REAL LASERS AND OTHER DEFORMED OBJECTS[1]

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Heard at the Second Harmonic Oscillator Conference:
"There are no harmonic oscillators in Nature." K.B. Wolf
"Tell that to the quantum opticians!" M.M. Nieto

Abstract

In this talk we re-examine three important properties of quantum laser systems:
(i) Photon counting statistics (ii) Squeezing (iii) Signal-to-Quantum Noise Ratio.
None of these phenomena depends on the choice of hamiltonian; indeed, we analyze them initially without restriction to any specific form of the commutation relations.

1 Introduction

Although most of the recent motivation for deforming the bosonic canonical commutation relations has been derived from considerations of theory, in this note we should like to take a different tack. To what extent does the assumption of modified (*deformed*) commutation relations lead to new, even non-intuitive, physical predictions? Ideally, such predictions should not be based on the choice of a specific hamiltonian, due to the additional ambiguity involved in such a choice; unfortunately, this rules out delicate tests involving frequency measurements, some of the most refined of physics. And, initially at any rate, it would be of interest to embark on the analysis without resorting to a specific form of deformed commutation relations, although ultimately any quantitative result will depend on a specific set.

With this *minimalist* philosophy in mind, let us consider the ingredients necessary for a theory of quantum photons. First of all, we need an operator a which annihilates photons one at a time; and its hermitian conjugate $a^\dagger$ which creates them. We also postulate a number operator N which counts photons; $N|n\rangle = n|n\rangle$. The set $\{|n\rangle; n = 0, \ldots,\}$ provides a denumerable basis for the Hilbert space (Fock space). Thus the number operator N satisfies $[N, a] = -a$, just as for the usual (non-deformed) boson operators. Necessarily, since the vacuum state $|0\rangle$ is defined to have no photons, $N|0\rangle = 0$ and $a|0>= 0$. Clearly the combination $a^\dagger a$ does not change the number of photons, so it commutes with N and must be a function of N. We write this function conventionally as $[N]$ (read "box N"). Thus we have

$$a^\dagger a = [N].$$

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Clearly $aa^\dagger$ is also a function of N; from evaluation of

$$a(a^\dagger a)|n> = (aa^\dagger)a|n> \tag{1}$$

this function may easily be seen to be

$$aa^\dagger = [N+1].$$

The generalized commutation relations may therefore be written

$$aa^\dagger - a^\dagger a = [N+1] - [N] \tag{2}$$

where [ ] is some (analytic) function.

For example, the two most commonly used deformations of the canonical commutation relations which have been considered are:

(a) "Maths" Boson:

$$aa^\dagger - qa^\dagger a = I. \tag{3}$$

This was introduced by Arik and Coon [1], who also described the corresponding q-coherent states. In the commutator form, this may be written as

$$aa^\dagger - a^\dagger a = q^N \tag{4}$$

where q is some real parameter. We refer to this deformed boson as a "Maths" (or M-) boson as the "basic" numbers (*cf.* Equation (15)) and special functions, q-functions, associated with this operator have been investigated in the mathematical literature for over 150 years; see, for example, [2].

(b) "Physics" Boson:

$$aa^\dagger - qa^\dagger a = q^{-N}. \tag{5}$$

In the commutator form, this may be written as

$$aa^\dagger - a^\dagger a = \cosh(2N+1)s/\cosh s \tag{6}$$

where $q = \exp(2s)$.

This deformation was introduced [3, 4] in order to provide a realization of the "quantum groups" [5] (non-cocommutative Hopf algebras) which arise naturally in the solution of certain lattice models [6].

