{pq}). \quad (34)$$

In the particular case (7)

$$\sigma_{qq}(\infty) = \frac{\hbar}{2m\omega} \coth \frac{\hbar\omega}{2kT}, \quad \sigma_{pp}(\infty) = \frac{\hbar m\omega}{2} \coth \frac{\hbar\omega}{2kT}, \quad \sigma_{pq}(\infty) = 0 \quad (35)$$

and the asymptotic state is a Gibbs state (6):

$$\rho_G(\infty) = 2 \sinh \frac{\hbar\omega}{2kT} \exp\left[-\frac{1}{kT} \left(\frac{1}{2m}\hat{p}^2 + \frac{m\omega^2}{2}\hat{q}^2\right)\right]. \quad (36)$$

Because of the presence of the exponential form in the density operator, the construction of the logarithmic density is straightforward. In view of the relations (25-27), the expectation value of the logarithmic density becomes

$$\langle \ln \rho \rangle = \ln \hbar - \frac{1}{2} \ln \left( \delta - \frac{\hbar^2}{4} \right) - \frac{\sqrt{\delta}}{\hbar} \ln \frac{2\sqrt{\sigma} + \hbar}{2\sqrt{\sigma} - \hbar}. \quad (37)$$

By putting  $\hbar\nu = \sqrt{\delta} - \hbar/2$ , we finally get the entropy in a closed form:

$$S(t) = k[(\nu + 1) \ln(\nu + 1) - \nu \ln \nu]. \quad (38)$$

It is worth noting that the entropy depends only upon the variance of the Wigner distribution. When time  $t \rightarrow \infty$ , the function  $\nu$  goes to  $s = \omega(d_2^2/\lambda^2 - |d_1|^2/(\lambda^2 + \Omega^2))^{1/2}/2\Omega - 1/2$  and the entropy relaxes to its equilibrium value  $S(\infty) = k[(s + 1) \ln(s + 1) - s \ln s]$ . It should also be noted that the expression (38) has the same form as the entropy of a system of harmonic oscillators in thermal equilibrium. In the later case  $\nu$  represents, of course, the average of the number operator [29]. While the formal expression (38) for the entropy has a well-known appearance, the form of the function  $\nu$  displays clearly a specific feature of the present entropy. We see that the time dependence of the entropy is represented by the damping factor  $\exp(-2\lambda t)$  and also by the oscillating function  $\sin^2(\Omega t)$ . The entropy relaxes to its equilibrium value  $S(\infty)$ .

## 5 Concluding remarks

Recently we assist to a revival of interest in quantum Brownian motion as a paradigm of quantum open systems. There are many motivations. The possibility of preparing systems in macroscopic quantum states led to the problems of dissipation in tunneling and of loss of quantum coherence (decoherence). These problems are intimately related to the issue of quantum-to-classical transition. All of them point the necessity of a better understanding of open quantum systems and all requires the extension of the model of quantum Brownian motion. The Lindblad theory provides a selfconsistent treatment of damping as a possible extension of quantum mechanics to open systems. In the present paper we have studied the one-dimensional harmonic oscillator with dissipation within the framework of this theory. From the master equation of the damped quantum oscillator we have derived the corresponding Fokker-Planck equation in the Wigner  $W$  representation. The obtained equation describes an Ornstein-Uhlenbeck process. By using the Wang-Uhlenbeck method we have solved this equation for the Wigner function, subject to either the Gaussian type or the  $\delta$ -function type of initial conditions and showed that the Wigner functions are two-dimensional Gaussians with different widths. Then we have obtained the density operator. The density operator in a Gaussian form is a function of  $\hat{q}, \hat{p}$  in addition to several time dependent factors. The explicit form of the density operator has been subsequently used to calculate the entropy. It relaxes to its equilibrium value.

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