The relativistic many body problem with an oscillator interaction

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Abstract

We start with the total energy E for a system of three scalar relativistic particles that, because of Einstein's relation, will have square roots of functions of the momenta. By taking powers of this relation, we finally get a fourth degree polynomial in E^2, where the square roots have disappeared, and which we can convert into a type of Schroedinger equation. To be in the center of mass frame we pass to Jacobi momenta and then replace them by creation and annihilation operators. We thus get an equation in terms of the generators of a U(2) group, which, in principle, we can solve in an elementary way. Finally we rewrite our equation in a Poincaré invariant form.

1 Introduction

In the II Harmonic Oscillator Conference I presented a paper dealing with systems of relativistic particles interacting through Dirac oscillators. The results were later applied to the mass spectra of baryons and mesons [1,2,3].

As all the results presented had already been published I prefer to deal in this paper with a new approach, restricted here to scalar particles, that seems to me a systematic way to attack many body problems with oscillator interactions.

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I will start by considering a non relativistic problem of n free particles and indicate the steps by which later it can be reduced to a system with oscillator interactions, which will serve as a model for the relativistic problem we wish to analyze.

2 The non-relativistic problem

Let us first start with a system of n free non-relativistic particles of the same mass m, and take units

$$\hbar = m = c = 1, \tag{2.1}$$

where the velocity of light will appear only in the next section, but we want to start from the beginning with units in which everything is dimensionless.

The classical total energy is then

$$E = (1/2) \sum_{s=1}^{n} \mathbf{p}_s \cdot \mathbf{p}_s \tag{2.2}$$

where $\mathbf{p}_s$ are the three dimensional classical momentum vectors of particle s.

From the beginning we would like to work in the center of mass frame, because our interest will be the internal energy of the system and not the contribution from its center of mass motion. The best way to achieve this is to pass to Jacobi coordinates [4] defined by the orthogonal transformation

$$\mathbf{p}'_s = [s(s+1)]^{-\frac{1}{2}} \left[\sum_{t=1}^{s} \mathbf{p}_t - s\mathbf{p}_{s+1} \right], s = 1, 2, \ldots n-1, \tag{2.3a}$$

$$\mathbf{p}'_n = n^{-\frac{1}{2}} \sum_{t=1}^{s} \mathbf{p}_s, \tag{2.3b}$$

Clearly $\mathbf{p}'_n$ is proportional to the total momentum and in the center of mass system it will vanish, so Eq. (2.2) reduces to

$$E = \tfrac{1}{2} \sum_{s=1}^{n-1} \mathbf{p}'_s \cdot \mathbf{p}'_s \tag{2.4}$$

The Schroedinger equations corresponding to (2.4) is obtained when we replace $\mathbf{p}'_s$ by the operator

$$\mathbf{p}'_s = \frac{1}{i} \frac{\partial}{\partial \mathbf{x}'_s}, \tag{2.5}$$

with $\mathbf{x}'_s$ being the corresponding Jacobi coordinate vector. As the $\mathbf{p}'_s$ are hermitian operators we can also write Schroedinger equation as

$$\frac{1}{2} \sum_{s=1}^{n-1} \left(\mathbf{p}'^\dagger_s \cdot \mathbf{p}'_s \right) \psi = E\psi \tag{2.6}$$

We can easily transform this into an hermitian oscillator operator equation if we make the replacement

$$\mathbf{p}'_s \rightarrow \mathbf{p}'_s - i\omega \mathbf{x}'_s, \tag{2.7a}$$

$$\mathbf{p}'^\dagger_s \rightarrow \mathbf{p}'_s + i\omega \mathbf{x}'_s, \tag{2.7b}$$

where the second equation follows from the first as *both* $\mathbf{p}'_s, \mathbf{x}'_s$ are hermitian operators.

Thus we now get a Schroedinger equation of the form

$$\frac{1}{2} \sum_{s=1}^{n-1} \left[p'^2_s + \omega^2 x'^2_s - (3/2)\omega(n-1) \right] \psi = E\psi, \tag{2.8}$$

whose eigenvalue for the energy E will be

$$E = \omega N, \tag{2.9}$$

with N being the total number of quanta *i.e.*

$$N = \sum_{s=1}^{n-1} \nu_s. \tag{2.10}$$

The previous analysis is standard except for the fact that we start from a system of n free particles. Furthermore our notation in terms of three vectors and Jacobi coordinates, avoids the worry about the Galilean invariance of the whole procedure. We will now consider a similar set of steps for a relativistic problem.