An alternate formulation of Equation 2 is [7]

$$aa^\dagger - f(N)a^\dagger a = 1 \tag{7}$$

with the correspondence [8]

$$\begin{aligned}[n] &= 1 + f(n-1) + f(n-1)f(n-2) + f(n-1)f(n-2)f(n-3) \\ &\quad + \cdots + f(n-1)f(n-2)\cdots f(2)f(1) \end{aligned} \tag{8}$$

$$= \sum_{k=0}^{n-1} \frac{f(n-1)!}{f(k)!}. \tag{9}$$

Following the pioneering work of Jackson, we may introduce a generalized calculus related to our general deformation characterized by the analytic function [ ]. We define an operator D_x such that

$$D_x = \frac{1}{x}\left[x\frac{d}{dx}\right]. \tag{10}$$

This acts as a generalized derivative operator, e.g.

$$D_x x^n = [n]x^{n-1}. \tag{11}$$

The eigenfunction $E(x)$ of D_x given by

$$E(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]!}. \tag{12}$$

is well–defined provided the function [ ] satisfies appropriate convergence criteria. This plays the role of a generalized exponential function.

A related generalized quantum optics may be described [8], starting with the generalized coherent states $|\lambda\rangle$ defined to satisfy

$$a|\lambda\rangle = \lambda|\lambda\rangle. \tag{13}$$

Since $a\,E(\lambda a^\dagger)|0\rangle = \lambda E(\lambda a^\dagger)|0\rangle$, we can use $E(x)$ to define analogues of coherent states as normalized eigenstates of the generalized annihilation operator.

$$|\lambda\rangle = \{E(|\lambda|^2)\}^{-\frac{1}{2}} E(\lambda a^\dagger)|0\rangle. \tag{14}$$

The q-coherent states associated with the special cases of the bosons described by Equation 3 and Equation 5 have been investigated by several authors e.g. [4, 9]. For these two special cases, $[n]$ is given by

$$[n] = \begin{cases} \dfrac{1-q^n}{1-q} & \text{M-case} \\[2ex] \dfrac{q^n - q^{-n}}{q - q^{-1}} & \text{P-case} \end{cases} \tag{15}$$

We now consider in turn each of three phenomena in quantum optics from our new generalized viewpoint:

- Photon counting statistics
- Squeezing
- Signal-to-Quantum Noise Ratio.

2 Photon Counting Statistics

The states of an ideal laser are conventionally described by Glauber coherent states [10]. However, real lasers do not strictly adhere to this description; in particular, the photon number statistics of real lasers are not exactly Poissonian [11]. Furthermore, various non-linear interactions give rise to well-defined deviations from the Poissonian distribution [12]. Recently, deformations of the commutation rules of boson operators have been considered as models for physical systems which deviate from the ideal cases [13]. We approach the problem of the "real" laser in this latter phenomenological spirit, and show that indeed a coherent state of the deformed boson (q-coherent state) provides a more accurate model of a non-ideal laser, at least as far as the photon number statistics is concerned.

An *ideal* laser may be described as a normalized eigenstate of the photon annihilation operator a, where a and its hermitian conjugate $a^\dagger$ (photon creation operator) satisfy

$$[a, a^\dagger] \equiv aa^\dagger - a^\dagger a = I. \tag{16}$$

The normalized eigenstate satisfying $a|\alpha >= \alpha|\alpha >$ is easily seen to be

$$|\alpha >= \exp(-\frac{|\alpha|^2}{2}) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n > . \tag{17}$$

The number eigenstates are $|n >$, and this coherent state gives rise to the Poisson distribution

$$P_n = | < n|\alpha > |^2 = \exp(-|\alpha|^2) \frac{|\alpha|^{2n}}{n!}. \tag{18}$$

The factorial moments of this distribution are

$$< n >= |\alpha|^2$$

$$< n(n-1) >= |\alpha|^4$$

$$< n(n-1)(n-2) >= |\alpha|^6$$

etc., from which the variance is found to be

$$\sigma^2 =< n^2 > - < n >^2 = |\alpha|^2$$

A convenient measure of the deviation of a distribution from the Poisson distribution is the Mandel parameter

$$Q = \frac{\sigma^2}{< n >} - 1 = \frac{< n(n-1) >}{< n >} - < n >$$

which vanishes for the Poisson distribution, is positive for a super-Poissonian distribution, and negative for a sub-Poissonian distribution.

