NOTES ON OSCILLATOR-LIKE INTERACTIONS OF VARIOUS SPIN RELATIVISTIC PARTICLES*

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Abstract

The equations for various spin particles with oscillator-like interactions are discussed in this talk. Contents: 1. Comment on "The Klein-Gordon Oscillator"; 2. The Dirac oscillator in quaternion form; 3. The Dirac-Dowker oscillator; 4. The Weinberg oscillator; 5. Note on the two-body Dirac oscillator.

1 Comment on "The Klein-Gordon Oscillator"

In connection with the publications of Moshinsky *et al.*, e. g. [1], the interest in the model with the $j = 1/2$ Hamiltonian that is linear in both momenta and coordinates has grown recently [2]. Analogous type of interaction has been considered for the case of $j = 0$ and $j = 1$ Duffin-Kemmer field [3] and for the case of $j = 0$ Klein-Gordon field [4].

In the paper [4] the operators $\vec{Q}$, coordinate, and $\vec{P}$, momentum, have been represented in $n \otimes n$ matrix form

$$\vec{Q} = \hat{\eta}\vec{q}, \qquad \vec{P} = \hat{\eta}\vec{p}, \tag{1.1}$$

with $\hat{\eta}^2 = 1$. The interaction in the Klein-Gordon equation has been introduced in the following way:

$$\vec{P} \rightarrow \vec{P} - im\hat{\gamma}\hat{\Omega} \cdot \vec{Q}, \tag{1.2}$$

where for the sake of completeness $\hat{\Omega}$ is chosen by $3 \otimes 3$ matrix with coefficients $\hat{\Omega}_{ij} = \omega_i \delta_{ij}$. The $\hat{\gamma}$ matrix obeys the following anticommutation relations $\{\hat{\gamma}, \hat{\eta}\} = 0$, $\hat{\gamma}^2 = 1$.

The Klein-Gordon equation for $\Psi(\vec{q}, t)$, the wave function which could be expanded in two-component form, is then

$$-\frac{\partial^2}{\partial t^2}\Psi(\vec{q}, t) = \left(\vec{p}^2 + m^2\vec{q} \cdot \hat{\Omega}^2 \cdot \vec{q} + m\hat{\gamma}\, tr\Omega + m^2\right)\Psi(\vec{q}, t), \tag{1.3}$$

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what gives the energy spectrum [4]

$$E^2_{(a)N_i} - m^2 = 2m\left(\omega_1 N_1 + \omega_2 N_2 + \omega_3 N_3\right), \qquad N_1, N_2, N_3 = 0, 1, 2\ldots$$

$$E^2_{(b)N_i} - m^2 = 2m\left(\omega_1(N_1 + 1) + \omega_2(N_2 + 1) + \omega_3(N_3 + 1)\right). \tag{1.4}$$

However, the physical sense of implementing the matrices $\hat{\eta}$ and $\hat{\gamma}$ in [4] is obscure. In this Section we try to attach some physical foundations to this procedure. It is well-known some ways to recast the Klein-Gordon equation in the Hamiltonian form e. g. [5, 6]. First of all, the Klein-Gordon equation could be re-written to the system of two coupled equations [6,p.98]

$$\frac{\partial \Psi}{\partial x^\alpha} = \kappa \Xi_\alpha, \qquad \frac{\partial \Xi^\alpha}{\partial x^\alpha} = -\kappa \Psi, \tag{1.5}$$

where $\kappa = mc/\hbar$ (in the following we use the system where $c = \hbar = 1$). By means of redefining the components they are easy to present in the matrix Hamiltonian form (cf. with [7])