3   The system of three relativistic particles

Rather than discuss the system of n relativistic particles, we shall restrict ourselves to $n = 3$, as we will see that the case is general enough, with only the algebraic steps becoming more complicated as n increases.

In our units the total energy for a system of three free relativistic particles can be written as

$$E = \pm\Pi_1 \pm \Pi_2 \pm \Pi_3, \tag{3.1}$$

where $\Pi_s, s = 1, 2, 3$ is defined as

$$\Pi_s = (p_s^2 + 1)^{\frac{1}{2}}. \tag{3.2}$$

It is very important to note that, in our units, Einstein relation is $E^2 = p^2 + 1$, and when reduced to the E itself gives both the square root in (3.2) and the $\pm$ signs in (3.1).

Obviously we can not get a Schroedinger equation from the relation (3.1), but we can take $\pm\Pi_3$ to the right hand side and square both sides. Then we can square again and again appropriately, and we easily arrive at the fact that (3.1) becomes an eight degree equation in E (actually of fourth degree in E^2) of the form

$$\Phi(E^2, \Pi_s^2) \equiv$$

$$E^8 - 4AE^6 + (4A^2 + 2B)E^4 - (4C^2 + 4AB)E^2 + B^2 = 0 \tag{3.3}$$

where A, B, C are functions of $\Pi_s^2, s = 1, 2, 3$ given by

$$A \equiv \Pi_1^2 + \Pi_2^2 + \Pi_3^2, \tag{3.4a}$$

$$B \equiv \Pi_1^4 + \Pi_2^4 + \Pi_3^4 - 2\Pi_1^2\Pi_2^2 - 2\Pi_2^2\Pi_3^2 - 2\Pi_1^2\Pi_3^2, \tag{3.4b}$$

$$C^2 \equiv 16\Pi_1^2\Pi_2^2\Pi_3^2. \tag{3.4c}$$

Now we can write an equation that does not have E as an eigenvalue, but in which it appears as a parameter, if we replace p_s by $-i\partial/\partial x_s$ as in (2.5), so that Π_s^2 become the operators

$$\hat{\Pi}_s^2 = (-\nabla_s^2 + 1), \tag{3.5}$$

and we get

$$\Phi(E^2, \hat{\Pi}_s^2)\psi = 0 \tag{3.6}$$

Thus far we have obtained nothing useful because p_1, p_2, p_3 considered as operators of the form (2.5), commute with the operator Φ and so are integrals of motion, so that ψ can be written as

$$\psi = \exp[i(p_i \cdot x_1 + p_2 \cdot x_2 + p_3 \cdot x_3)], \tag{3.7}$$

where now p_1, p_2, p_3 are ordinary numbers and we are returned to equation (3.3) whose eight roots for the energy E are obviously given by (3.1) with all the possible combination of the signs $\pm$.

Before proceeding further, along the lines of the previous section, we again remark that we would like to work in the center of mass frame, as our interest is restricted to the internal energy of the system. Thus we go, as in section 2, to the Jacobi momenta $\mathbf{p}'_s, s = 1, 2, 3$ which from (2.3) are given now by the matrix relation

$$\begin{pmatrix} \mathbf{p}'_1 \\ \mathbf{p}'_2 \\ \mathbf{p}'_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{pmatrix}. \tag{3.8}$$

As the matrix is orthogonal, transposing it we get $\mathbf{p}_s$ in terms of $\mathbf{p}'_s$, and as we want to be in the center of mass frame $\mathbf{p}'_3 = 0$, so we get finally

$$\mathbf{p}_1 = \frac{1}{\sqrt{2}}\mathbf{p}'_1 + \frac{1}{\sqrt{6}}\mathbf{p}'_2, \mathbf{p}'_2 = \frac{1}{\sqrt{2}}\mathbf{p}'_1 - \frac{1}{\sqrt{6}}\mathbf{p}'_2, \mathbf{p}_3 = -\sqrt{\frac{2}{3}}\mathbf{p}'_2, \tag{3.9}$$