In order to enter into the phenomenological spirit of our approach, and to compare with the experimental data, we need to specify the form of the commutation relations Equation 2; that is, specify a choice of the function [ ]. It is sufficient for our purposes here to compare the distributions arising from the M and P forms Equation 3 and Equation 5 respectively. One can easily check that the P-type q-Poisson distribution is sub-Poissonian ($Q \leq 0$) for all values of q, reducing to the conventional Poisson distribution for $q = 1$. On the other hand, the M-type q-Poisson distribution is super-Poissonian for $q < 1$ and sub-Poissonian for $q > 1$.

The q-Poissonian q-factorial moments are $< [n] >= |\alpha|^2$, $< [n][n-1] >= |\alpha|^4$, etc.

To evaluate the average number of photons and the Mandel parameter for the q-Poisson distribution we note that the corresponding factorial moments satisfy

$$< n > \; = \; \frac{x}{E_q(x)} \cdot \frac{\partial E_q(x)}{\partial x}\bigg|_{x=|\alpha|^2}$$

$$< n(n-1) > \; = \; \frac{x^2}{E_q(x)} \cdot \frac{\partial^2 E_q(x)}{\partial^2 x}\bigg|_{x=|\alpha|^2}$$

These expressions may be used to provide estimates of the q-Poissonian parameters q and $|\alpha|^2$ corresponding to a distribution which is specified in terms of given values of $< n >$ and Q . The values of q corresponding to given pairs of values of $< n >$ and Q , and the corresponding values of $|\alpha|^2$ were tabulated in reference [14].

For small deviations from a Poissonian distribution we define $q = e^{-s}$ and obtain in the M-case

$$s = \frac{2Q}{< n >}$$

which is positive (*i.e.,* $q < 1$) for a super-Poissonian distribution and negative ($q > 1$) for a sub-Poissonian distribution. In the P-case we obtain

$$s^2 = -\frac{3Q}{< n > (< n > + \frac{3}{2})}$$

so that only the sub-Poissonian distribution ($Q < 0$) corresponds to a real value of s (and q).

Another useful result is

$$\rho \equiv \lim_{<n> \to 0} \frac{Q}{< n >} = \begin{cases} \dfrac{1-q}{1+q} & \text{M-case} \\ \\ \dfrac{2}{q+q^{-1}} - 1 & \text{P-case} \end{cases} \tag{19}$$

In the M-case the range of ρ is $-1 < \rho < 1$, corresponding to a sub-Poissonian distribution for $\rho < 0$ and to a super-Poissonian distribution for $\rho > 0$. In the P-case the range of ρ is $-1 < \rho \leq 0$, exhibiting only a sub-Poissonian distribution.

From Equation (19) we obtain

$$q = \begin{cases} \dfrac{1-\rho}{1+\rho} & \text{M-case} \\ \\ \dfrac{1}{1+\rho} \pm \sqrt{\dfrac{1}{(1+\rho)^2} - 1} & \text{P-case} \end{cases} \tag{20}$$

Using the three highest peaks in the experimental data pertaining to the photon statistics of a He-Ne laser just above threshold [15] we obtain $\frac{P_2^2}{P_1 P_3} = \frac{[3]}{[2]} = 1.319$, which in the M-case is a quadratic equation in q, yielding $q = 0.747$. Note that the corresponding equation for the P-case can be shown to rule out the P-boson as a model of this system since for all real and positive q the inequality $\frac{[3]}{[2]} \geq \frac{3}{2}$ holds.

If one compares the best fit for the M-boson q-coherent state against the experimental data [15] and the ideal (Glauber) coherent state, one finds that the value of q corresponding to the best fit is 0.749, in very close agreement with the value estimated above using the highest three peaks. It is not surprising that a better fit is obtained with the q-coherent state, due to the extra parameter q. However, certain constraints are satisfied (for example, the convergence criterion for the M-type q-exponential function demands that $(1-q)|\alpha|^2 \leq 1$ and is satisfied here) and, as we have already remarked, the P-boson model is ruled out.