$$i\frac{\partial}{\partial t}\begin{pmatrix} \phi \\ \chi_1 \\ \chi_2 \\ \chi_3 \end{pmatrix} = \left[\begin{pmatrix} 0 & p_1 & p_2 & p_3 \\ p_1 & 0 & 0 & 0 \\ p_2 & 0 & 0 & 0 \\ p_3 & 0 & 0 & 0 \end{pmatrix} + m\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\right]\begin{pmatrix} \phi \\ \chi_1 \\ \chi_2 \\ \chi_3 \end{pmatrix} \tag{1.6}$$

provided that $\phi = i\partial_t \Psi + m\Psi$, $\chi_i = -i\nabla_i \Psi = \vec{p}_i \Psi$. Using matrices $\vec{\alpha}$ and β, corresponding to this case, and introducing interaction analogously [1a] we come to the equation for upper component

$$(E^2 - m^2)\phi = \left[\vec{p}^2 + m^2\omega^2\vec{r}^2 - 3m\omega\right]\phi \tag{1.7}$$

what coincides with Eq. (10a) of ref. [4] in the case $\omega_1 = \omega_2 = \omega_3$.

The similar formulation also originated from the Duffin-Kemmer approach. In this case the wave function $\Phi = column(\phi_1, \phi_2, \chi_i)$ is five-dimensional and its components are $\phi_1 = (i\partial_t \Psi + m\Psi)/\sqrt{2}$, $\phi_2 = (i\partial_t \Psi - m\Psi)/\sqrt{2}$, $\chi_i = -i\nabla_i \Psi = \vec{p}_i \Psi$. It satisfies the equation

$$i\frac{\partial \Phi}{\partial t} = \left(\vec{B}\vec{p} + m\beta_0\right)\Phi, \qquad B_\mu = [\beta_0, \beta_\mu]_- \tag{1.8}$$

(our choice of $5 \otimes 5$ dimension β-matrices corresponds to ref. [3]). As shown there, the substitution $\vec{p} \to \vec{p} - im\omega\eta_0\vec{r}$ leads to the equation (1.7) for both ϕ_1 and ϕ_2. Let us remark, in both the approach based on Eq. (1.6) and the Duffin-Kemmer approach, Eq. (1.8), we have the equation

$$(E^2 - m^2)\chi_i = (p_i - im\omega x^i)(p_j + im\omega x^j)\chi_j \tag{1.9}$$

for down component, which seems to not be reduced to oscillator-like equation.

Then, Sakata-Taketani approach, e. g. [5], is characterized by the equation which we write in the form:

$$i\frac{\partial}{\partial t}\Phi = \left\{\frac{\vec{p}(\tau_3 + i\tau_2)\vec{p}}{2m} + m\tau_3\right\}\Phi, \tag{1.10}$$

with τ_i being the Pauli matrices. $\Phi = column(\phi, \chi)$ is the two-component wave function with components which could be written as following: $\phi = (\Psi + \frac{i}{m}\partial_t \Psi)/\sqrt{2}$, $\chi = (\Psi - \frac{i}{m}\partial_t \Psi)/\sqrt{2}$.

From the previous experience we learned that in order to get the oscillator-like equation we need to do substitution with matrix which anticommutes with matrix structure of the momentum part of the equation. In our case the matrix which has this property is τ_1 matrix. Therefore, we do the substitution $\vec{p} \to \vec{p} - im\omega\tau_1\vec{r}$ and come to

$$E^2\xi = \left[\vec{p}^2 + m^2\omega^2\vec{r}^2 - 3m\omega + m^2\right]\xi, \tag{1.11}$$

where $\xi = \phi + \chi$, and to the analogous equation for $\eta = \phi - \chi = \frac{E}{m}(\phi + \chi)$. In the process of calculations we convinced ourselves that the interaction Hamiltonian

$$\mathcal{H} = \frac{1}{2m}\left(\vec{p}^2 + m^2\omega^2\vec{x}^2 - 3m\omega\right)(\tau_3 + i\tau_2) + m\tau_3 \tag{1.12}$$

is the same as in [3, formula (3.9)] since $\tau_1(\tau_3 + i\tau_2)\tau_1 = -(\tau_3 + i\tau_2)$ and $(\tau_3 + i\tau_2)\tau_1 = \tau_3 + i\tau_2$.

