1995116579 N95- 22996

Pseudoclassical description of the Dirac Oscillator.

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Abstract

In this paper we discuss the Dirac Oscillator wave equation in terms of a pseudoclassical language, using Grassmann variables to describe the internal degrees of freedom of the oscillator. Regarding the original wave equation as a classical constraint, we use the theory of constrained systems, to develop a reparametrization invariant lagrangian, which is the pseudoclassical equivalent of the quantum case. The consistency of the Hamiltonian formalism and the quantization procedure are also analized.

1. Introduction

As is well known, in the second decade of this century Dirac developed the square root method, to analyze the internal spin degrees of freedom in quantum mechanics. Dirac accomplished his task by means of a Clifford algebra for these degrees. Altought at that time the Grassmann algebras were already known, there existed no classical counterpart available for his approach. In present day point of view, however, we know that this old problem can be formulated using anticommuting, fermionic variables, to reproduce the behavior of the spin degrees of freedom at the classical level. Since the Grassmann variables have no direct physical meaning, the theories formulated with them are usually called pseudoclassical.

One important point of the approach formulated above is that, associated with the anticommuting variables and involving also the rest of the non-Grassmann dynamical variables of the theory, (called bosonic variables), there exist a supersymmetric gauge invariance in the formulation. One of the reasons for this supersymmetry is the fact that any quantum wave equation present in the theory is translated at the classical level as a first class constraint. According to Dirac conjecture, all first class constraints generate gauge transformations, but since in this case the Grassmann and the bosonic variables are mixed up by the gauge transformation they become, in fact, supersymmetric.

This way of reasoning has been analyzed by several authors [1,2,3,4] as a pevious step to quantization. The idea is in some sense based in Dirac's point of view that we should first try to fully understand a problem at a classical level, and only then try to quantize it [5]. One consequence of this procedure is that we can apply it to systems which we don't fully understand, for instance in the case of two body, or more, relativistic wave equations [1]. The interesting point here is that the time evolution of the dynamical variables are just the Euler Lagrange equations, which in principle are known, thus solving the dynamics of the problem, at least at the classical level.

2. The Dirac Oscillator

Let us begin with a simple introduction to the Dirac Oscillator. Some years ago, Moshinsky and Szczepaniak introduced a relativistic wave equation linear in momentum and in position which has an harmonic oscillator spectrum plus a strong spin-orbit coupling term [6]. This equation is obtained by the replacement of the momentum of the particle in the Dirac equatiom by

$$\mathbf{p} \rightarrow \mathbf{p} - im\omega\mathbf{r}\beta, \qquad 1)$$

where $\mathbf{p}$ is the momentum, m the mass of the particle and r is its position, ω is the frequency of the oscillator, and β the Dirac γ^0 matrix. Taking advantage of the frame dependence vector u^μ, it is easy to show that the Dirac oscillator equation can be put in the manifestly covariant form

$$(\theta \cdot P + m\theta_5)\,\psi = 0, \tag{2}$$

where

$$P^{\mu} \equiv p^{\mu} - 2im\omega x_{\perp}^{\mu}(\hat{u}\cdot\theta)\theta_5, \tag{3.a}$$

and

$$x_{\perp}^{\mu} \equiv x^{\mu} + (\hat{u}\cdot x)\hat{u}^{\mu} \tag{3.6}$$

The operators θ^{μ} and θ_5, are expressed in terms of the Dirac matrices in the following way

$$\begin{aligned}\theta^{\mu} &\equiv \frac{i}{\sqrt{2}}\gamma_5\gamma^{\mu}, \quad \mu = 0,1,2,3\\ \theta_5 &= \frac{i}{\sqrt{2}}\gamma_5,\end{aligned} \tag{3.c}$$

where $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. We use natural units in which $\hbar = c = 1$ and our metric is given by $(\eta_{\mu\nu}) = \text{diag.}(-1,1,1,1)$. We recover the original Dirac Oscillator in the frame in which $\hat{u}^{\mu} = (1,0,0,0)$. The solution, spectrum degeneracy, hidden supersymetry and other important properties are discussed in [8,9], (and references therein).

3. Pseudoclassical description

Now, the idea is to reformulate the problem in a pseudoclassical language. This is done in a natural way by translating Eq. 2) into a first class constraint

$$\mathcal{J} \equiv \theta \cdot P + m\theta_5 \approx 0, \tag{4}$$

where $\approx$ means weak equality and the dynamical variables become pseudoclassical ones

$$\begin{aligned}\{\theta^{\mu},\theta^{\nu}\} &= i\eta^{\mu\nu},\\ \{\theta_5,\theta_5\} &= i.\end{aligned} \tag{5.a}$$

Of course we know that

$$\{x^{\mu},p^{\nu}\} = \eta^{\mu\nu}. \tag{5.b}$$

The Poisson braket of the first class constraint $\mathcal{J}$ with itself, is the classical equivalent of the square of the Dirac Oscillator. In this way, thus, we generate another constraint which could be called the pseudoclassical analogue of the Klein Gordon equation. This new constraint is constructed in such a way to be again a first class, altought secondary, constraint. Of course, if there is any second class constraint we should replace the Poisson braket for the Dirac one, in order to get rid of them.