Experimental studies of the photon statistics of a laser at different intensities above the threshold were reported in refs. [16] and [17]. Since super-Poissonian statistics is exhibited, only M-type analysis is warranted. In both cases it is found that for counting times short relative to the intensity correlation time the distributions agree with q-Poissonian statistics, the value of q increasing from a value which could be close to zero at threshold to a value close to unity (Poissonian distribution) for intensities about an order of magnitude higher than the threshold intensity. At twice the threshold intensity values of q ranging between roughly 0.3 and 0.8 were obtained from the different sets of experimental data.

Another set of experimental data, exhibiting a sub-Poissonian distribution, involves the photons emitted by single-atom resonance fluorescence [18]. Using the data for P_0, P_1, P_2 we obtained in Reference [14] $q_M = 2.44$ or $q_P = 3.12$. This is in agreement with the estimate for q_M obtained using Equation (20) and the data reported in [18], $< n >= 6.23 \cdot 10^{-3}$ and $Q = -2.52 \cdot 10^{-3}$, from which $q_M = 2.36$.

The examples of this section illustrate cases from quantum optics where a more accurate model of a physical system may be obtained by use of quantum group ideas.

3 Squeezing

The electromagnetic field components x and p are given by

$$x = \frac{1}{\sqrt{2}}(a + a^\dagger) \quad \text{and} \quad p = \frac{1}{i\sqrt{2}}(a - a^\dagger). \tag{21}$$

As usual, we define the variances $(\Delta\, x)$ and $(\Delta\, p)$ by

$$(\Delta\, x)^2 = \langle x^2 \rangle - \langle x \rangle^2 \quad \text{and} \quad (\Delta\, p)^2 = \langle p^2 \rangle - \langle p \rangle^2. \tag{22}$$

In the vacuum state

$$(\Delta\, x)_0 = \frac{1}{\sqrt{2}} \quad \text{and} \quad (\Delta\, p)_0 = \frac{1}{\sqrt{2}}. \tag{23}$$

and so

$$(\Delta\, x)_0 (\Delta\, p)_0 = \frac{1}{2}. \tag{24}$$

The commutation relation Equation (16) for a and $a^\dagger$ leads to the following uncertainty principle

$$(\Delta\, x)(\Delta\, p) \geq \frac{1}{2} |\langle [x, p] \rangle| = \frac{1}{2}. \tag{25}$$

Thus the vacuum state attains the lower bound for the uncertainty, as do the coherent states.

While it impossible to lower the product $(\Delta x)(\Delta p)$ below the vacuum uncertainty value, it is nevertheless possible to define *squeezed* states [19] for which (at most) one quadrature lies below the vacuum value, i.e.

$$(\Delta x) < (\Delta x)_0 = \frac{1}{\sqrt{2}} \quad \text{or} \quad (\Delta p) < (\Delta p)_0 = \frac{1}{\sqrt{2}}. \tag{26}$$

If we now consider the generalized bosonic operators given by (2), using the same definitions for the the field quadratures, x and p, as in (21) we find that, just as in the conventional case, the vacuum uncertainty product $(\Delta x)_0(\Delta p)_0 = \frac{1}{2}$ is a lower bound for all *number* states.

However, unlike the conventional case, it is not a global lower bound.

Consider the quadrature values in eigenstates of the generalized annihilation operator.

Then

$$\langle x\rangle_\lambda = \langle\lambda|\frac{1}{\sqrt{2}}(a^\dagger + a)|\lambda\rangle = \frac{1}{\sqrt{2}}(\lambda + \bar{\lambda}) \tag{27}$$

and

$$\langle x^2\rangle_\lambda = \langle\lambda|\ \frac{1}{2}((a^\dagger)^2 + a^2 + a^\dagger a + aa^\dagger)\ |\lambda\rangle \tag{28}$$

$$= \frac{1}{2}\{(\bar{\lambda} + \lambda)^2 + 1 - \varepsilon_{f,\lambda}|\lambda|^2\} \tag{29}$$

where

$$\varepsilon_{f,\lambda} = 1 - \langle f(N+1)\rangle_\lambda. \tag{30}$$

If we choose $0 < f(n) < 1$, then it can be shown that $\varepsilon_{f,\lambda}|\lambda|^2 \in (0,1)$ for λ within the radius of convergence of the generalized exponential (12).