2 The Dirac oscillator in quaternion form

The quaternion (and conjugated to it) with real coefficient is defined as $q = q_0 + iq_1 + jq_2 + kq_3$, $\bar{q} = q_0 - iq_1 - jq_2 - kq_3$. The basis vectors satisfy the equations $i^2 = j^2 = k^2 = -1$ and $ij = -ji = k$ with cyclic permutations. Considering a two-component quaternionic spinor (or $SL(2, H)$ spinor) one could write the free Dirac equation as following, ref. [8c,d],

$$\tilde{\Gamma} \cdot \partial\Psi - m\tau_3\Psi k = 0. \tag{2.1}$$

Anticommutation relations for $\tilde{\Gamma}$ are given in [8d,p.222]. In Pauli representation ($i \to -\sqrt{-1}\tau_1$, $j \to -\sqrt{-1}\tau_2$ and $k \to -\sqrt{-1}\tau_3$) it goes through to usual Dirac equation and its complex conjugate. As mentioned in [8] it is convenient to diagonalize the matrices entering in Eq. (2.1) using matrix

$$T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -\sqrt{-1} \\ 1 & \sqrt{-1} \end{pmatrix}. \tag{2.2}$$

In such a way we come to biquaternionic formulation ($q_i \in \mathbb{C}$):

$$\begin{cases} \bar{\partial}\psi_L + im\bar{\psi}_R^\dagger = 0 \\ \partial\bar{\psi}_R^\dagger + im\psi_L = 0, \end{cases} \tag{2.3}$$

where $\psi_L \equiv \psi p_+$, $\psi_R \equiv \psi p_-$. This decomposition of Ψ into left ideals is carried out by means of the projection operators $p_\pm = (b_0 \pm b_3)/2$. New basis is $b_0 \equiv 1, b_1 \equiv \sqrt{-1}i, b_2 \equiv \sqrt{-1}j, b_3 \equiv \sqrt{-1}k$ and $\bar{b}_0 = b_0, \bar{b}_\alpha = -b_\alpha$. Introducing interaction in the form $\partial_i \to \partial_i + \tau_3 V_i(\vec{x})$, V is the compensating field for this type of $Sp(1, Q)$ transformations, and taking into account that the vectors of biquaternionic basis anticommute $b_\alpha \bar{b}_\beta + b_\beta \bar{b}_\alpha = -2\eta_{\alpha\beta}$, $\eta_{\alpha\beta} = diag(-1, 1, 1, 1)$, we come to the equations for the left and right spinor-quaternions in the following form:

$$(E^2 - m^2)\psi_L = \left[(\vec{p}^2 + k^2\vec{x}^2) - 3k - 2\epsilon_{ijk}b_k x_i p_j\right]\psi_L \tag{2.4}$$

$$(E^2 - m^2)\bar{\psi}_R^\dagger = \left[(\vec{p}^2 + k^2\vec{x}^2) + 3k + 2\epsilon_{ijk}b_k x_i p_j\right]\bar{\psi}_R^\dagger \tag{2.5}$$

if we choose $V_i(\vec{x}) = kx_i$. Eqs. (2.4) and (2.5) are the Dirac oscillator equations in the Pauli rep, $b_k \to \tau_k$. Analogous equations for ψ_R and $\bar{\psi}_L^\dagger$ could be obtained from (2.3) if one choose the opposite signs at the mass terms.