In our case the superalgebra obeyed by the constraints is given by

$$\begin{aligned} \{\mathcal{J},\mathcal{J}\} &= i\mathcal{H} = i(P^2 + m^2 - 4im\omega\lambda) \approx 0, \\ \{\mathcal{J},\mathcal{H}\} &= 0, \\ \{\mathcal{H},\mathcal{H}\} &= 0. \end{aligned} \qquad 6)$$

In this equation, λ is given by

$$\lambda \equiv (\theta \cdot x_\perp)\hat{u} \cdot (m\theta - \theta_5 P). \qquad 7)$$

Notice that the first of this constraints is precisely the pseudoclassical Klein Gordon equation. Is in this sense that we say that we translate the square root method into a classical language. We also note from Eq. 7) that $\mathcal{J}$ and $\mathcal{H}$ are first class.

Following the procedure described in Ref. [7], we can construct the Lagrangian of the problem. Since the whole dynamics of the theory is contained in the constraints, the Hamiltonian of the system is a linear combination of them

$$H = N\mathcal{H} + iM\mathcal{J} \equiv 0, \qquad 8)$$

where N and M are gauge fixing parameters. This in turn means that the Hamiltonian is weakly zero, implying a reparametrization invariant Lagrangian as a consequence. According to Ref. [7] the whole action is given by

$$\begin{aligned} S = \int_{\tau_2}^{\tau_1} d\tau \{ &-m\sqrt{-z^2}[1 - 2i\omega(\theta \cdot x_\perp)(\hat{u} \cdot \theta)] + i/2[\dot{\theta}^\mu \theta_\mu + \dot{\theta}_5 \theta_5] \\ &- 2im\omega(\theta \cdot x_\perp)\theta_5(\hat{u} \cdot z) + 2im\omega(z \cdot x_\perp)(\hat{u}\theta)\theta_5 \\ &- 2m\omega M(\theta \cdot x_\perp)(\hat{u} \cdot \theta)\theta_5 - imM\theta_5. \end{aligned} \qquad 9)$$

Where $z^\mu \equiv \dot{x}^\mu - iM\theta^\mu$.

It is not hard to prove that this action is the correct answer to our problem. From the Hamiltonian formalism, we know that the time evolution of any dynamical variable F is given by

$$\dot{F} = \{F, H\}. \qquad 10)$$

Hence, for the dynamical variables in our case we obtain that

$$\dot{x}^\mu = iM\theta^\mu + 2NP^\mu + 4im\omega N(\theta\cdot x_\perp)\,\theta_5 u^\mu$$
$$\dot{\theta}^\mu = 4m^2\omega N[x_\perp^\mu(\hat{u}\cdot\theta) - (\theta\cdot x_\perp)\,\hat{u}^\mu] + MP^\mu \qquad 11)$$
$$\dot{\theta}_5 = mM - 2imM\omega(\theta\cdot x_\perp)(\hat{u}\cdot\theta) - 4m\omega N(x_\perp\cdot P)(\hat{u}\cdot\theta) + 4m\omega N(\theta\cdot x_\perp)(\hat{u}\cdot P),$$

which are precisely the Euler-Lagrange equations of the action 9) provided

$$N(\tau) = \frac{\sqrt{-z^2}}{2m}[1 + 2i\omega(\theta\cdot x_\perp)(\hat{u}\cdot\theta)]. \qquad 12)$$

This result proves the complete agreement between the Hamiltonian and Lagrangian formalisms, and as a result we are confident that action 9) is the corect answer. What Eq. 12) tells us is that we can specify the gauge by giving a value to $\sqrt{-z^2}$, usually $\sqrt{-z^2} = -1$. In the same way we can construct the supergauge transformations for each dynamical variable F by means of the equations

$$\delta F = \{F, \epsilon^a(\tau)\phi_a\}, \quad \mathrm{a} = 1,2 \qquad 13)$$

where ϕ_a represent our two constraints, and $\epsilon^a(\tau)$ are two time dependent infinitesimal parameters. The result is too long to be given here, (see reference [7]), the only point we want to remark here is that, as we already mentioned, they express the full dynamics of the theory, as is suggested by the comparisson of Eqs. 10) and 13).

4. Conclusions

We note form Eq. 3.a) that the Dirac Oscillator interaction term is θ-dependent. In cases like this, the quantization procedure should be done carefully, since some properties of the Grassmann variables changes radically when quantized. For example, the θ_5 variable has the property that ${\theta_5}^2 = 0$ at the classical level, but at the quantum level (Eq. 3.c) ${\theta_5}^2 = -1/2$. Thus if we consider for instance, the Taylor expansion of a θ_5 dependent function, we obtain different results depending wether the quantization is done before or after the series expansion.

In the case of the system studied here, if we put in Eq. 6) the square of P^μ given by 3.a) and proceed to quantize by means of definitions (3.c), we obtain a wrong result. The central point here, is that we should regard P^μ as a basic quantity to quantize. If instead of developing the square of P^μ, we first quantize Eq. 6) and regard 3.a) as a quantum operator identity, we obtain a complete compatibilty with the Klein Gordon wave equation associated with the Dirac Oscillator. Finally, we would like to remark that our approach could be useful to problems that are not fully understood at the quantum level, but that have θ-dependent interaction terms, such as the afforementioned two body relativistic wave equations and some supergravity theories [7].

ACKNOWLEDGEMENTS

This work was partially suported by grants DGAPA IN 101593 and ESP 100191.

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