Hence

$$(\Delta x)^2_\lambda = \frac{1}{2}\{1 - \varepsilon_{f,\lambda}|\lambda|^2\}. \tag{31}$$

Evaluating the variance for the other component, we find that $(\Delta p)^2_\lambda = (\Delta x)^2_\lambda$ so

$$(\Delta x)_\lambda(\Delta p)_\lambda = \frac{1}{2}\{1 - \varepsilon_{f,\lambda}|\lambda|^2\} < \frac{1}{2}. \tag{32}$$

However, it can also be shown that

$$\frac{1}{2}\{1 - \varepsilon_{f,\lambda}|\lambda|^2\} = \frac{1}{2}|\langle[x,p]\rangle_\lambda| \tag{33}$$

so

$$(\Delta x)_\lambda(\Delta p)_\lambda = \frac{1}{2}|\langle[x,p]\rangle_\lambda| \tag{34}$$

Thus we see that these generalized q–coherent states satisfy a restricted form of the Minimum Uncertainty Property (M.U.P.) of the conventional coherent states. Additionally we see that there is a general noise reduction in both quadratures compared to their vacuum value. In conventional coherent states there is no noise reduction relative to the vacuum value. In conventional squeezed states, there is noise reduction in only one component.

We can apply the preceding analysis to the two usual forms of q-deformed bosons:

(a) 'Physics' q–bosons
First consider the q-bosons of Equation 5. The deformed commutation relation

$$aa^\dagger - q\,a^\dagger a = q^{-N}. \tag{35}$$

can be rewritten [20] as

$$aa^\dagger - f(N)\,a^\dagger a = 1 \tag{36}$$

where $f(N) = \frac{q^{N+2}+1}{q(q^N+1)}$.

In this case, for normalizable eigenstates, the function $\varepsilon_{f,\lambda}$ is negative and so simultaneous two–component noise reduction does not take place. This is in agreement with the findings of Katriel and Solomon [21] and Chiu et al [22]. However, it can be shown that ordinary *squeezing* i.e. noise reduction in one component compared to the vacuum (with a corresponding noise amplification in the other component) does take place [23, 24].

(b) 'Maths' q–bosons
We now consider the q-boson described by Arik and Coon [1]. which is characterised by the deformed commutation relation

$$aa^\dagger - q\,a^\dagger a = 1 \tag{37}$$

For $q \in (0,1)$, the Jackson q-exponential $E_q(|\lambda|^2)$ converges, provided

$$\varepsilon_q|\lambda|^2 = (1-q)|\lambda|^2 < 1.$$

Given this condition on λ, we have normalizable q-analogue coherent states satisfying (13) in which

$$(\Delta\,x)^2_\lambda = (\Delta\,p)^2_\lambda = (\Delta\,x)_\lambda(\Delta\,p)_\lambda = \frac{1}{2}\{1-\varepsilon_q|\lambda|^2\} < \frac{1}{2}. \tag{38}$$

Hence, for this type of q–boson, we do obtain noise reduction in both quadratures with respect to the vacuum value.

4 Signal-to-Quantum Noise Ratio

In a classic paper, Yuen [19] showed that for a radiation field of photons the maximum signal-to-quantum noise ratio ρ for fixed energy has the value $4n_s(n_s+1)$, where n_s gives the upper limit on the number of photons in the signal (effectively a maximum power per unit frequency constraint).

The only mathematical input to this result consisted of the canonical commutation relations for the photon annihilation operato r a, namely;

$$[a, a^\dagger] = 1 \tag{39}$$

with the photon number operator given by $N = a^\dagger a$.

The hermitian components x, p of the electromagnetic field corresponding to our *generalized* photons of Equation 2 (which we now write as $a_q^\dagger, a_q$ to distinguish from the conventional ones) satisfy

$$[x_q, p_q] = i([N+1] - [N]) \tag{40}$$

which reduces to the canonical commutation relation $[x, p] = i$ when $[N] = N$.