3 The Dirac-Dowker oscillator

In this Section we start from the equation for any spin given by Dirac [9] in the form written down by Corson, ref. [6,p.154], (here we use Corson's notation)

$$\begin{cases} \partial^{\dot{A}B} v_B(k + \tfrac{1}{2})\psi(k + \tfrac{1}{2}, l - \tfrac{1}{2}) - m \left(\tfrac{2k+1}{2l}\right)^{1/2} v^{\dot{A}}(l)\psi(k,l) = 0 \\ \partial_{\dot{A}B} v^{\dot{A}}(l)\psi(k,l) + m \left(\tfrac{2l}{2k+1}\right)^{1/2} v_B(k + \tfrac{1}{2})\psi(k + \tfrac{1}{2}, l - \tfrac{1}{2}) = 0, \end{cases} \tag{3.1}$$

where v_A and $v_{\dot{A}}$ are the rectangular spinor-matrices of $2k$ rows and $2k + 1$ columns (see, e. g., section 17b of ref. [6] for the details). The wave function $\psi(k,l)$ belongs to the (k,l) representation of the homogeneous Lorentz group. The choice $l = 1/2$ and $k = j - 1/2$, j is the spin of a particle, permits one to reduce a number of subsidiary conditions. Moreover, the equations (3.1) are shown by Dowker [10] to recast to the matrix form which is similar to the well-known Dirac equation for $j = 1/2$ particle

$$\begin{cases} \alpha^\mu \partial_\mu \Phi = m\Upsilon \\ \bar{\alpha}^\mu \partial_\mu \Upsilon = -m\Phi. \end{cases} \tag{3.2}$$

The $4j$- component function Φ could be identified with the wave function in $(j,0) \oplus (j-1,0)$ representation. Then, Υ, which also has $4j$ components, is written down

$$\Upsilon = (-1)^{2j}(2j)^{-\frac{1}{2}} \begin{pmatrix} v_{\dot{A}}(j - \tfrac{1}{2}) \otimes v^{\dot{A}}(\tfrac{1}{2}) \\ u_{\dot{A}}(j) \otimes v^{\dot{A}}(\tfrac{1}{2}) \end{pmatrix} \psi(j - \tfrac{1}{2}, \tfrac{1}{2}). \tag{3.3}$$

and it belongs to $(j - 1/2, 1/2)$ representation. The matrices α^μ and $\bar{\alpha}^\mu = \alpha_\mu$ obeys all the algebraic relations of the Pauli matrices $\bar{\alpha}^{(\mu}\alpha^{\nu)} = g^{\mu\nu}$, except for completeness.

Defining $p_\mu = -i\partial_\mu$ and the analogs of γ- matrices as following:

$$\gamma^\mu = \begin{pmatrix} 0 & -i\bar{\alpha}^\mu \\ i\alpha^\mu & 0 \end{pmatrix} \tag{3.4}$$

the set of equations (3.2) is written down to the form of the Dirac equation

$$(p_\mu \gamma^\mu - m) \begin{pmatrix} \Phi \\ \Upsilon \end{pmatrix} = 0. \tag{3.5}$$

However, let us not forget that Φ and Υ are 2-spinors only in the case of $j = 1/2$.

In the case of spin $j = 1/2$ it is well-known the set of γ- matrices is defined up to the unitary transformation and Eq. (3.5) could be recast to the Hamiltonian form given by Dirac (with α_k and β matrices) by means of the unitary matrix. It is easy to carry out the same procedure ($\alpha^k = S\gamma^0\gamma^k S^{-1}$ and $\beta = S\gamma^0 S^{-1}$) for γ- matrices, Eq. (3.4), and functions of arbitrary spin ($\Psi = S^{-1}\Phi$). For our aims it is convenient to chose the unitary matrix as following:

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1}_{4j\otimes 4j} & i\mathbb{1}_{4j\otimes 4j} \\ i\mathbb{1}_{4j\otimes 4j} & \mathbb{1}_{4j\otimes 4j} \end{pmatrix}. \tag{3.6}$$