As Eq. (3.6) contains only powers of $\hat{\Pi}^2_s, s = 1, 2, 3$, we can write the latter using the hermitian property of $\mathbf{p}'_s$, now considered as operators of the type (2.6), as

$$\hat{\Pi}^2_1 = \frac{1}{2}\mathbf{p}'^\dagger_1 \cdot \mathbf{p}'_1 + \frac{1}{6}\mathbf{p}'^\dagger_2 \cdot \mathbf{p}'_2 + \frac{1}{2\sqrt{3}}(\mathbf{p}'^\dagger_1 \cdot \mathbf{p}'_2 + \mathbf{p}'^\dagger_2 \cdot \mathbf{p}'_1) + 1, \tag{3.10a}$$

$$\hat{\Pi}^2_2 = \frac{1}{2}\mathbf{p}'^\dagger_1 \cdot \mathbf{p}'_1 + \frac{1}{6}\mathbf{p}'^\dagger_2 \cdot \mathbf{p}'_2 - \frac{1}{2\sqrt{3}}(\mathbf{p}'^\dagger_1 \cdot \mathbf{p}'_2 + \mathbf{p}'^\dagger_2 \cdot \mathbf{p}'_1) + 1, \tag{3.10b}$$

$$\hat{\Pi}^2_3 = \frac{2}{3}\mathbf{p}'^\dagger_2 \cdot \mathbf{p}'_2 + 1 \tag{3.10c}$$

The interesting point is to introduce the oscillator interaction, exactly as in the replacement we made in (2.7) in the non-relativistic problem. For notational purposes we introduce the creation and annihilation operators

$$\boldsymbol{\eta}_s \equiv \frac{1}{\sqrt{2}}(\omega^{\frac{1}{2}}\mathbf{x}'_s - i\omega^{-\frac{1}{2}}\mathbf{p}'_s), s = 1, 2, \tag{3.11a}$$

$$\boldsymbol{\xi}_s \equiv \frac{1}{\sqrt{2}}(\omega^{\frac{1}{2}}\mathbf{x}'_s + i\omega^{-\frac{1}{2}}\mathbf{p}'_s), s = 1, 2, \tag{3.11b}$$

so that the relations (2.7) can be written as

$$\mathbf{p}'_s \rightarrow -i\omega^{\frac{1}{2}}\sqrt{2}\boldsymbol{\xi}_s, \tag{3.12a}$$

$$\mathbf{p}'^\dagger_s \rightarrow i\omega^{\frac{1}{2}}\sqrt{2}\boldsymbol{\eta}_s, \tag{3.12b}$$

Under this replacement the $\hat{\Pi}^2_s$ operators become then

$$\hat{\Pi}^2_1 = \omega[C_{11} + \frac{1}{3}C_{22} + \frac{1}{\sqrt{3}}(C_{12} + C_{21})] + 1 \tag{3.13a}$$

$$\hat{\Pi}_2^2 = \omega[C_{11} + \frac{1}{3}C_{22} - \frac{1}{\sqrt{3}}(C_{12} + C_{21})] + 1 \qquad (3.13b)$$

$$\hat{\Pi}_3^2 = \frac{4}{3}\omega C_{22} + 1, \qquad (3.13c)$$

where the operator $C_{st}, s, t = 1, 2$ are defined by

$$C_{st} = \boldsymbol{\eta}_s \cdot \boldsymbol{\xi}_t \qquad (3.14)$$

From the fact that

$$[\xi_{it}, \eta_{js}] = \delta_{ij}\delta_{st} \; ; \; i, j = 1, 2, 3 \; ; \; s = 1, 2, \qquad (3.15)$$

we have the commutation relations

$$[C_{st}, C_{s't'}] = C_{st'}\delta_{s't} - C_{s't}\delta_{st'}, \qquad (3.16)$$

and thus they are generators [5] of a U(2) group. Therefore the operators $\hat{\Pi}_1^2, \hat{\Pi}_2^2, \hat{\Pi}_3^2$, appearing in the equation (3.6), are linear functions of the generators of this group.