We now consider a state, which we write as $<, >$, although everything which follows applies equally to a general state described by a density function. Introducing the hermitian operators

$$X \equiv x_q - < x_q >, \quad P \equiv p_q - < p_q >,$$

the quantum dispersion (quantum noise) in each of the components is measured by the quantities $(\Delta x_q)^2 \equiv < X^2 >$ and $(\Delta p_q)^2 \equiv < P^2 >$. The positivity of the number $< A(t)A^\dagger(t) >$ for all t, where $A(t) \equiv tX + iP$, leads immediately to the modified uncertainty principle

$$(\Delta x_q)^2 (\Delta p_q)^2 \;\; \geq \;\; \frac{1}{4} < [N+1] - [N] >^2 . \tag{41}$$

This uncertainty product exceeds the conventional value of $\frac{1}{4}$ in the "Physics" case (5), and in the "Maths" case (3) for $q \geq 1$.

The signal-to-quantum noise ratio

$$\rho_q \equiv < x_q >^2 / (\Delta x_q)^2$$

must be maximized subject to the constraint

$$< a_q^\dagger a_q > \;\; \leq \;\; [n_s] \tag{42}$$

where n_s is the maximum number of q-photons for the frequency under consideration, and inequality (41) above. We may rewrite constraint (42) as

$$< x_q >^2 + < p_q >^2 + (\Delta x_q)^2 + (\Delta p_q)^2 - < [N+1] - [N] > \;\; \leq \;\; 2[n_s] \tag{43}$$

where we have substituted

$$< x_q^2 > = < x_q >^2 + (\Delta x_q)^2, \quad < p_q^2 > = < p_q >^2 + (\Delta p_q)^2.$$

Consideration of (43) leads us to infer that it is favourable to use all the available energy; that is, $< N > = n_s$: and to use it in the x-component alone, so that $< p_q > = 0$. The inequality thus becomes the equation

$$< x_q >^2 + (\Delta x_q)^2 + (\Delta p_q)^2 = [n_s] + [n_s + 1]. \tag{44}$$

It is a straightforward exercise in the calculus to show that the ratio ρ_q is maximized, subject to the constraints (41) and (44), at a value

$$\rho_q = 4[n_s][n_s + 1]/([n_s + 1] - [n_s])^2. \tag{45}$$

Given two types of "photon" described by $[\ ]_1$ and $[\ ']_2$, it is a straightforward exercise in inequalities to show that the corresponding signal-to-quantum noise ratios ρ_1, ρ_2 satisfy

$$\rho_1 \quad \leq \rho_2 \quad \text{if} \quad \frac{[n+1]_1}{[n]_1} \quad \geq \frac{[n+1]_2}{[n]_2}.$$

Taking $[n]_2 = n$ ("ordinary" photons) and $[n]_1$ as the q-photons defined by Equations (3) and (5) in turn, we obtain:

$$\rho^M_{q\leq 1} \ \geq \ \rho \ \ \geq \rho^P \ \geq \rho^M_{q\geq 1}$$

on comparing with Yuen's result for the conventional case

$$\rho = 4n_s(n_s + 1). \tag{46}$$

Therefore states based on the usual q-photons Equation (5), and Equation (3) for $q \geq 1$, (which are the more physical cases satisfying the conventional uncertainty principle) will not lead to an enhanced signal-to-quantum noise ratio over the conventional photon case.

5 Conclusions

In this talk we have given three examples where we are able to model physically observable properties of real photons by means of deformed photons satisfying very general deformations of the canonical commutation relations. The viewpoint we have adopted is the phenomenological one; we do not assume that "real" photons satisfy other than the conventional commutation relations. Rather, we have shown that simple models involving "dressed" photons, satisfying very general constraints, may be invoked to describe observed, and sometimes non-intuitive, phenomena.

This by no means addresses the still open question as to whether deformed commutation relations describe real particles, whatever that means.

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