After standard substitution $\vec{p} \to \vec{p} - im\omega\gamma^0\vec{r}$ we obtain

$$E\phi = -i\left[\alpha_0(\vec{\alpha}\vec{p}) + im\omega(\vec{\alpha}\vec{r})\right]v + m\alpha_0\phi, \tag{3.7}$$

$$Ev = i\left[\alpha_0(\vec{\alpha}\vec{p}) - im\omega(\vec{\alpha}\vec{r})\right]\phi - m\alpha_0 v. \tag{3.8}$$

Since it follows from the anticommutation relations that $\alpha_i\alpha_0 = \alpha_0\alpha_i$ we have the equations which coincide with Eq. (8) of ref. [1a] or Eqs. (3.6) and (3.12) of ref. [1d] except for $\tau_\mu \to \alpha_\mu$, i. e. their explicit forms,

$$(E^2 - m^2)\phi = \left[\vec{p}^2 + m\omega\vec{r}^2 - 3\alpha_0 m\omega - m\omega\alpha_0\alpha^{[i}\bar{\alpha}^{j]}r^i\nabla_j\right]\phi \tag{3.9}$$

$$(E^2 + m^2)v = \left[\vec{p}^2 + m\omega\vec{r}^2 + 3\alpha_0 m\omega + m\omega\alpha_0\alpha^{[i}\bar{\alpha}^{j]}r^i\nabla_j\right]v. \tag{3.10}$$

Thus, we convinced ourselves that we got the same oscillator-like interaction and the similar spectrum as for the case of $j = 1/2$ particles in [1a].

4 The Weinberg oscillator

The principal equation of $2(2j + 1)$- component approach [11] in the case of spin $j = 1$ is

$$(\gamma_{\mu\nu}p_\mu p_\nu + M^2)\Psi^{(j=1)}(x) = 0, \tag{4.1}$$

with $\gamma_{\alpha\beta}$ being $6 \otimes 6$ covariantly defined matrices. The $j = 1$ Hamiltonian has been given in refs. [11b,c]:

$$\mathcal{H} = \frac{2E^2}{2E^2 - M^2}(\vec{\alpha}\vec{p}) + \beta\left[E - \frac{2E}{2E^2 - M^2}(\vec{\alpha}\vec{p})^2\right], \tag{4.2}$$

where

$$\vec{\alpha} \equiv \begin{pmatrix} \vec{S} & 0 \\ 0 & -\vec{S} \end{pmatrix}, \qquad \beta \equiv \begin{pmatrix} 0 & \mathbb{1}_{3\otimes3} \\ \mathbb{1}_{3\otimes3} & 0 \end{pmatrix}.$$

(S_i are the spin matrices for a vector particle).

In general, the upper and down components of 6- component wave function do not uncouple neither under the interaction $\vec{p} \to \vec{p} - im\omega\beta\vec{r}$ nor under $\gamma_{5,\mu\nu}u_\mu r_\nu$. However, if we introduce the Dirac oscillator interaction so that the conditions of the longitudity of $\Psi = column(\phi_i, \chi_i)$ respective to $\vec{r}$, i. e. $\vec{r}\times\vec{\phi} = 0$, $\vec{r}\times\vec{\chi} = 0$ are fulfilled, we come to the equations more simple

$$(2E^2 - M^2)\xi = E(\vec{S}\vec{p})\eta + \left[(\vec{S}\vec{p}) - k(\vec{S}\vec{r})\right](\vec{S}\vec{p})\xi, \tag{4.3}$$

$$E(\vec{S}\vec{p})\xi = \left[(\vec{S}\vec{p}) + k(\vec{S}\vec{r})\right](\vec{S}\vec{p})\eta \tag{4.4}$$