To obtain from Eq.(3.6) the eigenvalues of the energy for this relativistic oscillator problem we can proceed as follows: First we note that the first order Casimir operator of U(2) group is

$$\hat{N} = C_{11} + C_{22}, \qquad (3.17)$$

and that it has an SU(2) subgroup whose generators are

$$\hat{F}_+ \equiv C_{12}, \qquad (3.18a)$$

$$\hat{F}_o \equiv (\tfrac{1}{2})(C_{11} - C_{22}), \qquad (3.18b)$$

$$\hat{F}_- \equiv C_{21}, \qquad (3.18c)$$

with a corresponding Casimir operator of the form

$$\hat{F}^2 \equiv \hat{F}_-\hat{F}_+ + \hat{F}_o(\hat{F}_o + 1). \qquad (3.19)$$

The $\hat{N}, \hat{F}^2$ by definition commute with all C_{st} and among themselves, so from (3.13), they will be integrals of motion of the operator $\Phi(E^2, \Pi_s^2)$. Thus the eigenstates of the Eq. (3.6) can be characterized by the eigenvalues of $\hat{N}, \hat{F}^2$ which we denote respectively by

$$N, f(f+1) \qquad (3.20)$$

with f taking the values $(N/2, (N/2) - 1$, down to $1/2$ or 0 depending on whether N is odd or even.

Another operator that commutes with $\hat{N}, \hat{F}^2$ is obviously $\hat{F}_o$ and we shall designate its eigenvalue by

$$\nu = f, f - 1, \ldots, -f, \tag{3.21}$$

so the eigenstates associated with $\hat{N}, \hat{F}^2, \hat{F}_o$ could be represented by the ket

$$|Nf\nu>, \tag{3.22}$$

and the solution ψ of Eq. (3.6) is necessarily a linear combination of these kets $i.e.$

$$\psi = \sum_{\nu=-f}^{f} a_\nu |Nf\nu>, \tag{3.23}$$

as $\hat{N}, \hat{F}^2$ are integrals of motion.

To obtain the eigenvalues of the internal energy E as function of N, f we need first to consider the matrix elements of the operator Φ of (3.6) in the basis (3.22) $i.e.$

$$< Nf\nu'|\Phi(E^2, \hat{\Pi}_s^2)|Nf\nu>, \tag{3.24}$$

which from (3.13), (3.17), (3.18) is a straightforward, but laborious, calculation of the type familiar in angular momentum theory, as the group there is also SU(2).

To get the internal energy

$$E(N, f, \alpha), \tag{3.25}$$

with α indicating the rest of the indices, we need to evaluate the determinant of the $(2f + 1) \times (2f + 1)$ matrix whose elements are (3.24) and equate it to zero. This gives us a numerical equation of degree $4(2f + 1)$ in the variable E^2 and its solution provides us with values indicated symbolically in (3.25).

As our purpose is to provide the *method* of solution for the internal energy of relativistic three body oscillator problem, we will only carry the calculation of (3.24) for the single case when

$$N = f = \nu = 0 \tag{3.26}$$

which implies that

$$< 000|\hat{\Pi}_s^2|000 >= 1, \quad s = 1, 2, 3 \tag{3.27}$$

and so A, B, C^2 in (3.4) become respectively

$$A = 3, \qquad B = -3, \qquad C^2 = 16, \tag{3.28}$$

and the equation for the energy is given by

$$E^8 - 12E^6 + 30E^4 - 28E^2 + 9 = 0, \qquad (3.29)$$

whose four roots for E^2 are $E^2 = 1$, repeated three times, and $E^2 = 9$, with $E = \pm 3$ and ± 1 as we expect from (3.1).

So far we have discussed, and given a method for solving, the equation related with a three body relativistic problem with an oscillator interaction. In the next section we proceed to show that we can formulate it in a Poincaré invariant form.

4 Poincaré invariance of the three body relativistic equation with oscillator interactions

To express Eq. (3.6) in a Poincaré invariant form we start with definition of the total four momentum for the three particle problem $i.e.$

$$P_\mu = p_{\mu 1} + p_{\mu 2} + p_{\mu 3}, \qquad (4.1)$$

where $\mu = 0, 1, 2, 3$ with $p_{os}, s = 1, 2, 3$, being the time like component while $p_{is}, i = 1, 2, 3$, are the space like components of the vector $\mathbf{p}_s$ of the previous section.

We shall require also a unit time like four vector u_μ which we shall define as

$$u_\mu = P_\mu (-g^{\sigma\tau} P_\sigma P_\tau)^{-\frac{1}{2}}, \qquad (4.2)$$

where repeated indices σ, τ are summed over 0,1,2,3 and our metric tensor is taken as

$$g^{\sigma\tau} = 0 \quad \text{if} \quad \sigma \neq \tau, \quad g_{11} = g_{22} = g_{33} = -g_{oo} = 1 \qquad (4.3)$$

Clearly in the center of mass frame where $P_i = 0, i = 1, 2, 3, u_\mu$ takes the value

$$(u_\mu) = (1, 0, 0, 0) \qquad (4.4)$$

The operators $\boldsymbol{\eta}_s, \boldsymbol{\xi}_s, s = 1, 2$, defined in (3.11) are space like three component vectors which could be denoted by $\eta_{is}, \xi_{is}, i = 1, 2, 3$. A time like component could be added through the definition (3.11) just by putting p'_{os}, x'_{os} instead of p'_{is}, x'_{is} and thus we would get η_{os}, ξ_{os} which, together with η_{is}, ξ_{is}, form the four vectors

$$\eta_{\mu s}, \xi_{\mu s}; \quad \mu = 0, 1, 2, 3; \quad s = 1, 2. \qquad (4.5)$$

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We do not want to use these operators directly in the definition of the C_{st} of (3.14), but rather utilize their transversal parts defined by

$$\eta_{\mu s}^{\perp} \equiv \eta_{\mu s} + (g^{\sigma\tau}\eta_{\sigma s}u_{\tau})u_{\mu}, \qquad (4.6a)$$

$$\xi_{\mu s}^{\perp} \equiv \xi_{\mu s} + (g^{\sigma\tau}\xi_{\sigma s}u_{\tau})u_{\mu}. \qquad (4.6b)$$

These transverse operators have the property that in the center of mass frame where $(u_{\mu}) = (1000)$ we have, because of the matrix (4.3), that

$$\eta_{os}^{\perp} = 0, \quad \xi_{os}^{\perp} = 0, \quad \eta_{is}^{\perp} = \eta_{is}, \quad \xi_{is}^{\perp} = \xi_{is}. \qquad (4.7)$$

Thus now the generator $C_{st}, s, t = 1, 2$, appearing in the definitions (3.13) of $\hat{\Pi}_1^2, \hat{\Pi}_2^2, \hat{\Pi}_3^2$ can be expressed in a Lorentz invariant way by

$$C_{st} = g^{\sigma\tau}\eta_{\sigma s}^{\perp}\xi_{\tau t}^{\perp}, \qquad (4.8)$$

as in the center of mass frame it takes the form (3.14) i.e. $C_{st} = \boldsymbol{\eta}_s \cdot \boldsymbol{\xi}_t$.

As for the energy E^2 appearing in Eq. (3.6) it can be substituted by the operator

$$E^2 \rightarrow (-g^{\sigma\tau}P_{\sigma}P_{\tau}), \qquad (4.9)$$

because in the center of mass frame $P_i = 0, i = 1, 2, 3$, and from the metric tensor (4.3), we see that the parenthesis in (4.9) reduces to P_o^2, which is the time like component of the four momentum vector squared and thus corresponds to the square of the total energy of the system.

With the definitions (4.8) of C_{st} and (4.9) of E^2 substituted in Eq.(3.6) we get a Poincaré invariant equation for our problem, as C_{st}, given in terms of Jacobi coordinates and momenta, is also invariant under translation in space time, and thus commutes with P_{μ}.

We have then arrived at a procedure for deriving a Poincaré invariant equation for a three particle system with oscillator interactions which, in the center of mass reference frame, can be solved by a simple group theoretical procedure, which leads eventually to algebraic equations of degree $4(2f + 1)$ for E^2, that can be solved numerically to give the spectrum of the problem.

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