($\xi = \phi - \chi$, $\eta = \phi + \chi$), which could be uncoupled to the following form ($k = im\omega$)

$$(\vec{S}\vec{p})(E^2 - M^2)(\vec{S}\vec{p})\xi = (\vec{S}\vec{p})\left[\vec{p}^2 + m\omega^2\vec{r}^2 + 3m\omega + 4m\omega\vec{S}[\vec{r}\times\vec{p}]\right](\vec{S}\vec{p})\xi \tag{4.5}$$

$$(\vec{S}\vec{p})(E^2 - M^2)(\vec{S}\vec{p})\eta = (\vec{S}\vec{p})\left[\vec{p}^2 + m\omega^2\vec{r}^2 - 3m\omega - 4m\omega\vec{S}[\vec{r}\times\vec{p}]\right](\vec{S}\vec{p})\eta -$$
$$-im\omega(2E^2 - M^2)(\vec{S}\vec{r})(\vec{S}\vec{p})\eta \tag{4.6}$$

These equations can be considered as the extension of the equations with Dirac oscillator interaction to the $j = 1$ case, for the components $(\vec{S}\vec{p})\xi$ and $(\vec{S}\vec{p})\eta$. However, remark that one has the additional spin-orbit term acting as earlier at η.

5 Note on the two-body Dirac oscillator

The two-body Dirac Hamiltonian with oscillator-like interaction is given by (see, e. g., ref. [1c])

$$\mathcal{H}\psi == \left[\frac{1}{\sqrt{2}}(\vec{\alpha}_1 + \vec{\alpha}_2)\cdot\vec{P} + \frac{1}{\sqrt{2}}(\vec{\alpha}_1 - \vec{\alpha}_2)\cdot\vec{p} - \frac{i}{\sqrt{2}}(\vec{\alpha}_1 - \vec{\alpha}_2)\cdot\vec{r}B + m(\beta_1 + \beta_2)\right]\psi. \tag{5.1}$$

In the c.m.s. it is possible to equate $\vec{P} = 0$. The matrices are given by the direct products

$$\vec{\alpha}_1 = \begin{pmatrix} 0 & \vec{\sigma}_1 \\ \vec{\sigma}_1 & 0 \end{pmatrix} \otimes \begin{pmatrix} \mathbb{1}_{2\otimes2} & 0 \\ 0 & \mathbb{1}_{2\otimes2} \end{pmatrix}, \quad \vec{\alpha}_2 = \begin{pmatrix} \mathbb{1}_{2\otimes2} & 0 \\ 0 & \mathbb{1}_{2\otimes2} \end{pmatrix} \otimes \begin{pmatrix} 0 & \vec{\sigma}_2 \\ \vec{\sigma}_2 & 0 \end{pmatrix}, \tag{5.2}$$

$$B = \beta_1 \otimes \beta_2 = \begin{pmatrix} \mathbb{1}_{2\otimes2} & 0 \\ 0 & -\mathbb{1}_{2\otimes2} \end{pmatrix} \otimes \begin{pmatrix} \mathbb{1}_{2\otimes2} & 0 \\ 0 & -\mathbb{1}_{2\otimes2} \end{pmatrix}, \tag{5.3}$$

$$\Gamma_5 = \gamma_1^5 \otimes \gamma_2^5 = \begin{pmatrix} 0 & \mathbb{1}_{2\otimes2} \\ \mathbb{1}_{2\otimes2} & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & \mathbb{1}_{2\otimes2} \\ \mathbb{1}_{2\otimes2} & 0 \end{pmatrix}. \tag{5.4}$$

Now we apply the same procedure like that was used for transformation the Bargmann-Wigner equation to the Proca equations. The 16- component wave function of the two-body Dirac equation could be expanded on the complete set of matrices: $(\gamma^\mu C)$, $(\sigma^{\mu\nu}C)$ and C, $(\gamma^5 C)$, and $(\gamma^5\gamma^\mu C)$. We consider the system multiplied by C, the matrix of charge conjugation, in order to trace for the symmetric properties under oscillator-like potentials. The wave function is decomposed in symmetric and antisymmetric parts using the above-mentioned complete system of matrices:

$$\psi_{\{\alpha\beta\}} = \gamma^\mu_{\alpha\eta}C_{\eta\beta}A_\mu + \sigma^{\mu\nu}_{\alpha\eta}C_{\eta\beta}F_{\mu\nu} \tag{5.5}$$

$$\psi_{[\alpha\beta]} = C_{\alpha\beta}\phi + \gamma^5_{\alpha\eta}C_{\eta\beta}\tilde{\phi} + \gamma^5_{\alpha\epsilon}\gamma^\mu_{\epsilon\eta}C_{\eta\beta}\tilde{A}_\mu. \tag{5.6}$$

In such a way we obtain the set of equations:

$$EA_0 = 0, \quad E\tilde{A}_0 = -2m\tilde{\phi}, \quad E\phi = 2i\sqrt{2}(\vec{p}_i - i\vec{r}^i)F^{i0} \tag{5.7}$$

$$E\tilde{\phi} = -2m\tilde{A}_0 + \sqrt{2}\epsilon_{ijk}(\vec{p}_i + i\vec{r}^i)F^{jk} \tag{5.8}$$

$$E\tilde{A}^i = -i\sqrt{2}\epsilon_{ijk}(\vec{p}_j \mp i\vec{r}^j)A^k \tag{5.9}$$

$$EA^i = 4imF^{0i} + i\sqrt{2}\epsilon_{ijk}(\vec{p}_j \pm i\vec{r}^j)\tilde{A}^k \tag{5.10}$$

$$EF^{0i} = -2imA^i + i\sqrt{2}(\vec{p}_i + i\vec{r}^i)\phi \tag{5.11}$$

$$EF_{jk} = \frac{1}{\sqrt{2}}\epsilon_{ijk}(\vec{p}_i - i\vec{r}^i)\tilde{\phi} \tag{5.12}$$

Let us mention that for another type of Dirac oscillator-like interaction $\sim (\vec{\alpha}_1 - \vec{\alpha}_2)B\Gamma_5$ the only changes are the sign changes at the term $i\vec{r}$ in Eqs. (5.9) and (5.10) of the above system. The two-body Dirac oscillator equations in the form (5.7)-(5.12) could be uncoupled on the set containing only functions ϕ, $\tilde{\phi}$ and $\tilde{A}_\mu$ and the another one containing only A_μ and $F_{\mu\nu}$:

$$\text{1)} \qquad (E^2 - 8m^2)\phi = 4(\vec{p}_i - i\vec{r}^i)(\vec{p}_i + i\vec{r}^i)\phi - \frac{1}{E}\left\{\begin{pmatrix} 16m\epsilon_{ijk}\vec{r}^i\vec{p}_j \\ 0 \end{pmatrix}\right\}\tilde{A}^k \tag{5.13}$$

$$(E^2 - 4m^2)\tilde{\phi} = 2(\vec{p}_i + \vec{r}^i)(\vec{p}_i - i\vec{r}^i)\tilde{\phi} \tag{5.14}$$

$$E\tilde{A}_0 = -2m\tilde{\phi} \tag{5.15}$$

$$(E^2 - 8m^2)\tilde{A}^i = 2(\vec{p}_j \mp i\vec{r}^{\,j})(\vec{p}_i \pm i\vec{r}^{\,i})\tilde{A}^j - 2(\vec{p}_j \mp i\vec{r}^{\,j})(\vec{p}_j \pm i\vec{r}^{\,i})\tilde{A}^i +$$

$$+ \frac{1}{E}\left\{ \begin{pmatrix} 16m\epsilon_{ijk}\vec{r}^{\,j}\vec{p}_k \\ 0 \end{pmatrix} \right\}\phi \tag{5.16}$$

2) $$EA_0 = 0 \tag{5.17}$$

$$(E^2 - 8m^2)F^{0i} = 4(\vec{p}_i + i\vec{r}^{\,i})(\vec{p}_j - i\vec{r}^{\,j})F^{0j} -$$

$$-4i\frac{m}{E}(\vec{p}_j \pm i\vec{r}^{\,j})(\vec{p}_i \mp i\vec{r}^{\,i})A^j + 4i\frac{m}{E}(\vec{p}_j \pm i\vec{r}^{\,j})(\vec{p}_j \mp i\vec{r}^{\,j})A^i \tag{5.18}$$

$$E^2 A^i = 2(\vec{p}_j \pm i\vec{r}^{\,j})(\vec{p}_i \mp i\vec{r}^{\,i})A^j - 2(\vec{p}_j \pm i\vec{r}^{\,j})(\vec{p}_j \mp i\vec{r}^{\,j})A^i + 4imEF^{0i}$$

$$(E^2 - 4m^2)F^{jk} = \epsilon_{ijk}\epsilon_{lmn}(\vec{p}_i - i\vec{r}^{\,i})(\vec{p}_l + i\vec{r}^{\,l})F^{mn}. \tag{5.19}$$

This fact proves the Dirac oscillator interaction, like the case of introduction of electrodynamic interaction in the Proca or the Bargmann-Wigner equations, does not mix $S = 1$ and $S = 0$ states.

Next, the interaction term of the following form:

$$\mathcal{V}^{int} = \frac{1}{r}\frac{dV(r)/dr}{1 - [V(r)]^2}(\vec{\alpha}_1 - \vec{\alpha}_2)B\Gamma_5\vec{r} \tag{5.20}$$

has been deduced [12] from the equation of Relativistic Quantum Constraint Dynamics (RQCR) or N- particle Barut equation. In [12] it proved to lead to the Dirac oscillator-like interactions provided that the definite choice of the function $V(r)$. In connection with that let us remark the curious behavior of the another potential $V(r)$ which has been proposed in ref. [13b,c]:

$$V(r) = -g^2\frac{coth\,(rm\pi)}{4\pi r} = -g^2\frac{coth\,(\kappa r)}{4\pi r}. \tag{5.21}$$

It could be deduced from the one-boson exchange quasipotential $V(\vec{p}, \vec{k}; E) = -g^2(p - k)^{-2}$ by means of the transformation into the relativistic configurational representation (RCR) using the complete set of Shapiro plane-wave functions: $\xi(\vec{\Delta}, \vec{r}) = (\Delta_0 - \vec{\Delta}\vec{n}/m)^{-1-irm}$, $\Delta_0 = \sqrt{\vec{\Delta}^2 + m^2}$, $\vec{n} = \vec{r}/|\vec{r}|$.

In the case of the quasipotential (5.21) the interaction term $\mathcal{V}$, Eq. (5.20), has the different asymptotic behavior in three regions $(g^2/(4\pi) = 1)$. Namely,

$$\mathcal{V}^{int} \simeq \frac{1}{r(r^2 - 1)}(\vec{\alpha}_1 - \vec{\alpha}_2)B\Gamma_5\vec{r} \simeq$$

$$\simeq \begin{cases} (1/r^3)(\vec{\alpha}_1 - \vec{\alpha}_2)B\Gamma_5\vec{r}, & \text{if} \quad r \gg \frac{1}{\kappa} \quad \text{and} \quad r > 1 \\ -(1/r)(\vec{\alpha}_1 - \vec{\alpha}_2)B\Gamma_5\vec{r}, & \text{if} \quad \frac{1}{\kappa} \ll r < 1, \end{cases} \tag{5.22}$$

in the infrared region $(r \gg \frac{1}{\kappa}$, large distances); and

$$\mathcal{V}^{int} \simeq -2\kappa(\vec{\alpha}_1 - \vec{\alpha}_2)B\Gamma_5\vec{r}, \qquad \text{if} \quad r \ll \frac{1}{\kappa}, \tag{5.23}$$

in the ultraviolet region (small distances). In one of the regions one has the Dirac oscillator-like behavior.

Acknowledgments

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